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PROPERTIES OF BIHARMONIC SUBMANIFOLDS IN SPHERES*

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Abstract. In the present paper we survey the most recent classification results for proper biharmonic submanifolds in unit Euclidean spheres. We also obtain some new results concerning geometric properties of proper biharmonic constant mean curvature submanifolds in spheres.

1. Introduction

Biharmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are critical points of the **bienergy functional**

$$E_2(\phi) = \frac{1}{2}\int_M \|\tau(\phi)\|^2\, v_g$$

where $\tau(\phi) = \operatorname{tr} \nabla \mathrm{d}\phi$ is the tension field of ϕ that vanishes for harmonic maps (see [17]). The Euler-Lagrange equation corresponding to E_2 is given by the vanishing of the **bitension field**

$$\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = -\Delta\tau(\phi) - \operatorname{tr} R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi$$

where J^{ϕ} is formally the Jacobi operator of ϕ (see [24]). The operator J^{ϕ} is linear, thus any harmonic map is biharmonic. We call **proper biharmonic** the non-harmonic biharmonic maps. Geometric and analytic properties of proper biharmonic maps were studied, for example, in [2, 25, 27].

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The submanifolds with non-harmonic (non-minimal) biharmonic inclusion map are called **proper biharmonic submanifolds**. Initially encouraged by the non-existence results for proper biharmonic submanifolds in non-positively curved space forms (see, for example [8, 13, 16, 21]), the study of proper biharmonic submanifolds in spheres constitutes an important research direction in the theory of proper biharmonic submanifolds.

The present paper is organized as follows.

Section 2 is devoted to the main examples of proper biharmonic submanifolds in spheres and to their geometric properties, mainly regarding the type and the order in the sense of Chen. Also, it gathers the most recent classification results for such submanifolds (for detailed proofs see [3]).

In Section 3 we present a series of new results concerning geometric properties of proper biharmonic constant mean curvature submanifolds in spheres. We begin with some identities which hold for proper biharmonic submanifolds with parallel mean curvature vector field (Propositions 2 and 3). We then obtain some necessary conditions that must be fulfilled by proper biharmonic constant mean curvature submanifolds (Corollary 1) and we end this section with a refinement, for hypersurfaces, of a result giving an estimate on the mean curvature of proper biharmonic submanifolds in spheres (Theorem 17).

The fourth section presents two open problems concerning the classification of proper biharmonic hypersurfaces and the mean curvature of proper biharmonic submanifolds in spheres.

In the last section we briefly present an interesting link between proper biharmonic hypersurfaces and II-minimal hypersurfaces in spheres.

Other results on proper biharmonic submanifolds in spaces of non-constant sectional curvature can be found, for example, in [10, 15, 18, 19, 22, 30, 31].

2. Proper Biharmonic Submanifolds in Spheres

The attempt to obtain classification results for proper biharmonic submanifolds in spheres was initiated with the following characterization theorem.

Theorem 1 ([28]). i) *The canonical inclusion $\phi : M^m \to \mathbb{S}^n$ of a submanifold M in an n-dimensional unit Euclidean sphere is biharmonic if and only if*

$$\begin{aligned} \Delta^{\perp} H + \operatorname{tr} B(\cdot, A_H(\cdot)) - mH &= 0 \\ 4 \operatorname{tr} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + m \operatorname{grad}(\|H\|^2) &= 0 \end{aligned} \tag{1}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $\nabla^{\perp}$ and $\Delta^{\perp}$ are the connection and the Laplacian in the normal bundle of M in $\mathbb{S}^n$, and grad *denotes the gradient on M.*

If M is a submanifold with parallel mean curvature vector field in $\mathbb{S}^n$, then M is biharmonic if and only if $\operatorname{tr} B(\cdot, A_H(\cdot)) = mH$.

ii) *A hypersurface M with nowhere zero mean curvature vector field in $\mathbb{S}^{m+1}$ is biharmonic if and only if*

$$\begin{aligned} &\Delta^{\perp} H - (m - \|A\|^2)H = 0 \\ &2A(\operatorname{grad}(\|H\|)) + m\|H\| \operatorname{grad}(\|H\|) = 0. \end{aligned} \tag{2}$$

If M is a non-zero constant mean curvature hypersurface in $\mathbb{S}^{m+1}$, then M is proper biharmonic if and only if $\|A\|^2 = m$.

We note that the compact minimal, i.e., $H = 0$, hypersurfaces with $\|A\|^2 = m$ in $\mathbb{S}^{m+1}$ are just the Clifford tori $\mathbb{S}^k(\sqrt{k/m}) \times \mathbb{S}^{m-k}(\sqrt{(m-k)/m})$, $1 \leq k \leq m-1$ (see [14]).

Before presenting some basic examples of proper biharmonic hypersurfaces in spheres, together with some of their geometric properties, we recall the following definition (see, for example, [12]), which shall be used throughout the paper.

Definition 1. *An isometric immersion of a compact manifold M in $\mathbb{R}^n$ specified by the map $\varphi : M \to \mathbb{R}^n$, is called of k-type if its spectral decomposition contains exactly k non-zero terms, excluding the center of mass $\varphi_0 = \frac{1}{\operatorname{Vol}(M)} \int_M \varphi \, v_g$. More precisely*

$$\varphi = \varphi_0 + \sum_{t=p}^{q} \varphi_t$$

where $\Delta \varphi_t = \lambda_t \varphi_t$ and $0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty$.

The pair $[p, q]$ is called the order of the immersion $\varphi : M \to \mathbb{R}^n$.

2.1. The Main Examples of Proper Biharmonic Submanifolds in Spheres

The Hypersphere $\mathbb{S}^m(1/\sqrt{2}) \subset \mathbb{S}^{m+1}$.

Consider $\mathbb{S}^m(a) = \left\{(x^1, \ldots, x^m, x^{m+1}, b) \in \mathbb{R}^{m+2}; \|x\| = a\right\} \subset \mathbb{S}^{m+1}$, where $a^2 + b^2 = 1$. If H is the mean curvature vector field of $\mathbb{S}^m(a)$ in $\mathbb{S}^{m+1}$, one gets $\nabla^{\perp} H = 0$, $\|H\| = \frac{|b|}{a}$ and $\|A\|^2 = m\frac{b^2}{a^2}$.

Theorem 1 implies that $\mathbb{S}^m(a)$ is proper biharmonic in $\mathbb{S}^{m+1}$ if and only if $a = 1/\sqrt{2}$ (see [9]).

The Generalized Clifford Torus $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2}) \subset \mathbb{S}^{m+1}$.

The generalized Clifford torus, $M = \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$, was the first example of proper biharmonic submanifold in $\mathbb{S}^{m+1}$ (see [24]).

Consider

$$\begin{aligned} M &= \left\{(x^1,\dots,x^{m_1+1},y^1,\dots,y^{m_2+1})\in\mathbb{R}^{m+2};\ \|x\|=a_1,\|y\|=a_2\right\} \\ &= \mathbb{S}^{m_1}(a_1)\times\mathbb{S}^{m_2}(a_2)\subset\mathbb{S}^{m+1} \end{aligned}$$

where $a_1^2+a_2^2=1$. Then $\nabla^{\perp}H=0$, $\|H\|=\frac{1}{a_1a_2m}|a_2^2m_1-a_1^2m_2|$ and $\|A\|^2=\left(\frac{a_2}{a_1}\right)^2m_1+\left(\frac{a_1}{a_2}\right)^2m_2$.

From Theorem 1 it follows that M is a proper biharmonic in $\mathbb{S}^{m+1}$ if and only if $a_1=a_2=1/\sqrt{2}$ and $m_1\neq m_2$ (see, also, [8]).

Inspired by these basic examples, two methods for constructing proper biharmonic submanifolds of codimension higher than one in $\mathbb{S}^n$ were given.

Theorem 2 ([8]). *Let M be a minimal submanifold of $\mathbb{S}^{n-1}(a)\subset\mathbb{S}^n$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if $a=1/\sqrt{2}$.*

Remark 1. i) This result, called the **composition property**, proved to be quite useful for the construction of proper biharmonic submanifolds in spheres. For instance, it implies the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$ (see [8]).

ii) All minimal submanifolds of $\mathbb{S}^{n-1}(1/\sqrt{2})\subset\mathbb{S}^n$ are pseudo-umbilical, i.e., $A_H=\|H\|^2\,\mathrm{Id}$, with parallel mean curvature vector field in $\mathbb{S}^n$ and $\|H\|=1$.

iii) Denote by $\phi:\mathbb{S}^m(1/\sqrt{2})\to\mathbb{S}^{m+1}$ the inclusion of $\mathbb{S}^m(1/\sqrt{2})$ in $\mathbb{S}^{m+1}$ and by $\mathbf{i}:\mathbb{S}^{m+1}\to\mathbb{R}^{m+2}$ the canonical inclusion. Let $\varphi:\mathbb{S}^m(1/\sqrt{2})\to\mathbb{R}^{m+2}$, $\varphi=\mathbf{i}\circ\phi$, be the inclusion of $\mathbb{S}^m(1/\sqrt{2})$ in $\mathbb{R}^{m+2}$. Then

$$\varphi=\varphi_0+\varphi_p \tag{3}$$

where $\varphi_0,\varphi_p:\mathbb{S}^m(1/\sqrt{2})\to\mathbb{R}^{m+2}$, $\varphi_0(x,1/\sqrt{2})=(0,1/\sqrt{2})$, $\varphi_p(x,1/\sqrt{2})=(x,0)$ and $\Delta\varphi_p=2m\varphi_p$.

Thus $\mathbb{S}^m(1/\sqrt{2})$ is a 1-type submanifold of $\mathbb{R}^{m+2}$ with center of mass in $\varphi_0=(0,1/\sqrt{2})$ and eigenvalue $\lambda_p=2m$, which is the first eigenvalue of the Laplacian on $\mathbb{S}^m(1/\sqrt{2})$, i.e., $p=1$.

Moreover, it is not difficult to verify that all minimal submanifolds in $\mathbb{S}^m(1/\sqrt{2})\subset\mathbb{S}^{m+1}$, as submanifolds in $\mathbb{R}^{m+2}$, have the spectral decomposition given by (3).

Non pseudo-umbilical examples were also produced by proving the following **product composition property**.

Theorem 3 ([8]). *Let $M_1^{m_1}$ and $M_2^{m_2}$ be two minimal submanifolds of $\mathbb{S}^{n_1}(a_1)$ and $\mathbb{S}^{n_2}(a_2)$, respectively, where $n_1+n_2=n-1$, $a_1^2+a_2^2=1$. Then $M_1\times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if $a_1=a_2=1/\sqrt{2}$ and $m_1\neq m_2$.*

Remark 2. i) The proper biharmonic submanifolds of $\mathbb{S}^n$ constructed as above are not pseudo-umbilical, but they still have parallel mean curvature vector field, thus constant mean curvature, and $\|H\| = \frac{|m_2-m_1|}{m_1+m_2} \in (0,1)$.
ii) Let $\varphi : \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2}) \to \mathbb{R}^{m+2}$ be the inclusion of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$ in $\mathbb{R}^{m+2}$, $m_1 < m_2$, $m_1 + m_2 = m$. Then

$$\varphi = \varphi_p + \varphi_q \tag{4}$$

where $\varphi_p, \varphi_q : \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2}) \to \mathbb{R}^{m+2}$, $\varphi_p(x,y) = (x,0)$, $\varphi_q(x,y) = (0,y)$ and $\Delta\varphi_p = 2m_1\varphi_p$, $\Delta\varphi_q = 2m_2\varphi_q$.
Thus $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$ is a 2-type submanifold of $\mathbb{R}^{m+2}$ with eigenvalues $\lambda_p = 2m_1$ and $\lambda_q = 2m_2$, and it is mass-symmetric, i.e., its center of mass is the origin.
Since the eigenvalues of the torus $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$ are obtained as sums of the eigenvalues of the spheres $\mathbb{S}^{m_1}(1/\sqrt{2})$ and $\mathbb{S}^{m_2}(1/\sqrt{2})$, we conclude that $p = 1$. Also, $q = 2$, i.e., $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$ has order $[1,2]$ in $\mathbb{R}^{m+2}$, if and only if $m_2 \leq 2(m_1+1)$. Note that this holds, for example, for $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2}) \subset \mathbb{S}^4$.
Moreover, it can be easily proved that all proper biharmonic submanifolds in $\mathbb{S}^{m+1}$ obtained by means of the product composition property, as submanifolds in $\mathbb{R}^{m+2}$, have the spectral decomposition given by (4).

Other Examples of Proper Biharmonic Immersed Submanifolds in Spheres

In [32] and [1] the authors studied the proper biharmonic Legendre immersed surfaces and the proper biharmonic three-dimensional anti-invariant immersed submanifolds in Sasakian space forms. They obtained the explicit representations of such submanifolds in the unit Euclidean five-dimensional sphere $\mathbb{S}^5$.

Theorem 4 ([32])**.** *Let $\phi : M^2 \to \mathbb{S}^5$ be a proper biharmonic Legendre immersion. Then the position vector field $\varphi = \imath \circ \phi = \varphi(u,v)$ of M in $\mathbb{R}^6$ is given by*

$$\varphi(u,v) = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}u}, \mathrm{i}\mathrm{e}^{-\mathrm{i}u}\sin\sqrt{2}v, \mathrm{i}\mathrm{e}^{-\mathrm{i}u}\cos\sqrt{2}v)$$

where $\imath : \mathbb{S}^5 \to \mathbb{R}^6$ is the canonical inclusion.

Remark 3. The map ϕ is a full proper biharmonic Legendre embedding of a two-dimensional flat torus $\mathbb{R}^2/\Lambda$ into $\mathbb{S}^5$, where the lattice Λ is generated by $(2\pi, 0)$ and $(0, \sqrt{2}\pi)$. It has constant mean curvature $\|H\| = 1/2$, it is not pseudo-umbilical and its mean curvature vector field is not parallel. Moreover, $\varphi = \varphi_p + \varphi_q$, where

$$\varphi_p(u,v) = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}u}, 0, 0)$$

$$\varphi_q(u,v) = \frac{1}{\sqrt{2}}(0, \mathrm{i}e^{-\mathrm{i}u}\sin\sqrt{2}v, \mathrm{i}e^{-\mathrm{i}u}\cos\sqrt{2}v)$$

and $\Delta\varphi_p = \varphi_p$, $\Delta\varphi_q = 3\varphi_q$. Thus φ is a 2-type immersion in $\mathbb{R}^6$ with eigenvalues 1 and 3. In this case, $p = 1$ and $q = 3$, i.e., φ is a $[1,3]$-order immersion in $\mathbb{R}^6$.

Theorem 5 ([1]). *Let $\phi : M^3 \to \mathbb{S}^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $\varphi = \imath \circ \phi = \varphi(u,v,w)$ of M in $\mathbb{R}^6$ is given by*

$$\varphi(u,v,w) = \frac{1}{\sqrt{2}}e^{\mathrm{i}w}(e^{\mathrm{i}u}, \mathrm{i}e^{-\mathrm{i}u}\sin\sqrt{2}v, \mathrm{i}e^{-\mathrm{i}u}\cos\sqrt{2}v).$$

Remark 4. The map ϕ is a full proper biharmonic anti-invariant immersion from a three-dimensional flat torus $\mathbb{R}^3/\Lambda$ into $\mathbb{S}^5$, where the lattice Λ is generated by $(2\pi, 0, 0)$, $(0, \sqrt{2}\pi, 0)$ and $(0, 0, 2\pi)$. It has constant mean curvature $\|H\| = 1/3$, and it is not pseudo-umbilical, but its mean curvature vector field is parallel. Moreover, $\varphi = \varphi_p + \varphi_q$, where

$$\varphi_p(u,v,w) = \frac{1}{\sqrt{2}}e^{\mathrm{i}w}(e^{\mathrm{i}u}, 0, 0)$$

$$\varphi_q(u,v,w) = \frac{1}{\sqrt{2}}e^{\mathrm{i}w}(0, \mathrm{i}e^{-\mathrm{i}u}\sin\sqrt{2}v, \mathrm{i}e^{-\mathrm{i}u}\cos\sqrt{2}v)$$

and $\Delta\varphi_p = 2\varphi_p$, $\Delta\varphi_q = 4\varphi_q$. Thus φ is a 2-type submanifold of $\mathbb{R}^6$ with eigenvalues 2 and 4. It is easy to verify that φ is a $[2,4]$-order immersion in $\mathbb{R}^6$.

Since the immersion ϕ has parallel mean curvature vector field, one could ask whether its image arises by means of the product composition property. Indeed, it can be proved that, up to an orthogonal transformation of $\mathbb{R}^6$ which commutes with the usual complex structure, ϕ covers twice the proper biharmonic submanifold $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/2) \times \mathbb{S}^1(1/2) \subset \mathbb{S}^5$.

2.2. Classification Results

Some of the techniques used in order to obtain non-existence results in the case of non-positively curved space forms were adapted to the study of proper biharmonic submanifolds in spheres. Thus, in order to approach the classification problem for proper biharmonic hypersurfaces in spheres, the study was divided according to the number of principal curvatures. For submanifolds of higher codimension, supplementary conditions on the mean curvature vector field were imposed. This led to a series of rigidity results, which we enumerate below.

2.2.1. Proper Biharmonic Hypersurfaces

First, if M is a proper biharmonic umbilical hypersurface in $\mathbb{S}^{m+1}$, i.e., all of its principal curvatures are equal, then it is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

Afterwards, proper biharmonic hypersurfaces with at most two distinct principal curvatures were considered.

Theorem 6 ([6]). *Let M be a hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. If M is proper biharmonic in $\mathbb{S}^{m+1}$, then it has constant mean curvature.*

By using this result, the classification of such hypersurfaces was obtained.

Theorem 7 ([6]). *Let M^m be a hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then M is proper biharmonic if and only if it is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Next is the case of biharmonic hypersurfaces with at most three distinct principal curvatures. In order to solve this problem, the following property of proper biharmonic hypersurfaces in spheres was needed.

Proposition 1 ([6]). *Let M be a proper biharmonic hypersurface with constant mean curvature $\|H\|$ in $\mathbb{S}^{m+1}$ and $m \geq 2$. Then M has positive constant scalar curvature $s = m^2(1 + \|H\|^2) - 2m$.*

First a non-existence result was obtained.

Theorem 8 ([5]). *There exist no compact proper biharmonic hypersurfaces of constant mean curvature with three distinct principal curvatures everywhere in the unit Euclidean sphere.*

The proof relies on the fact that such hypersurfaces are isoparametric, i.e., they have constant principal curvatures with constant multiplicities, and explicit results for their principal curvatures.

We note that, in [23], the authors classified the isoparametric proper biharmonic hypersurfaces in spheres.

Theorem 9 ([23]). *Let M^m be an isoparametric hypersurface in $\mathbb{S}^{m+1}$. Then M is proper biharmonic if and only if it is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Compact proper biharmonic hypersurfaces in $\mathbb{S}^4$ were fully classified.

Theorem 10 ([5]). *The only compact proper biharmonic hypersurfaces in $\mathbb{S}^4$ are the hypersphere $\mathbb{S}^3(1/\sqrt{2})$ and the torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$.*

The proof uses the fact that a proper biharmonic hypersurface in $\mathbb{S}^4$ has constant mean curvature, and thus constant scalar curvature, and a result in [11].

2.2.2. Proper Biharmonic Submanifolds of Codimension Higher Than One

In higher codimension, it was proved that in spheres the proper biharmonic pseudo-umbilical submanifolds, of dimension different from four, have constant mean curvature. This result led to the classification of proper biharmonic pseudo-umbilical submanifolds of codimension two.

Theorem 11 ([6]). *Let M^m be a pseudo-umbilical submanifold in $\mathbb{S}^{m+2}$, $m \neq 4$. Then M is proper biharmonic in $\mathbb{S}^{m+2}$ if and only if it is minimal in $\mathbb{S}^{m+1}(1/\sqrt{2})$.*

Surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ were also investigated.

Theorem 12 ([6]). *Let M^2 be a surface with parallel mean curvature vector field in $\mathbb{S}^n$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if it is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$.*

The above two results allowed the classification of proper biharmonic constant mean curvature surfaces in $\mathbb{S}^4$.

Theorem 13 ([4]). *The only proper biharmonic constant mean curvature surfaces in $\mathbb{S}^4$ are the minimal surfaces in $\mathbb{S}^3(1/\sqrt{2})$.*

Proof: The key of the proof is to show that $\nabla^\perp H = 0$, in order to be able to apply Theorem 12.
We assume that $\nabla^\perp H \neq 0$ and consider $\{E_1, E_2\}$ tangent to M and $\{E_3, E_4 = \frac{H}{\|H\|}\}$ normal to M, such that $\{E_1, E_2, E_3, E_4\}$ constitutes a local orthonormal frame field on $\mathbb{S}^4$. Using the connection one-forms w.r.t. $\{E_1, E_2, E_3, E_4\}$ and the tangent part of the biharmonic equation (1), we get $A_4 = 0$, where A_4 is the shape operator in direction of E_4. Then we identify two cases:

i) If $A_3 = \|H\| \,\mathrm{Id}$, then M is pseudo-umbilical and, by Theorem 11, it is minimal in $\mathbb{S}^3(1/\sqrt{2})$. This implies that $\nabla^\perp H = 0$, and we have a contradiction.
ii) If $A_3 \neq \|H\| \,\mathrm{Id}$, then the Gauss and Codazzi equations lead us to a contradiction and we conclude.

□

3. Properties of Proper Biharmonic Submanifolds in Spheres

We begin this section by presenting some general properties of proper biharmonic submanifolds with parallel mean curvature vector field in spheres, which are consequences of (1) and of the Codazzi and Gauss equations, respectively.

Proposition 2. *Let M be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$. Then*

i) $\|A_H\|^2 = m\|H\|^2$, *and it is constant*

ii) $\operatorname{tr} \nabla A_H = 0$

iii) $\langle \operatorname{tr}(\nabla^{\perp} B)(X, \cdot, A_H(\cdot)), H\rangle = \langle \operatorname{tr}(\nabla^{\perp} B)(\cdot, X, A_H(\cdot)), H\rangle = 0$, *for all* $X \in C(TM)$.

Proposition 3. *Let M be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$. Let p be an arbitrary point of M and consider $\{e_i\}_{i=1}^m$ to be an orthonormal basis of eigenvectors for A_H in T_pM . Denote by $\{a_i\}_{i=1}^m$ the eigenvalues of A_H at p. Then, at p*

i) $m\|H\|^2 = \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} (a_i)^2$

ii) $(2m-1)m\|H\|^2 = \frac{1}{2}\sum_{i,j=1}^{m} (a_i + a_j)(K_{ij} + \|B(e_i, e_j)\|^2)$

iii) $(m - 1 + m\|H\|^2)m\|H\|^2 = \sum_{i,j=1}^{m} a_i a_j (K_{ij} + \|B(e_i, e_j)\|^2)$ *and* $K_{ii} = 0$

where K_{ij} denotes the sectional curvature of the two-plane tangent to M generated by e_i and e_j.

For what concerns proper biharmonic constant mean curvature submanifolds in spheres, a partial classification result was obtained.

Theorem 14 ([29]). *Let M be a proper biharmonic submanifold with constant mean curvature in $\mathbb{S}^n$. Then $\|H\| \in (0, 1]$. Moreover, if $\|H\| = 1$, then M is a minimal submanifold of the hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

Also, the properties regarding the type of the main examples previously presented are not casual. In fact, Theorem 14 was extended by establishing a general link between compact proper biharmonic constant mean curvature submanifolds in spheres and finite type submanifolds in the Euclidean space.

Theorem 15 ([6]). *Let M^m be a compact constant mean curvature, $\|H\| \in (0,1]$, submanifold in $\mathbb{S}^n$. Then M is proper biharmonic if and only if*

either

i) $\|H\| = 1$ *and M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$ and center of mass of norm equal to $1/\sqrt{2}$*

or

ii) $\|H\| \in (0,1)$ *and M is a mass-symmetric 2-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalues $\lambda_p = m(1 - \|H\|)$ and $\lambda_q = m(1 + \|H\|)$.*

This can be further used in order to obtain some necessary conditions which compact proper biharmonic submanifolds with constant mean curvature in spheres must fulfill.

Corollary 1. *Let M^m be a compact proper biharmonic constant mean curvature, $\|H\| \in (0,1)$, submanifold in $\mathbb{S}^n$. Then*

i) $\lambda_1 \leq m(1-\|H\|)$, *where λ_1 is the first non-zero eigenvalue of the Laplacian on M.*

ii) *if* $\mathrm{Ricci(X,X)} \geq \mathrm{cg(X,X)}$, *for all $X \in C(TM)$, where $c > 0$, then $c \leq (m-1)(1-\|H\|)$.*

Proof: i) From Theorem 15 it follows that the inclusion map of M in $\mathbb{R}^{n+1}$, $\varphi : M \to \mathbb{R}^{n+1}$, decomposes as $\varphi = \varphi_p + \varphi_q$, where $\Delta\varphi_p = \lambda_p\varphi_p$, $\Delta\varphi_q = \lambda_q\varphi_q$, $\lambda_p = m(1-\|H\|)$ and $\lambda_q = m(1+\|H\|)$. Conclusively, $m(1-\|H\|)$ is a non-zero eigenvalue of the Laplacian on M, and thus $\lambda_1 \leq m(1-\|H\|)$.
ii) The condition $\mathrm{Ricci(X,X)} \geq \mathrm{cg(X,X)}$, for all $X \in C(TM)$, implies, by a well-known result of Lichnerowicz (see [7]), that $\lambda_1 \geq \frac{m}{m-1}c$. This, together with item i), leads to the conclusion. □

We shall need the following result in order to obtain a refinement of Theorem 14.

Theorem 16 ([26]). *Let M be a compact hypersurface with constant normalized scalar curvature $r = \frac{s}{m(m-1)}$ in $\mathbb{S}^{m+1}$. If*

i) $r \geq 1$

ii) *the squared norm $\|B\|^2$ of the second fundamental form of M satisfies*

$$m(r-1) \leq \|B\|^2 \leq (m-1)\frac{m(r-1)+2}{m-2} + \frac{m-2}{m(r-1)+2} \qquad (5)$$

then either $\|B\|^2 = m(r-1)$ and M is a totally umbilical hypersurface, or

$$\|B\|^2 = (m-1)\frac{m(r-1)+2}{m-2} + \frac{m-2}{m(r-1)+2}$$

and $M = \mathbb{S}^1(\sqrt{1-c^2}) \times \mathbb{S}^{m-1}(c)$, with $c^2 = \frac{m-2}{mr}$.

Thus we get the following theorem.

Theorem 17. *Let M^m, $m \geq 4$, be a compact proper biharmonic constant mean curvature hypersurface in $\mathbb{S}^{m+1}$. Then $\|H\| \in (0, \frac{m-2}{m}] \cup \{1\}$. Moreover*

i) $\|H\| = 1$ *if and only if* $M = \mathbb{S}^m(1/\sqrt{2})$

and

ii) $\|H\| = \frac{m-2}{m}$ *if and only if* $M = \mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^{m-1}(1/\sqrt{2})$.

Proof: Since M is proper biharmonic with constant mean curvature $\|H\|$, Theorem 1 implies that

$$\|B\|^2 = \|A\|^2 = m. \tag{6}$$

We shall denote, for convenience, $t = m\|H\|^2 - 1$.

Suppose that $\|H\| \in (\frac{m-2}{m}, 1)$, which is equivalent to $t \in \left(\frac{(m-4)(m-1)}{m}, m-1\right)$.

By using Proposition 1, we obtain that

$$r = 1 + \frac{t}{m-1}. \tag{7}$$

Condition i) of Theorem 16 is equivalent to $t \geq 0$, which is satisfied. Also, using (6), since $t < m - 1$, the first inequality of (5) is satisfied. The second inequality of (5) becomes

$$0 \leq mt^2 - (m^2 - 6m + 4)t - (m-4)(m-1)$$

and it is satisfied since $t > \frac{(m-4)(m-1)}{m}$. According to the hypotheses of Theorem 16 we get $r = 2$, i.e., $\|H\| = 1$, or $r = \frac{2(m-2)}{m}$, i.e., $\|H\| = \frac{m-2}{m}$, thus we have a contradiction. Conclusively, we obtain $\|H\| \in (0, \frac{m-2}{m}] \cup \{1\}$.

Case i) is given by Theorem 14. It can also be proved by using Theorem 16.

For the ii) part as we have already seen, if $M = \mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^{m-1}(1/\sqrt{2})$, then $\|H\| = \frac{m-2}{m}$. Conversely, if $\|H\| = \frac{m-2}{m}$, then $r = \frac{2(m-2)}{m}$, and we are in the hypotheses of Theorem 16, thus we conclude. □

4. Open Problems

In view of the all above results the following conjectures were proposed.

Conjecture 1 ([6]). *The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(1/\sqrt{2})$ or of generalized Clifford tori $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Conjecture 2 ([6]). *Any proper biharmonic submanifold in $\mathbb{S}^n$ has constant mean curvature.*

5. Further Remarks

There is an interesting link between the proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ and the II-minimal hypersurfaces, although the variational problems which generate them are different. While the bienergy functional is defined on the infinite dimensional manifold of the smooth maps between two given Riemannian manifolds (the metrics are fixed), the area functional of the second fundamental form is defined on the set of all Riemannian immersions of a given submanifold into a Riemannian manifold (the domain metric varies with the immersion). This reminds the well-known result which states that a Riemannian immersion is harmonic if and only if it is minimal.

We briefly recall here the notion of II-minimal hypersurfaces (see [20]). We denote by $\mathcal{E}$ the set of all hypersurfaces in a semi-Riemannian manifold (N, h) for which the first, as well as the second, fundamental form is a semi-Riemannian metric. The critical points of the area functional of the second fundamental form

$$\text{Area}_{II} : \mathcal{E} \to \mathbb{R}, \qquad \text{Area}_{II}(M) = \int_M \sqrt{|\det A|}\, v_g$$

are called II-minimal. According to [20], we have

Proposition 4. *Let $\mathbb{S}^m(a)$ be the hypersphere of radius $a \in (0,1)$ in $\mathbb{S}^{m+1}$. The following are equivalent*

i) *$\mathbb{S}^m(a)$ is proper biharmonic*

ii) *$\mathbb{S}^m(a)$ is II-minimal*

iii) *$a = 1/\sqrt{2}$.*

Proposition 5. *Let $M = \mathbb{S}^{m_1}(a_1) \times \mathbb{S}^{m_2}(a_2)$, where $a_1 \in (0,1)$, $a_1^2 + a_2^2 = 1$, be the generalized Clifford torus in $\mathbb{S}^{m+1}$, $m_1 + m_2 = m$. The following statements are equivalent*

i) *M is proper biharmonic*

ii) *M is II-minimal and non-minimal*

iii) *$a_1 = a_2 = 1/\sqrt{2}$ and $m_1 \neq m_2$.*

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