Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 244–251

STAR PRODUCTS AND APPLICATIONS*

MARI IIDA and AKIRA YOSHIOKA

Department of Mathematics, Tokyo University of Science, 162-8601 Tokyo, Japan

Abstract. A family of star products parametrized by complex matrices is defined. Especially commutative associative star products are treated, and star exponentials with respect to these star products are considered. Jacobi's theta functions are given as infinite sums of star exponentials. As application, several concrete identities are obtained by properties of the star exponentials.

1. Star Products

Using an arbitrary complex symmetric matrix, we can define a star product, which gives a family of star products parameterized by complex matrices [4–6]. In particular for symmetric matrices we obtain a family of commutative associative star products [1, 2].

In this note, as a special case we consider a family of star product algebras of functions of one variable. Using star exponentials of these algebras we describe Jacobi's theta and its basic identities (cf. [1, 2, 6]).

First we consider a star product given by an arbitrary complex matrix. For simplicity, we consider star products of two variables (u_1, u_2). The general case for $(u_1, u_2, \cdots, u_{2m})$ is similar.

For any 2×2 complex matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in M_2(\mathbb{C})$, we have a biderivation in the space of polynomials

$$p_1 \overleftarrow{\partial} \Lambda \overrightarrow{\partial} p_2 = p_1 \left(\sum_{\alpha\beta} \lambda_{\alpha\beta} \overleftarrow{\partial_\alpha} \overrightarrow{\partial_\beta} \right) p_2 = \sum_{\alpha\beta} \lambda_{\alpha\beta} \partial_\alpha p_1 \, \partial_\beta p_2, \quad p_1, p_2 \in \mathcal{P}(\mathbb{C}^2).$$

Then we define a star product by the formula

*Reprinted from J. Geom. Symm. Phys. **20** (2010) 49–56.

$$p_1 *_{\Lambda} p_2 = p_1 \exp\left(\tfrac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)p_2 = p_1\left(\sum_{n=0}^{\infty}\tfrac{1}{n!}\left(\tfrac{i\hbar}{2}\right)^n\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^n\right)p_2$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n p_1\left(\overleftarrow{\partial}\wedge\overrightarrow{\partial}\right)^n p_2$$

where $\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^k$ is the k-th power of the derivation such that

$$p_1\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^k p_2 = p_1\underbrace{\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)\cdots\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)}_{k}p_2$$
$$= \sum\lambda_{\alpha_1\beta_1}\cdots\lambda_{\alpha_k\beta_k}\partial_{\alpha_1}\cdots\partial_{\alpha_k}p_1\partial_{\beta_1}\cdots\partial_{\beta_k}p_2.$$

In this setting we have

Proposition 1. For any $\Lambda \in M_2(\mathbb{C})$, the product $*_{\Lambda}$ is well-defined and associative on $\mathcal{P}(\mathbb{C}^2)$.

2. Star Products on Functions

The star products are well defined on the space of polynomials. In this section we look for their extension to certain class of functions on $\mathbb{C}^2$. We introduce a system of semi-norms and then its topology in $\mathcal{P}(\mathbb{C}^2)$. We take the completion to obtain a space of functions on which the star products are well defined. On this space we can consider star exponentials.

Now we define a topology. Let ρ be a positive number. For every $s > 0$ we define a semi-norm for polynomials by

$$|p|_s = \sup_{u\in\boldsymbol{C}^2}|p(u_1,u_2)|\exp\left(-s|u|^{\rho}\right).$$

Then the system of semi-norms $\{|\cdot|_s\}_{s>0}$ defines a locally convex topology $\mathcal{T}_\rho$ on $\mathcal{P}(\boldsymbol{C}^2)$.

By taking the completion of $\mathcal{P}(\boldsymbol{C}^2)$ with respect to the topology $\mathcal{T}_\rho$ we obtain the Fréchet space $\mathcal{E}_\rho(\boldsymbol{C}^2)$.

Proposition 2. For a positive number ρ, the **Fréchet space** $\mathcal{E}_\rho$ consists of entire functions on the complex plane $\boldsymbol{C}^2$ with finite semi-norm for every $s > 0$, namely,

$$\mathcal{E}_\rho(\boldsymbol{C}^2) = \left\{f \in \mathcal{H}(\boldsymbol{C}^2);\ |f|_s < \infty,\ \text{for all } s > 0\right\}.$$

As to the continuity of star products, the space $\mathcal{E}_\rho(\boldsymbol{C}^2)$, $0 < \rho \leq 2$ is very suitable, namely, we have the following

Theorem 1. *On the space $\mathcal{E}_\rho(\boldsymbol{C}^2)$ for $0 < \rho \leq 2$, every product $*_\Lambda$ is continuous.*

For the spaces $\mathcal{E}_\rho(\boldsymbol{C}^2)$ where $\rho > 2$, the situation is not so good, but we still can rely on the following result.

Theorem 2. *For $\rho > 2$, take $\rho' > 0$ such that $\frac{1}{\rho'} + \frac{1}{\rho} = 1$. Then every star product $*_\Lambda$ defines a continuous bilinear product*

$$*_\Lambda : \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2), \quad \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \times \mathcal{E}_\rho(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2).$$

This means that $(\mathcal{E}_\rho(\boldsymbol{C}^2), *_\Lambda)$ is a continuous $\mathcal{E}_{\rho'}(\boldsymbol{C}^2)$-bimodule.

Let us introduce the Fréchet space

$$\mathcal{E}_{\rho+}(\boldsymbol{C}^2) = \cap_{\lambda > \rho} \mathcal{E}_\lambda(\boldsymbol{C}^2)$$

and consider the exponential element

$$\exp_{*_\Lambda} t\left(\frac{H}{\mathrm{i}\hbar}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{H}{\mathrm{i}\hbar} *_\Lambda \cdots *_\Lambda \frac{H}{\mathrm{i}\hbar}}_{n}$$

in $\mathcal{E}_\rho(\boldsymbol{C}^2)$. The right hand side is not convergent in general. Hence for a polynomial $H \in \mathcal{P}(\boldsymbol{C}^2)$, we define the star exponential $\exp_{*_\Lambda} t(H/\mathrm{i}\hbar)$ by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp_{*_\Lambda} t\left(\frac{H}{\mathrm{i}\hbar}\right) = \frac{H}{\mathrm{i}\hbar} *_\Lambda \exp_* t\left(\frac{H}{\mathrm{i}\hbar}\right), \qquad \exp_{*_\Lambda} t\left(\frac{H}{\mathrm{i}\hbar}\right)|_{t=0} = 1.$$

When $H \in \mathcal{P}(\boldsymbol{C}^2)$ is a linear element, then $\exp_* t\left(\frac{H}{\mathrm{i}\hbar}\right)$ belongs to the good space $\mathcal{E}_{1+}(\subset \mathcal{E}_2)$. In this case, the star exponentials are obtained directly by the formula $\sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{H}{\mathrm{i}\hbar} *_\Lambda \cdots *_\Lambda \frac{H}{\mathrm{i}\hbar}}_{n}$.

On the other hand, we remark here that the most interesting case is given by quadratic form $H \in \mathcal{P}(\boldsymbol{C}^2)$, which case $\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{H}{\mathrm{i}\hbar} *_\Lambda \cdots *_\Lambda \frac{H}{\mathrm{i}\hbar}$ is not convergent and we need the differential equation to define the star exponentials. The star exponential belongs to the space $\mathcal{E}_{2+}(\boldsymbol{C}^2)$, which is difficult to treat at present.

3. Theta Functions

In this section, we consider the star product for the simple case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we can see easily that the star product is commutative and explicitly given by $p_1 *_\Lambda p_2 = p_1 \exp\left(\frac{\mathrm{i}\hbar\rho}{2} \overleftarrow{\partial_{u_1}} \overrightarrow{\partial_{u_1}}\right) p_2$. This means that the algebra is essentially reduced to the space of functions of one variable u_1. Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ for which we define a commutative **star product** $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w).$$

3.1. Star Theta Functions

In this section we consider the Jacobi's theta functions as an example of star exponentials.

A direct calculation gives

$$\exp_{*_\tau} itw = \exp(itw - (\tau/4)t^2).$$

Hence for $\Re\tau > 0$, the **star exponential** $\exp_{*_\tau} niw = \exp(niw - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then we can consider summations for τ satisfying $\Re\tau > 0$

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw = \sum_{n=-\infty}^{\infty} \exp\left(2niw - \tau n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niw}, \quad q = e^{-\tau}.$$

This is Jacobi's theta function $\theta_3(w, \tau)$ (cf. [1]). Then we have expression of the **theta functions** as

$$\theta_{1*_\tau}(w) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*_\tau} (2n+1)iw, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty} \exp_{*_\tau} (2n+1)iw$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw, \qquad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*_\tau} 2niw.$$

Remark that $\theta_{k*_\tau}(w)$ are the Jacobi's theta functions $\theta_k(w, \tau)$, $k = 1, 2, 3, 4$ respectively. This is obvious by the exponential law

$$2 \exp_{*_\tau} 2iw *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w), \qquad k = 2, 3$$
$$\exp_{*_\tau} 2iw *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w), \qquad k = 1, 4.$$

Then using $\exp_{*_\tau} 2iw = e^{-\tau} e^{2iw}$ and the product formula directly we have

$$2e^{2iw-\tau} \theta_{k*_\tau}(w + i\tau) = \theta_{k*_\tau}(w), \qquad k = 2, 3$$
$$e^{2iw-\tau} \theta_{k*_\tau}(w + i\tau) = -\theta_{k*_\tau}(w), \qquad k = 1, 4.$$

Following Toda's idea [3] we obtain the following formulas with the help of the above expressions. In what follows we use as a variable v instead of w given by the relation $\pi v = w$.

Lemma 1.

$$\theta^2_{1*_\tau}(v) = \left(\sum_\lambda^e \sum_\mu^o - \sum_\lambda^o \sum_\mu^e\right) M(\lambda,\mu)$$

$$\theta^2_{2*_\tau}(v) = \left(\sum_\lambda^e \sum_\mu^o + \sum_\lambda^o \sum_\mu^e\right) M(\lambda,\mu)$$

$$\theta^2_{3*_\tau}(v) = \left(\sum_\lambda^e \sum_\mu^e + \sum_\lambda^o \sum_\mu^o\right) M(\lambda,\mu)$$

$$\theta^2_{4*_\tau}(v) = \left(\sum_\lambda^e \sum_\mu^e - \sum_\lambda^o \sum_\mu^o\right) M(\lambda,\mu)$$

where $M(\lambda,\mu) = \mathrm{e}^{\frac{\tau}{2}(\lambda^2-\mu^2)\pi^2} \mathrm{e}_{*_\tau}^{2\lambda\pi \mathrm{i}v}$ *and* $\sum\limits_\lambda^e$ *means that* λ *runs through all even integers etc.*

Proof: By a direct calculation we have

$$\theta^2_{1*_\tau}(v) = -\sum_{n,m}(-1)^{n+m} \mathrm{e}_{*_\tau}^{(2n+1)\pi \mathrm{i}v} \mathrm{e}_{*_\tau}^{(2m+1)\pi \mathrm{i}v}.$$

We notice that

$$\begin{aligned} \mathrm{e}_{*_\tau}^{(2n+1)\pi \mathrm{i}v} \mathrm{e}_{*_\tau}^{(2m+1)\pi \mathrm{i}v} &= -\mathrm{e}^{-\frac{\tau}{2}(2n+1)\pi \mathrm{i}(2m+1)\pi \mathrm{i}} \mathrm{e}_{*_\tau}^{(2n+1)\pi \mathrm{i}v} *_\tau \mathrm{e}_{*_\tau}^{(2m+1)\pi \mathrm{i}v} \\ &= -\mathrm{e}^{\frac{\tau}{2}(2n+1)(2m+1)\pi^2} \mathrm{e}_{*_\tau}^{2(n+m+1)\pi \mathrm{i}v}. \end{aligned}$$

The introduction of $\lambda = n+m+1$ and $\mu = n-m$ gives after some work the following formula

$$\theta^2_{1*_\tau}(v) = \sum_{\lambda,\mu}(-1)^\lambda \mathrm{e}^{\frac{\tau}{2}(\lambda^2-\mu^2)\pi^2} \mathrm{e}_{*_\tau}^{2\lambda\pi \mathrm{i}v}.$$

Cancellation in the summation yields

$$\theta^2_{1*_\tau}(v) = \left(\sum_\lambda^e \sum_\mu^o - \sum_\lambda^o \sum_\mu^e\right) \mathrm{e}^{\frac{\tau}{2}(\lambda^2-\mu^2)\pi^2} \mathrm{e}_{*_\tau}^{2\lambda\pi \mathrm{i}v}$$

which produces the desired result. Other identities are obtained in a similar manner. □

Using Lemma 1 we easily obtain

Proposition 3.

$$\theta^4_{1*_\tau}(v) + \theta^4_{3*_\tau}(v) = \theta^4_{2*_\tau}(v) + \theta^4_{4*_\tau}(v).$$

Proof: We have

$$\theta^4_{1_{*_\tau}}(v) = \left(\sum_l^e\sum_k^o - \sum_l^o\sum_k^e\right) M(l,k)\left(\sum_\lambda^e\sum_\mu^o - \sum_\lambda^o\sum_\mu^e\right) M(\lambda,\mu)$$
$$= \left(\sum_l^e\sum_k^o M(l,k) - \sum_l^o\sum_k^e M(l,k)\right)\left(\sum_\lambda^e\sum_\mu^o M(\lambda,\mu)\right.$$
$$\left. - \sum_\lambda^o\sum_\mu^e M(\lambda,\mu)\right) = (A-B)(C-D).$$

Similarly we have $\theta^4_{2_{*_\tau}}(v) = (A+B)(C+D)$. By the same manner we see

$$\theta^4_{3_{*_\tau}}(v) = \left(\sum_l^e\sum_k^e + \sum_l^o\sum_k^o\right) M(l,k)\left(\sum_\lambda^e\sum_\mu^e + \sum_\lambda^o\sum_\mu^o\right) M(\lambda,\mu)$$
$$= (E+F)(G+H)$$

and $\theta^4_{4_{*_\tau}}(v) = (E-F)(G-H)$. Therefore

$$\theta^4_{1_{*_\tau}}(v) + \theta^4_{3_{*_\tau}}(v) - \left(\theta^4_{2_{*_\tau}}(v) + \theta^4_{0_{*_\tau}}(v)\right) = (A-B)(C-D)$$
$$+ (E+F)(G+H) - \{(A+B)(C+D) + (E-F)(G-H)\}$$
$$= 2\left(-AD - BC + EH + FG\right).$$

Arranging the summation we have

$$2\left(-\sum_l^e\sum_k^o\sum_\lambda^o\sum_\mu^e - \sum_l^o\sum_k^e\sum_\lambda^e\sum_\mu^o + \sum_l^e\sum_k^e\sum_\lambda^o\sum_\mu^o + \sum_l^o\sum_k^o\sum_\lambda^e\sum_\mu^e\right)$$
$$\times\, \mathrm{e}^{\frac{\tau}{2}(l^2-k^2+\lambda^2-\mu^2)\pi^2}\mathrm{e}_{*_\tau}^{2\pi\mathrm{i}v}\mathrm{e}_{*_\tau}^{2\lambda\pi\mathrm{i}v}$$
$$= 2\left(\sum_l^o\sum_\lambda^e\left(\sum_k^o\sum_\mu^e - \sum_k^e\sum_\mu^o\right) - \sum_l^e\sum_\lambda^o\left(\sum_k^o\sum_\mu^e - \sum_k^e\sum_\mu^o\right)\right)$$
$$\times\, \mathrm{e}^{\frac{\tau}{2}(l^2-k^2+\lambda^2-\mu^2)\pi^2}\mathrm{e}_{*_\tau}^{2\pi\mathrm{i}v}\mathrm{e}_{*_\tau}^{2\lambda\pi\mathrm{i}v} = 0$$

since $\sum_k^o\sum_\mu^e \mathrm{e}^{-\frac{\tau}{2}(k^2+\mu^2)} = \sum_k^e\sum_\mu^o \mathrm{e}^{-\frac{\tau}{2}(k^2+\mu^2)}$. □

Proposition 4. For a complex parameter $a\in\mathbb{C}$ we have the identity

$$\theta^2_{3_{*_\tau}}(v)\,\theta^2_{3_{*_\tau}}(a) + \theta^2_{1_{*_\tau}}(v)\,\theta^2_{1_{*_\tau}}(a) = \theta^2_{3_{*_\tau}}(0)\,\theta_{3_{*_\tau}}(v+a)\,\theta_{3_{*_\tau}}(v-a).$$

Proof: By a similar manner as in Lemma 1 we have

$$\theta_{3_{*_\tau}}(v+a)\,\theta_{3_{*_\tau}}(v-a) = \left(\sum_l^e\sum_\lambda^e + \sum_l^o\sum_\lambda^o\right)\mathrm{e}^{\frac{\tau}{2}(l^2-\lambda^2)\pi^2}\mathrm{e}^{2\lambda\pi\mathrm{i}a}\mathrm{e}_{*_\tau}^{2l\pi\mathrm{i}v}.$$

Since $e_{*_\tau}^{2\lambda\pi iv}|_{v=0} = e^{-\tau\lambda^2\pi^2} e^{2\lambda\pi iv}|_{v=0} = e^{-\tau\lambda^2\pi^2}$ we have also

$$\theta_{3_{*_\tau}}^2(0) = \left(\sum_k^e\sum_\mu^e + \sum_k^o\sum_\mu^o\right) e^{-\frac{\tau}{2}(k^2+\mu^2)\pi^2}$$

$$\theta_{3_{*_\tau}}^2(a) = \left(\sum_k^e\sum_\mu^e + \sum_k^o\sum_\mu^o\right) e^{-\frac{\rho}{2}(k^2+\mu^2)\pi^2} e^{2\lambda\pi ia}$$

$$\theta_{1_{*_\tau}}^2(a) = \left(\sum_\lambda^e\sum_\mu^o - \sum_\lambda^o\sum_\mu^e\right) e^{-\frac{\tau}{2}(\lambda^2+\mu^2)\pi^2} e^{2\lambda\pi ia}.$$

Then we obtain

$$\begin{aligned}
&\theta_{3_{*_\tau}}^2(v)\theta_{3_{*_\tau}}^2(a) + \theta_{1_{*_\tau}}^2(v)\,\theta_{1_{*_\tau}}^2(a) \\
&= \Bigg(\sum_l^e\sum_k^e\sum_\lambda^e\sum_\mu^e + \sum_l^e\sum_k^e\sum_\lambda^o\sum_\mu^o + \sum_l^o\sum_k^o\sum_\lambda^e\sum_\mu^e + \sum_l^o\sum_k^o\sum_\lambda^o\sum_\mu^o \\
&\quad + \sum_l^e\sum_k^o\sum_\lambda^e\sum_\mu^o - \sum_l^e\sum_k^o\sum_\lambda^o\sum_\mu^e - \sum_l^o\sum_k^e\sum_\lambda^e\sum_\mu^o + \sum_l^o\sum_k^e\sum_\lambda^o\sum_\mu^e\Bigg) \\
&\quad \times e^{\frac{\tau}{2}(l^2-\lambda^2-k^2-\mu^2)\pi^2} e^{2\lambda\pi ia} e_{*_\tau}^{2l\pi iv} \\
&= \left(\sum_k^e\sum_\mu^e + \sum_k^o\sum_\mu^o\right) e^{-\frac{\tau}{2}(k^2+\mu^2)\pi^2} \left(\sum_l^e\sum_\lambda^e + \sum_l^o\sum_\lambda^o\right) \\
&\quad \times e^{\frac{\tau}{2}(l^2-\lambda^2)\pi^2} e^{2\lambda\pi ia} e_{*_\rho}^{2l\pi iv} = \theta_{3_{*_\tau}}^2(0)\,\theta_{3_{*_\tau}}(v+a)\,\theta_{3_{*_\tau}}(v-a).
\end{aligned}$$

□

Acknowledgements

The second author is partially supported by the Grant-in-Aid for Scientific Research (# 21540096). The authors are thankful to Professors Ivaïlo Mladenov and Gaetano Vilasi for discussions and encouragement in Tokyo. The second author is grateful for the hospitality of the Institute of Biophysics of the Bulgarian Academy of Science during his stay there.

References

[1] Omori H, Maeda Y., Miyazaki N. and Yoshioka A., *Orderings and Non-formal Deformation Quantization*, Lett. Math. Phys. **82** (2007) 153–175.

[2] Omori H, Maeda Y., Miyazaki N. and Yoshioka A., *Deformation of Expression of Algebras*, (preprint)

[3] Toda M., *Introduction to Elliptic Functions* (in Japanese), Nihon Hyoronshya, Tokyo, 2001.

[4] Tomihisa T. and Yoshioka A., *Star Products and Star Exponentials*, J. Geom. Symm. Phys. **19** (2010) 99–111.
[5] Yoshioka A., *A Family of Star Products and its Application*, In: Proceedings XXVI Workshop on Geometrical Methods in Physics, P. Kielanowski, A. Odzijewicz, M. Schlichenmaier and T. Voronov (Eds), AIP Conference Proceedings vol. **956**, Melville, New York 2007, pp 37–42.
[6] Yoshioka A., *Examples of Star Exponentials*, In: Proceedings XXVIII Workshop on Geometrical Methods in Physics, P. Kielanowski, S. T. Ali, A. Odzijewicz, M. Schlichenmaier and T. Voronov (Eds), AIP Conference Proceedings vol. **1191**, Melville, New York 2009, pp 188–193.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 252–262

MOTION OF CHARGED PARTICLES IN TWO-STEP NILPOTENT LIE GROUPS*

OSAMU IKAWA

Department of General Education, Fukushima National College of Technology
Iwaki, Fukushima, 970-8034 JAPAN

Abstract. We formulate the equation of motion of a charged particle in a Riemannian manifold with a closed two form. Since a two-step nilpotent Lie group has natural left-invariant closed two forms, it is natural to consider the motion of a charged particle in a simply connected two-step nilpotent Lie groups with a left invariant metric. We study the behavior of the motion of a charged particle in the above spaces.

1. Introduction

Let Ω be a closed two-form on a connected Riemannian manifold $(M, \langle\ ,\ \rangle)$, where $\langle\ ,\rangle$ is a Riemannian metric on M. We denote by $\bigwedge^m(M)$ the space of m-forms on M. We denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from a vector field X on M, and by $\mathcal{L} : T(M) \to T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ over M onto the cotangent bundle $T^*(M)$ over M, which is defined by

$$\mathcal{L} : T(M) \to T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v\rangle\ , \quad u, v \in T(M). \tag{1}$$

A curve $x(t)$ in M is referred as a *motion of a charged particle under electromagnetic field* Ω, if it satisfies the following second order differential equation

$$\nabla_{\dot{x}}\dot{x} = \mathcal{L}^{-1}(\iota(\dot{x})\Omega) \tag{2}$$

where ∇ is the Levi-Civita connection of M. Here $\nabla_{\dot{x}}\dot{x}$ means the acceleration of the charged particle. Since $\mathcal{L}^{-1}(\iota(\dot{x})\Omega)$ is perpendicular to the direction $\dot{x}$ of the movement, $\mathcal{L}^{-1}(\iota(\dot{x})\Omega)$ means the **Lorentz force**. The speed $|\dot{x}|$ is a conservative constant for a charged particle. When $\Omega = 0$, then the motion of a

*Reprinted from J. Geom. Symm. Phys. **20** (2010) 57–67.

charged particle is nothing but a geodesic. The equation (2) originated in the theory of relativity (see [2] for details).

In this paper, we deal with the motion of a charged particles in a simply connected two-step nilpotent Lie group N with a left invariant Riemannian metric. Since a two-step nilpotent Lie group has a non-trivial center Z, we can construct a left-invariant closed two form Ω_{a_0} from an element $a_0 \in Z$ specified below and consider the motion of a charged particle under the electromagnetic field Ω_{a_0}. H. Naitoh and Y. Sakane proved that there are no closed geodesics in a simply connected nilpotent Lie group. In contrast with geodesics, there exist motions of charged particles which are periodic. Kaplan defined a H-type Lie group, which is a kind of two-step nilpotent Lie groups. We study the motion of a charged particle in a H-type Lie group more explicitly than in a general two-step nilpotent Lie group.

2. Charged Particles in Two-step Nilpotent Lie Groups

Let N be a simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric $\langle\, ,\rangle$. Denote by   the vector space consisting of all left-invariant vector fields on N. Since   is two-step nilpotent,   has a non-trivial center . Let $= \oplus\ ^{\perp}$ be an orthogonal direct sum decomposition of , then $^{\perp}$, $^{\perp} \subset$. For $a_0 \in$, we define a linear transformation ϕ_{a_0} on $^{\perp}$ by

$$\langle \phi_{a_0}(X), Y\rangle = \langle a_0,\ X, Y\ \rangle, \qquad X, Y \in\ ^{\perp}.$$

We extend ϕ_{a_0} to a linear transformation on   by $\phi = 0$ on , which is also denoted by ϕ_{a_0}. We can regard ϕ_{a_0} as a left-invariant $(1,1)$-tensor on N. Then ϕ_{a_0} is skew-symmetric with respect to the left-invariant Riemannian metric $\langle\, ,\rangle$ since

$$\langle \phi_{a_0}(X), Y\rangle + \langle X, \phi_{a_0}(Y)\rangle = \langle a_0,\ X, Y\ \rangle + \langle a_0,\ Y, X\ \rangle = 0$$

for any left invariant vector fields $X, Y \in$. If we define a left-invariant two-form Ω_{a_0} on N by

$$\Omega_{a_0}(X, Y) = \langle X, \phi_{a_0}(Y)\rangle, \qquad X, Y \in$$

then a simple calculation implies that Ω_{a_0} is closed. In fact, for any X_1, X_2 and X_3 in   we have

$$\begin{aligned} 3!(d\Omega_{a_0})(X_1, X_2, X_3) &= \quad \Omega_{a_0}(\ X_1, X_2\ , X_3) \\ &= \quad \langle\ X_1, X_2\ , \phi_{a_0}(X_3)\rangle = 0 \end{aligned}$$

where we denote by   the cyclic sum, and the last equality follows from the fact that $X_1, X_2 \in$ and $\phi(X_3) \in\ ^{\perp}$. The equation of motion of the charged particle under the electromagnetic field Ω_{a_0} is

$$\nabla_{\dot{x}}\dot{x} = \phi_{a_0}(\dot{x}). \tag{3}$$

Here a curve in a manifold is **simple** if it is a simply closed periodic curve, or if it does not intersect itself. Since N is simply connected, the one dimensional de-Rham cohomology group vanishes. Hence we get the following theorem using Theorem 9 in [2].

Theorem 1. *The motion of a charged particle* (3) *in a simply connected two-step nilpotent Lie group is simple.*

Now we will construct explicitly a simply connected two step nilpotent Lie group with a left-invariant Riemannian metric from an (abstract) two-step nilpotent Lie algebra with an inner product $\langle\,,\rangle$. In order to do this, we recall a **Hausdorff formula** for a Lie group (see [1, p. 106]), which states that

$$\exp X \exp Y = \exp\left(X + Y + \frac{1}{2}\ X, Y\ + \cdots\right).$$

If the Lie group is two-step nilpotent, then the higher terms $+\cdots$ on the right hand side vanish. Based on the Hausdorff formula, we define a Lie group structure on itself by

$$X \cdot Y = X + Y + \frac{1}{2}\ X, Y\ , \qquad X, Y \in\ .$$

The identity element in this group is 0, and the inverse element of $x \in$ is equal to x. We denote by $N = (\ ,\cdot)$ the so obtained Lie group. The center of N coincides with . Denote by Lie(N) the Lie algebra consisting of all left-invariant vector fields on N. Then Lie(N) is identified with as a Lie algebra as mentioned below. Since N is a Euclidean space as a manifold, we can identify $T_0(N)$ with as vector spaces. The identification induces a Lie algebra structure on $T_0(N)$. For $X \in T_0(N)$, we denote by $\tilde{X} \in \mathrm{Lie}(N)$ the left-invariant vector field on N such that $\tilde{X}_0 = X$. The mapping defined by $= T_0(N) \to \mathrm{Lie}(N)$, $X \mapsto \tilde{X}$ gives an isomorphism as Lie algebras. Hence N is a simply connected two-step nilpotent Lie group whose Lie algebra is . The inner product $\langle\,,\rangle$ on induces a left-invariant Riemannian metric $\langle\,,\rangle$ on N. Using this notation, we have

$$\Omega_{a_0}(\tilde{X}, \tilde{Y}) = \langle \tilde{X}, \phi\tilde{Y} \rangle = \langle \tilde{a}_0,\ \tilde{Y}, \tilde{X}\ \rangle = \langle a_0,\ Y, X\ \rangle.$$

The exponential mapping $\exp :\ \to N$ is equal to identity mapping. Hence for $X \in T_0(N)$, we have

$$\tilde{X}_x = \frac{d}{dt}(x \cdot tX)_{|t=0} = \frac{d}{dt}\left(x + tX + \frac{t}{2}\ x, X\ \right)_{|t=0} \in T_x(N).$$

Since the Riemannian metric on N is left-invariant, the left action of N on N itself is isometric. Hence $X \in T_0(N)$ induces a Killing vector field X^* on N by

$$X^*_x = \frac{d}{dt}(\exp tX)x_{|t=0} = \frac{d}{dt}(tX + x + \frac{t}{2}\ X, x\ )_{|t=0} \in T_x(N).$$

The Killing vector field X^* is right-invariant.

Lemma 1. *The mapping defined by*

$$\to \ , \qquad X \mapsto X + \frac{1}{2} \ X, x$$

is a linear isomorphism.

Proof: Since the mapping is clearly linear, it is sufficient to prove that it is injective. In order to do this, we study the kernel of the mapping. Suppose that $X \in$ satisfy the condition $X + \frac{1}{2} \ X, x \ = 0$. Decompose X as $X = X_1 + X_2$ where $X_1 \in \ ^{\perp}$ and $X_2 \in$, then $X_1 + (X_2 + \frac{1}{2} \ X_1, x \ ) = 0$. This implies $X_1 = 0$ and $X_2 + \frac{1}{2} \ X_1, x \ = 0$. Hence we have $X_2 = 0$, hence, $X = 0$. □

By the lemma above, we have $T_x(N) = \mathrm{span}\{X^*_x \, ; \, X \in \ \}$ for any x in N. The Killing vector field X^* is an infinitesimal automorphism of ϕ.

Lemma 2. *Let* X *be in* $T_0(N) = \ $. *For a fixed* $x \in N$, *we have* $X^*_x = \tilde{W}_x$, *where we set* $W = X + \ X, x \ $.

Proof: Since

$$\begin{aligned} \tilde{W}_x &= \frac{d}{dt}\left(x + tX + t \ X, x \ + \frac{t}{2} \ x, X + \ X, x \ \right)_{|t=0} \\ &= \frac{d}{dt}\left(x + tX + \frac{t}{2} \ X, x\right)_{|t=0} = X^*_x \end{aligned}$$

we have the assertion. □

Lemma 3. *Define a one-form* η_{a_0} *on* N *by*

$$\eta_{a_0}(X^*_x) = \langle \ x, X \ , a_0 \rangle, \qquad X \in \ .$$

Then $\iota(X^*)\Omega_{a_0} = d(\eta_{a_0}(X^*))$ *for any* X *in* .

Proof: Let X and Y be in . By Lemma 2, we have

$$\begin{aligned} (\iota(X^*_x)\Omega_{a_0})(\tilde{Y}_x) &= \Omega_{a_0}(X^*_x, \tilde{Y}_x) \\ &= \Omega_{a_0}(\tilde{W}_x, \tilde{Y}_x) \\ &= \Omega_{a_0}(X + \ X, x \ ), Y) \\ &= \langle a_0, \ Y, X + \ X, x \ \ \rangle = \langle a_0, \ Y, X \ \rangle. \end{aligned}$$

Using the above equation, we have also

$$\begin{aligned} d(\eta_{a_0}(X^*))(\tilde{Y}_x) &= \tilde{Y}_x(\eta_{a_0}(X^*)) \\ &= \frac{d}{dt}\eta_{a_0}(X^*_{x+tY+\frac{t}{2}[x,Y]})_{|t=0} \\ &= \frac{d}{dt}\langle\ x + tY + \frac{t}{2}\ x, Y\ , X\ , a_0\rangle \\ &= \langle\ Y, X\ , a_0\rangle = (\iota(X^*_x)\Omega_{a_0})(\tilde{Y}_x). \end{aligned}$$

Hence we get $d(\eta_{a_0}(X^*)) = \iota(X^*)\Omega_{a_0}$. □

Denote by $T_x(N) \to T_0(N); v \mapsto v_n$ the usual parallel translation in the Euclidean space : Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then $v_n = \frac{d}{dt}(c(t)\quad x)_{|t=0}$. The following lemma gives a relation between the two linear isomorphisms $L_x^{-1} : T_x(N) \to T_0(N)$ and $T_x(N) \to T_0(N), v \mapsto v_n$.

Lemma 4. $L_x^{-1}v = v_n \quad \frac{1}{2}\ x, v_n$ *for* $v \in T_x(N)$.

Proof: Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then

$$\begin{aligned} L_x^{-1}v = L_{-x}v &= \frac{d}{dt}\left(\ x + c(t) \quad \frac{1}{2}\ x, c(t)\ \right)_{|t=0} \\ &= \frac{d}{dt}\left(c(t) \quad x \quad \frac{1}{2}\ x, c(t) \quad x\ \right)_{|t=0} = v_n \quad \frac{1}{2}\ x, v_n\ . \end{aligned}$$

Hence we have the assertion. □

Similarly we define $T_y(\ ^{\perp}) \to T_0(\ ^{\perp}), u \mapsto u_{\mathfrak{z}^\perp}$ and $T_z(\ ) \to T_0(\ ), w \mapsto w_{\mathfrak{z}}$. Since is abelian, we have $L_z^{-1}w = w_{\mathfrak{z}}$ for $w \in T_z(\ )$. Hence we can write $w = w_{\mathfrak{z}}$. Let $x \in$ and $v \in T_x(\ )$. Expressing x and v as $x = y + z$ and $v = v_1 + v_2$, where $y \in\ ^{\perp}, z \in\ , v_1 \in T_y(\ ^{\perp})$ and $v_2 \in T_z(\ )$ we obtain

$$L_x^{-1}v = (v_1)_{\mathfrak{z}^\perp} + \left(v_2 \quad \frac{1}{2}\ y, (v_1)_{\mathfrak{z}^\perp}\ \right). \tag{4}$$

Proposition 1. Let $x(t) = y(t) + z(t)$ be a curve in , where $y(t) \in\ ^{\perp}$ and $z(t) \in$. Assume that $y(0) = 0$. Then $x(t)$ describes the motion of a charged particle (3) if and only if

$$\dot{y}(t)_{\mathfrak{z}^\perp} \quad \phi_{\dot{z}(0)+a_0}y(t) = \dot{y}(0), \qquad \dot{z}(t) \quad \frac{1}{2}\ y(t), \dot{y}(t)_{\mathfrak{z}^\perp}\ = \dot{z}(0). \tag{5}$$

Proof: Taking the inner product of (3) and the Killing vector field X^* for $X \in$, we have

$$\frac{d}{dt}\langle\dot{x}, X^*\rangle = \Omega(X^*, \dot{x}) = (\iota(X^*)\Omega)(\dot{x}).$$

Using Lemma 3 we find

$$\frac{d}{dt}\langle \dot{x}, X^*\rangle = (d(\eta(X^*)))(\dot{x}) = \frac{d}{dt}\eta(X^*_{x(t)}).$$

Since $T_x(N) = \text{span}\{X^*_x;\, X \in \quad\}$, the equation (3) is equivalent to

$$\frac{d}{dt}(\langle \dot{x}(t), X^*_{x(t)}\rangle \quad \eta(X^*_{x(t)})) = 0.$$

By the definition of η, we have

$$\eta(X^*_{x(t)}) = \langle\ x(t), X\ , a_0\rangle = \langle \phi_{a_0}(y(t)), X\rangle.$$

Since $\langle\, , \rangle$ is left invariant

$$\begin{aligned}\langle \dot{x}, X^*_{x(t)}\rangle &= \langle L_x^{-1}\dot{x}, L_x^{-1}X^*_x\rangle \\ &= \left\langle \dot{y}_{\mathfrak{z}^\perp} + (\dot{z} \quad \frac{1}{2}\ y, \dot{y}_{\mathfrak{z}^\perp}\ ), X + \ X, x\ \right\rangle \\ &= \langle \dot{y}_{\mathfrak{z}^\perp}, X\rangle + \left\langle \dot{z} \quad \frac{1}{2}\ y, \dot{y}_{\mathfrak{z}^\perp}\ , X + \ X, x\ \right\rangle\end{aligned}$$

where we have used Lemma 2 and equation (4). Hence the equation (3) is equivalent to

$$\frac{d}{dt}\left(\langle \dot{y}_{\mathfrak{z}^\perp} \quad \phi_{a_0}(y), X\rangle + \langle \dot{z} \quad \frac{1}{2}\ y, \dot{y}_{\mathfrak{z}^\perp}\ , X + \ X, y\ \rangle\right) = 0.$$

Taking $X \in \quad$, we have

$$\dot{z}(t) \quad \frac{1}{2}\ y(t), \dot{y}(t)_{\mathfrak{z}^\perp}\ = \dot{z}(0)$$

where we have used the initial condition $y(0) = 0$. Next, taking $X \in \ ^\perp$, we have

$$\frac{d}{dt}\left(\langle \dot{y}_{\mathfrak{z}^\perp} \quad \phi_{a_0}(y), X\rangle + \langle \dot{z}(0),\ X, y\ \rangle\right) = 0.$$

Taking into account the initial condition $y(0) = 0$, we finally have

$$\dot{y}(t)_{\mathfrak{z}^\perp} \quad \phi_{\dot{z}(0)+a_0} y(t) = \dot{y}(0).$$

□

Proposition 2. The motion of a charged particle (3) with $y(0) = 0$ is given by the equations

$$\begin{aligned}y(t) &= \exp t\phi_{\dot{z}(0)+a_0} \int_0^t \exp(\quad t\phi_{\dot{z}(0)+a_0})\dot{y}(0)dt \\ z(t) &= z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t\ y(t), (\exp t\phi_{\dot{z}(0)+a_0})\dot{y}(0)\ dt.\end{aligned}$$

Proof: Using the first equation of (5) with $y(0) = 0$, we have

$$y(t) = \exp t\phi_{\dot z(0)+a_0} \int_0^t \exp(\quad t\phi_{\dot z(0)+a_0})\dot y(0)dt.$$

Hence

$$\phi_{\dot z(0)+a_0} y(t) = (\exp t\phi_{\dot z(0)+a_0} \quad 1)\dot y(0)$$

which implies that

$$\phi_{\dot z(0)+a_0} y(t) + \dot y(0) = (\exp t\phi_{\dot z(0)+a_0})\dot y(0).$$

Using the second and the first equation from (5)

$$\begin{aligned} z(t) &= z(0) + t\dot z(0) + \frac{1}{2}\int_0^t \; y(t), \dot y(t)_{\mathfrak{z}^\perp} \; dt \\ &= z(0) + t\dot z(0) + \frac{1}{2}\int_0^t \; y(t), \phi_{\dot z(0)+a_0} y(t) + \dot y(0) \; dt \\ &= z(0) + t\dot z(0) + \frac{1}{2}\int_0^t \; y(t), (\exp t\phi_{\dot z(0)+a_0})\dot y(0) \; dt. \end{aligned}$$

Hence we get the assertion. □

When $\phi_{\dot z(0)+a_0} = 0$, then, using the above Proposition, we get $y(t) = t\dot y(0)$ and

$$z(t) = z(0) + t\dot z(0) + \frac{1}{2}\int_0^t \; t\dot y(0), \dot y(0) \; dt = z(0) + t\dot z(0).$$

Lemma 5. *The equation of motion* (3) *implies the following relation*

$$\frac{d}{dt}(\langle z(t), \dot z(0) + a_0\rangle + \frac{1}{2}\langle y(t), \dot y(0)\rangle) = |\dot z(0)|^2 + \langle \dot z(0), a_0\rangle + \frac{1}{2}|\dot y_{\mathfrak{z}^\perp}|^2.$$

Proof: Taking the inner product of the second equation of (5) with $\dot z(0) + a_0$, we have

$$\langle \dot z, \dot z(0) + a_0\rangle \quad \frac{1}{2}\langle \; y, \dot y_{\mathfrak{z}^\perp} \; , \dot z(0) + a_0\rangle = |\dot z(0)|^2 + \langle \dot z(0), a_0\rangle.$$

Using equation (5) again produces

$$\begin{aligned} \langle \; y, \dot y_{\mathfrak{z}^\perp} \; , \dot z(0) + a_0\rangle &= \langle \phi_{\dot z(0)+a_0} y, \dot y_{\mathfrak{z}^\perp}\rangle \\ &= \langle \dot y_{\mathfrak{z}^\perp} \quad \dot y(0), \dot y_{\mathfrak{z}^\perp}\rangle \\ &= |\dot y_{\mathfrak{z}^\perp}|^2 \quad \langle \dot y_{\mathfrak{z}^\perp}, \dot y(0)\rangle = |\dot y_{\mathfrak{z}^\perp}|^2 \quad \frac{d}{dt}\langle y(t), \dot y(0)\rangle. \end{aligned}$$

Hence

$$\frac{d}{dt}(\langle z(t), \dot z(0) + a_0\rangle + \frac{1}{2}\langle y(t), \dot y(0)\rangle) = |\dot z(0)|^2 + \langle \dot z(0), a_0\rangle + \frac{1}{2}|\dot y_{\mathfrak{z}^\perp}|^2.$$

□

Applying the lemma above for geodesics, we can re-demonstrate the following theorem of Naitoh-Sakane.

Theorem 2. (Naitoh-Sakane [4, Corrolary 3.3]) *Every geodesic in any simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric does not intersect itself.*

Proof: Let $x(t) = y(t) + z(t) \in N$ be a geodesic with $y(0) = 0$. Applying Lemma 5 with $a_0 = 0$

$$\frac{d}{dt}\left(\langle z(t), \dot{z}(0)\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle\right) = |\dot{z}(0)|^2 + \frac{1}{2}|\dot{y}_{\mathfrak{z}^\perp}|^2 > 0.$$

Hence $\langle z(t), \dot{z}(0)\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle$ is monotone increasing. Thus $x(t)$ is not periodic. Since we have already proved that $x(t)$ is simple by Theorem 1, we get the assertion. □

The author thinks that the above proof is easier than the original proof of Naitoh-Sakane.

3. Charged Particles in H-type Lie Groups

In this section, we study the motion of a charged particle in a simply connected H-type Lie group. First we review the definition of H-type Lie algebra according to Kaplan. Let $(U, \langle\ ,\ \rangle)$ and $(V, \langle\ ,\ \rangle)$ be finite-dimensional real vector spaces with inner products $\langle\ ,\ \rangle$. Denote by $\mathrm{End}(V)$ the vector space consisting of all linear transformations on V. We assume that there exists a linear mapping $j : U \to \mathrm{End}(V)$ such that

$$j(a)^2 = \quad |a|^2 I, \qquad |j(a)x| = |a||x|, \qquad a \in U, \quad x \in V. \tag{6}$$

In other words, V is a Clifford module over the Clifford algebra generated by U. By (6) we have

$$\langle j(a)x, j(b)x\rangle = \langle a, b\rangle |x|^2, \qquad \langle j(a)x, j(a)y\rangle = |a|^2\langle x, y\rangle$$
$$\langle j(a)x, y\rangle + \langle x, j(a)y\rangle = 0, \qquad a, b \in U, \quad x, y \in V.$$

Define a bi-linear mapping $\ ,\ : V \times V \to U$ via the formula

$$\langle a,\ x, y\ \rangle = \langle j(a)x, y\rangle, \qquad a \in U, \quad x, y \in V. \tag{7}$$

Then $\ ,\ $ is alternative. Substituting $j(b)x$ into y, we have

$$\langle a,\ x, j(b)x\ \rangle = \langle j(a)x, j(b)x\rangle = \langle a, b\rangle |x|^2.$$

Hence

$$x, j(b)x\ = |x|^2 b, \qquad b \in U, \quad x \in V. \tag{8}$$

We denote by $= \mathfrak{U} \oplus V$ the orthogonal direct sum of $\mathfrak{U}$ and V, and define a Lie algebra structure on by

$$a + x, b + y \;=\; x, y \;\in \mathfrak{U}, \qquad a, b \in \mathfrak{U}, \quad x, y \in V.$$

Then the Lie algebra is called **H-type**. Since the H-type Lie algebra is a kind of two-step nilpotent Lie algebra with an inner product, we can define a Lie group structure on with a left-invariant Riemannian metric, whose Lie algebra is itself as we mentioned in the previous section. For $a_0 \in \mathfrak{U}$, we consider the equation

$$\nabla_{\dot{x}}\dot{x} = j(a_0)\dot{x} \tag{9}$$

of motion of a charged particle. If we express its trajectory as $x(t) = y(t) + z(t)$ where $y(t) \in V, z(t) \in \mathfrak{U}$, then (9) is equivalent to

$$\dot{y}(t)_V \quad j(\dot{z}(0) + a_0)y(t) = \dot{y}(0) \tag{10}$$

where $T_y(V) \to V, w \mapsto w_V$ denotes the usual parallel translation in V. Here we have used equation (5).

Theorem 3. *Let* $x(t) = y(t) + z(t) \in N$ $($*where* $y(t) \in V, z(t) \in \mathfrak{U})$ *is a motion of a charged particle* (9) *with* $x(0) = 0$.

1) *When* $\dot{z}(0) + a_0 = 0$, *then* $x(t) = t\dot{x}(0)$.

2) *When* $\dot{z}(0) + a_0 \neq 0$, *then*

$$y(t) = \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|}\dot{y}(0) + \frac{1 \quad \cos(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|^2} j(\dot{z}(0) + a_0)\dot{y}(0)$$

$$z(t) = t\dot{z}(0) + \frac{t|\dot{y}(0)|^2}{2|\dot{z}(0) + a_0|^2}(\dot{z}(0) + a_0) \quad \frac{\sin(t|\dot{z}(0) + a_0|)}{2|\dot{z}(0) + a_0|^3}|\dot{y}(0)|^2(\dot{z}(0) + a_0).$$

The curve $y(t)$ *is a circle in* V. *The motion of a charged particle is periodic if and only if*

$$a_0 = \quad \left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|^2} + 1\right)\dot{z}(0).$$

In this case $x(t)$ *is an elliptic motion.*

Remark 1. When $x(t)$ is a geodesic, the condition $a_0 = 0$ implies the theorem of Kaplan [3].

Proof: 1) is clear from (10). We will show 2). Using the first equation of (10), we have

$$y(t) = \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|}\dot{y}(0) + \frac{1 \quad \cos(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|^2} j(\dot{z}(0) + a_0)\dot{y}(0)$$

which implies that

$$\dot{y}(t)_V = \cos(t|\dot{z}(0) + a_0|)\dot{y}(0) + \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|} j(\dot{z}(0) + a_0)\dot{y}(0).$$

Using the equation above, we have

$$y(t)_V, \dot{y}(t) \;=\; \frac{1 \quad \cos(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|^2}\; \dot{y}(0), j(\dot{z}(0) + a_0)\dot{y}(0) \; .$$

Further the second equation of (10) gives

$$\begin{aligned} \dot{z}(t) &= \dot{z}(0) + \frac{1 \quad \cos(t|\dot{z}(0) + a_0|)}{2|\dot{z}(0) + a_0|^2}\; \dot{y}(0), j(\dot{z}(0) + a_0)\dot{y}(0) \\ &= \dot{z}(0) + \frac{1 \quad \cos(t|\dot{z}(0) + a_0|)}{2|\dot{z}(0) + a_0|^2}(\dot{z}(0) + a_0)|\dot{y}(0)|^2 \end{aligned} \tag{11}$$

where we have used the equation (8). Since

$$\begin{aligned} y(t) \quad \frac{1}{|\dot{z}(0) + a_0|} j\left(\frac{\dot{z}(0) + a_0}{|\dot{z}(0) + a_0|}\right)\dot{y}(0) &= \frac{\sin(|\dot{z}(0) + a_0|t)}{|\dot{z}(0) + a_0|}\dot{y}(0) \\ &\quad \frac{\cos(|\dot{z}(0) + a_0|t)}{|\dot{z}(0) + a_0|} j\left(\frac{\dot{z}(0) + a_0}{|\dot{z}(0) + a_0|}\right)\dot{y}(0) \end{aligned}$$

the curve $y(t)$ is a circle in V whose center is $\frac{1}{|\dot{z}(0)+a_0|} j\left(\frac{\dot{z}(0)+a_0}{|\dot{z}(0)+a_0|}\right)\dot{y}(0)$ and the radius is $\frac{|\dot{y}(0)|}{|\dot{z}(0)+a_0|}$. The periodic condition is as follows

$$\begin{aligned} x(t) \text{ is periodic} \Leftrightarrow \dot{z}(0) + \frac{|\dot{y}(0)|^2}{2|\dot{z}(0) + a_0|^2}(\dot{z}(0) + a_0) = 0 \\ \Leftrightarrow a_0 = \left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|^2} + 1\right)\dot{z}(0). \end{aligned}$$

In this case, since

$$\begin{aligned} x(t) + \frac{2|\dot{z}(0)|}{|\dot{y}(0)|^2} j\left(\frac{\dot{z}(0)}{|\dot{z}(0)|}\right)\dot{y}(0) = \frac{2|\dot{z}(0)|}{|\dot{y}(0)|^2}\Bigg(&\sin\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|}t\right)(\dot{y}(0) + \dot{z}(0)) \\ &+ \cos\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|}t\right) j\left(\frac{\dot{z}(0)}{|\dot{z}(0)|}\right)\dot{y}(0)\Bigg) \end{aligned}$$

the curve $x(t)$ is an elliptic such that the ratio of the long axis to the short axis is equal to $\sqrt{|\dot{y}(0)|^2 + |\dot{z}(0)|^2}/|\dot{y}(0)|$. □

Acknowledgments

The author was partially supported by the Japan Society for Promotion of Science through a Grant-in-Aid for Scientific Research (# 22540108).

References

[1] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[2] Ikawa O., *Motion of Charged Particles From the Geometric View Point*, J. Geom. Symm. Phys. **18** (2010) 23–47.

[3] Kaplan A., *Riemannian Nilmanifiolds Attached to Clifford Modules*, Geometriae Dedicata **11** (1981) 127–136.

[4] Naitoh H. and Sakane Y., *On Conjugate Points of a Nilpotent Lie Group*, Tsukuba J. Math. **5** (1981) 143–152.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 263–289

MODULAR FORMS ON BALL QUOTIENTS OF NON-POSITIVE KODAIRA DIMENSION*

AZNIV KASPARIAN

Department of Mathematics, Kliment Ohridski University, Sofia 1164, Bulgaria

Abstract. The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of an arithmetic ball quotient admits projective embeddings by Γ-modular forms of sufficiently large weight. We are interested in the target and the rank of the projective map Φ, determined by Γ-modular forms of weight one. This paper concentrates on the finite H-Galois quotients $\mathbb{B}/\Gamma_H$ of a specific $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, birational to an abelian surface A_{-1}. Any compactification of $\mathbb{B}/\Gamma_H$ has non-positive Kodaira dimension. The rational maps Φ^H of $\widehat{\mathbb{B}/\Gamma_H}$ are studied by means of the H-invariant abelian functions on A_{-1}.

CONTENTS

*Reprinted from J. Geom. Symm. Phys. **20** (2011) 69–96.

1. Introduction

The modular forms of sufficiently large weight are known to provide projective embeddings of the arithmetic quotients of the **two-ball**

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 ; |z_1|^2 + |z_2|^2 < 1\} \simeq \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1).$$

The present work studies the projective maps, given by the modular forms of weight one on certain Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_H}$ of Kodaira dimension $\kappa(\widehat{\mathbb{B}/\Gamma_H}) \leq 0$. More precisely, we start with a fixed smooth Picard modular surface $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ with abelian minimal model $A_{-1} = E_{-1} \times E_{-1}$, $E_{-1} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$. Any automorphism group of A'_{-1}, preserving the toroidal compactifying divisor $T' = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \setminus \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)$ acts on A_{-1} and lifts to a ball lattice Γ_H, normalizing $\Gamma_{-1}^{(6,8)}$. The ball quotient compactification $A'_{-1}/H = \overline{\mathbb{B}/\Gamma_H}$ is birational to A_{-1}/H. We study the Γ_H-modular forms $[\Gamma_H, 1]$ of weight one by realizing them as H-invariants of $[\Gamma_{-1}^{(6,8)}, 1]$. That allows to transfer $[\Gamma_H, 1]$ to the H-invariant abelian functions, in order to determine $\dim_{\mathbb{C}}[\Gamma_H, 1]$ and the transcendence dimension of the graded $\mathbb{C}$-algebra, generated by $[\Gamma_H, 1]$. The last one is exactly the rank of the projective map $\Phi : \widetilde{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$.

2. The Transfer of Modular Forms to Meromorphic Functions is Inherited by the Finite Galois Quotients

Definition 1. *Let* $\Gamma < \mathrm{SU}(2,1)$ *be a lattice, i.e., a discrete subgroup, whose quotient* $\mathrm{SU}(2,1)/\Gamma$ *has finite invariant measure. A* Γ*-modular form of weight* n *is a holomorphic function* $\delta : \mathbb{B} \to \mathbb{C}$ *with transformation law*

$$\gamma(\delta)(z) = \delta(\gamma(z)) = [\det \mathrm{Jac}(\gamma)]^{-n}\delta(z), \qquad \gamma \in \Gamma, \quad z \in \mathbb{B}.$$

Bearing in mind that a biholomorphism $\gamma \in \mathrm{Aut}(\mathbb{B})$ acts on a differential form $dz_1 \wedge dz_2$ of top degree as a multiplication by the Jacobian determinant $\det \mathrm{Jac}(\gamma)$, one constructs the linear isomorphism

$$j_n : [\Gamma, n] \longrightarrow H^0(\mathbb{B}, (\Omega^2_{\mathbb{B}})^{\otimes n})^{\Gamma}$$

with the Γ-invariant holomorphic sections of the canonical bundle $\Omega^2_{\mathbb{B}}$ of $\mathbb{B}$. Thus, the graded $\mathbb{C}$-algebra of the Γ-modular forms can be viewed as the tensor algebra of the Γ-invariant volume forms on $\mathbb{B}$. For any $\delta_1, \delta_2 \in [\Gamma, n]$ the quotient $\frac{\delta_1}{\delta_2}$ is a correctly defined holomorphic function on $\mathbb{B}/\Gamma$. In such a way, $[\Gamma, n]$ and $j_n[\Gamma, n]$ determine a projective map

$$\Phi_n : \mathbb{B}/\Gamma \longrightarrow \mathbb{P}([\Gamma, n]) = \mathbb{P}(j_n[\Gamma, n]).$$

The Γ-cusps $\partial_\Gamma \mathbb{B}/\Gamma$ are of complex co-dimension two, so that Φ_n extends to the Baily-Borel compactification

$$\Phi_n : \widehat{\mathbb{B}/\Gamma} \longrightarrow \mathbb{P}([\Gamma, n]).$$

If the lattice $\Gamma < \mathrm{SU}_{2,1}$ is torsion-free then the toroidal compactification $X' = (\mathbb{B}/\Gamma)'$ is a smooth surface. Denote by $\rho : X' = (\mathbb{B}/\Gamma)' \to \widehat{X} = \widehat{\mathbb{B}/\Gamma}$ the contraction of the irreducible components T_i' of the toroidal compactifying divisor T' to the Γ-cusps $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$. The tensor product $\Omega^2_{X'}(T')$ of the canonical bundle $\Omega^2_{X'}$ of X' with the holomorphic line bundle $\mathcal{O}(T')$, associated with the toroidal compactifying divisor T' is the logarithmic canonical bundle of X'. In [2] Hemperly has observes that

$$H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_n[\Gamma, n] \simeq [\Gamma, n].$$

Let $K_{X'}$ be the canonical divisor of X'

$$\mathcal{L}_{X'}(nK_{X'} + nT') = \{f \in \quad (X'); \ (f) + nK_{X'} + nT' \geq 0\}$$

be the linear system of the divisor $n(K_{X'} + T')$ and s be a global meromorphic section of $\Omega^2_{X'}(T')$. Then

$$s^{\otimes n} : \mathcal{L}_{X'}(nK_{X'} + nT') \longrightarrow H^0(X', \Omega^2_{X'}(T')^{\otimes n})$$

is a $\mathbb{C}$-linear isomorphism. Let $\xi : X' \to X$ be the blow-down of the (-1)-curves on $X' = (\mathbb{B}/\Gamma)'$ to its minimal model X. The Kobayashi hyperbolicity of $\mathbb{B}$ requires X' to be the blow-up of X at the singular locus T^{sing} of $T = \xi(T')$. The canonical divisor $K_{X'} = \xi^* K_X + L$ is the sum of the pull-back of K_X with the exceptional divisor L of ξ. The birational map ξ induces an isomorphism $\xi^* : \quad (X) \to \quad (X')$ of the meromorphic function fields. In order to translate the condition $\xi^*(f) + nK_{X'} + nT' \geq 0$ in terms of $f \in \quad (X)$, let us recall the notion of a multiplicity of a divisor $D \subset X$ at a point $p \in X$. If $D = \sum_i n_i D_i$ is the decomposition of D into irreducible components then $m_p(D) = \sum_i n_i m_p(D_i)$, where

$$m_p(D_i) = \begin{cases} 1 & \text{for } p \in D_i \\ 0 & \text{for } p \notin D_i. \end{cases}$$

Let $L = \sum_{p \in T^{\mathrm{sing}}} L(p)$ for $L(p) = \xi^{-1}(p)$ and $f \in \quad (X)$. The condition $\xi^*(f) + nL \geq 0$ is equivalent to $m_p(f) + n \geq 0$ for all $p \in T^{\mathrm{sing}}$. Thus, $\mathcal{L}_{X'}(nK_{X'} + nT')$ turns to be the pull-back of the subspace

$$\mathcal{L}_X(nK_X + nT, nT^{\mathrm{sing}})$$

$$= \{f \in \quad (X); \ (f) + nK_X + nT \geq 0, \ \ m_p(f) + n \geq 0, \ \ p \in T^{\mathrm{sing}}\}$$

of the linear system $\mathcal{L}_X(nK_X + nT)$. The $\mathbb{C}$-linear isomorphism

$$\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_X(nK_X + nT, nT^{\mathrm{sing}})$$

introduced by Holzapfel in [3], is called **transfer of modular forms**.

Bearing in mind Hemperly's result $H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_1[\Gamma, n]$ for a fixed point free Γ, we refer to

$$\Phi_n^H : \widehat{\mathbb{B}/\Gamma_H} \longrightarrow \mathbb{P}([\Gamma_H, n]) = \mathbb{P}(j_n[\Gamma_H, n])$$

as the n-th logarithmic-canonical map of $\widehat{\mathbb{B}/\Gamma_H}$, regardless of the ramifications of $\mathbb{B} \to \mathbb{B}/\Gamma_H$.

The next lemma explains the transfer of modular forms on finite Galois quotients $\mathbb{B}/\Gamma_H$ of $\mathbb{B}/\Gamma$ to meromorphic functions on X/H. In general, the toroidal compactification $X'_H = (\mathbb{B}/\Gamma_H)'$ is a normal surface. The logarithmic-canonical bundle is not defined on a singular X'_H, but there is always a logarithmic-canonical Weil divisor on X'_H.

Lemma 1. *Let $A' = (\mathbb{B}/\Gamma)'$ be a neat toroidal compactification with an abelian minimal model A and H be a subgroup of $G = \mathrm{Aut}(A, T) = \mathrm{Aut}(A', T')$. Then*

i) *the transfer* $\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})$ *of Γ-modular forms to abelian functions induces a linear isomorphism*

$$\mathrm{Trans}_n^H : [\Gamma_H, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H$$

of Γ_H-modular forms with rational functions on A/H, called also a transfer

ii) *the projective maps*

$$\Phi_n^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, n]), \quad \Psi_n^H : A/H \dashrightarrow \mathbb{P}(\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H)$$

coincide on an open Zariski dense subset.

Proof: i) Note that $j_n[\Gamma_H, n] = j_n[\Gamma, n]^H$. The inclusion $j_n[\Gamma_H, n] \subseteq j_n[\Gamma, n]$ follows from $\Gamma \leq \Gamma_H$. If $\Gamma_H = \cup_{j=1}^n \gamma_j \Gamma$ is the coset decomposition of Γ_H modulo Γ, then $H = \{h_i = \gamma_i \Gamma;\ 1 \leq i \leq n\}$. A Γ-modular form $\omega \in j_n[\Gamma, n]$ is Γ_H-modular exactly when it is invariant under all γ_i, which amounts to the invariance under all h_i.

One needs a global meromorphic G-invariant section s of $\Omega^2_{A'}(T')$, in order to obtain a linear isomorphism

$$(\xi^*)^{-1} s^{\otimes(-n)} = \mathrm{Trans}_n^H j_n^{-1} : j_n[\Gamma_H, n] = j_n[\Gamma, n]^H \to \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H.$$

The global meromorphic sections of the logarithmic-canonical line bundle $\Omega^2_{A'}(T')$ are in a bijective correspondence with the families $(f_\alpha, U_\alpha)_{\alpha \in S}$ of local meromorphic defining equations $f_\alpha : U_\alpha \to \mathbb{C}$ of the logarithmic-canonical divisor $L + T'$. We construct local meromorphic G-invariant equations $g_\alpha : V_\alpha \to \mathbb{C}$ of T and pull-back to $(f_\alpha = \xi^* g_\alpha, U_\alpha = \xi^{-1}(V_\alpha))_{\alpha \in S}$. Let $F_A : \widetilde{A} = \mathbb{C}^2 \to A$ be the universal covering map of A. Then for any point $p \in A$ choose a lifting $\widetilde{p} \in F_A^{-1}(p)$ and a sufficiently small neighborhood $\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$, which is contained in the interior of a $\pi_1(A)$-fundamental domain on $\widetilde{A}$, centered at $\widetilde{p}$. The G-invariant open neighborhood $W = \cap_{g \in G} g\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$ intersects $F_A^{-1}(T)$ in lines with local equations $l_j(u,v) = a_j(\widetilde{p})u + b_j(\widetilde{p})v + c_j(\widetilde{p}) = 0$. The holomorphic function $g(u,v) = \prod_{g \in G} \prod_j (l_j(u,v))$ on W is G-invariant and can be viewed as a G-invariant local defining equation of T on $V = F_A(W)$. Note that F_A is locally biholomorphic, so that $V \subset A$ is an open subset, after an eventual shrinking of $\widetilde{W}$. The family $(g, V)_{p \in A}$ of local G-invariant defining equations of T pullbacks to a family $(f = \xi^* g, U = \xi^{-1}(V))_{p \in A}$ of local G-invariant sections of $\Omega^2_A(T')$.

ii) Towards the coincidence $\Psi_n^H|_{[(A\setminus T)/H]} \equiv \Phi_n^H|_{[(\mathbb{B}/\Gamma_H)\setminus(L/H)]}$, let us fix a basis $\{\omega_i;\ 1 \leq i \leq d\}$ of $j_n[\Gamma_H, n]$ and apply i), in order to conclude that the set $\{f_i = \mathrm{Trans}_n^H j_n^{-1}(\omega_i);\ 1 \leq i \leq d\}$ is a basis of $\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H$. Tensoring by $s^{\otimes(-n)}$ does not alter the ratios $\frac{\omega_i}{\omega_j}$. The isomorphism $\xi :\quad (A) \to \quad (A')$ is identical on $(A \setminus T)/H$. □

3. Preliminaries

In order to specify $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ let us note that the blow-down $\xi : A'_{-1} \to A_{-1}$ of the (-1)-curves maps T' to a divisor $T = \xi(T')$ with smooth elliptic irreducible components T_i. Such T are called multi-elliptic divisors. Any irreducible component T_i of T lifts to a $\pi_1(A_{-1})$-orbit of complex lines on the universal cover $\widetilde{A'_{-1}} = \mathbb{C}^2$. That allows to represent

$$T_j = \{(u (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}), v (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}));\ a_j u + b_j v + c_j = 0\}.$$

If T_j is defined over the field $\mathbb{Q}(\mathrm{i})$ of Gauss numbers, there is no loss of generality in assuming $a_j, b_j \in \mathbb{Z}[\mathrm{i}]$ to be Gaussian integers.

Theorem 1 (Holzapfel [4]). *Let $A_{-1} = E_{-1} \times E_{-1}$ be the Cartesian square of the elliptic curve $E_{-1} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\mathrm{i}$, $\omega_1 = \frac{1}{2}$, $\omega_2 = i\omega_1$, $\omega_3 = \omega_1 + \omega_2$ be half-periods,*

$$Q_0 = 0(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}),\ \ Q_1 = \omega_1(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}),\ \ Q_2 = \mathrm{i}Q_1,\ \ Q_3 = Q_1 + Q_2$$

be the two-torsion points on E_{-1}, $Q_{ij} = (Q_i, Q_j) \in A_{-1}^{2-\mathrm{tor}}$ and

$$T_k = \{(u(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i});\ u - i^k v = 0\} \ \ \textit{with} \ \ 1 \leq k \leq 4,$$

$$T_{4+m} = \{u(\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}\mathrm{i});\ u - \omega_m = 0\} \ \textit{for} \ \ 1 \leq m \leq 2 \ \ \textit{and}$$

$$T_{6+m} = \{u(\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}\mathrm{i});\ v - \omega_m = 0\} \ \textit{for} \ \ 1 \leq m \leq 2.$$

Then the blow-up of A_{-1} *at the singular locus* $\left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}} = Q_{00} + Q_{33} + \sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij}$ *of the multi-elliptic divisor* $T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ *is a neat toroidal ball quotient compactification* $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$.

Theorem 2 (Kasparian and Kotzev [6])**.** *The group* $G_{-1} = \mathrm{Aut}(A_{-1}, T_{-1}^{(6,8)}) = \mathrm{Aut}(A'_{-1}, T')$ *of order* 64 *is generated by the translation* τ_{33} *with* Q_{33}*, the multiplications*

$$I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}, \quad \textit{respectively} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

with $\mathrm{i} \in \mathbb{Z}[\mathrm{i}]$ *on the first, respectively, the second factor* E_{-1} *of* A_{-1} *and the transposition*

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of these factors.

Throughout, we use the notations from Theorem 1 and Theorem 2, without mentioning this explicitly. With a slight abuse of notation, we speak of Kodaira-Enriques classification type, irregularity and geometric genus of A_{-1}/H, $H \leq G_{-1}$, referring actually to a smooth minimal model Y of A_{-1}/H.

Theorem 3 (Kasparian and Nikolova [7])**.** *Let*

$$\mathcal{L} : G_{-1} \to \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}]) = \{g \in \mathbb{Z}[\mathrm{i}]_{2\times 2};\ \det(g) \in \mathbb{Z}[\mathrm{i}]^* = \langle \mathrm{i} \rangle\}$$

be the homomorphism, associating to $g \in G_{-1}$ *its linear part* $\mathcal{L}$ *and*

$$L_1(G_{-1}) = \{g \in G_{-1};\ \mathrm{rk}(\mathcal{L}(g) - I_2) = 1\}$$

$$= \{\tau_{33}^n I^k, \tau_{33}^n J^k, \tau_{33}^n I^l J^{-l}\theta;\ 0 \leq n \leq 1,\ \ 1 \leq k \leq 3,\ \ 0 \leq l \leq 3\}.$$

Then

i) A_{-1}/H *is an abelian surface for* $H = \langle \tau_{33} \rangle$

ii) A_{-1}/H *is a hyperelliptic surface for* $H = \langle \tau_{33} I^2 \rangle$ *or* $H = \langle \tau_{33} J^2 \rangle$

iii) A_{-1}/H *is a ruled surface with an elliptic base for*

$$H = \langle h \rangle,\ \ h \in L_1(G_{-1}) \setminus \{\tau_{33} I^2, \tau_{33} J^2\} \quad \textit{or} \quad H = \langle \tau_{33}, h_o \rangle,\ \ h_o \in \mathcal{L}(L_1(G_{-1}))$$

iv) A_{-1}/H *is a* K3 *surface for* $\langle \tau_{33}^n \rangle \neq H \leq K = \ker\det\mathcal{L}$*, where*

$$K = \{\tau_{33}^n I^k J^{-k}, \tau_{33}^n I^k J^{2-k}\theta;\ 0 \leq n \leq 1,\ \ 0 \leq k \leq 3\}$$

v) A_{-1}/H *is an Enriques surface for* $H = \langle I^2 J^2, \tau_{33} I^2 \rangle$

vi) *A_{-1}/H is a rational surface for*

$$\langle h\rangle \leq H,\ \ h\in\{\tau_{33}^n IJ, \tau_{33}^n I^2J, \tau_{33}^n IJ^2;\, 0\leq n\leq 1\}\ \ \textit{or}\ \ \langle \tau_{33}^n I^2J^2, h_1\rangle \leq H$$

$$h_1\in\{I^{2m}J^{2-2m}, \tau_{33}^m I, \tau_{33}^m J, \tau_{33}^m I^l J^{-l}\theta;\, 0\leq m\leq 1,\ \ 0\leq l\leq 3\},\ \ 0\leq n\leq 1.$$

The following lemma specifies some known properties of Weierstrass σ-function over Gaussian integers $\mathbb{Z}[\mathrm{i}]$.

Lemma 2. *Let* $\sigma(z) = z \prod_{\lambda\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}} (1-\frac{z}{\lambda})^{\frac{z}{\lambda}+\frac{1}{2}(\frac{z}{\lambda})^2}$ *be the* ***Weierstrass σ-function****, associated with the lattice* $\mathbb{Z}[\mathrm{i}]$ *of* $\mathbb{C}$*. Then*

i) $\sigma(\mathrm{i}^k z) = \mathrm{i}^k\sigma(z), \quad z\in\mathbb{C}, \quad 0\leq k\leq 3$

ii) $\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda)e^{-\pi\overline{\lambda}z-\frac{\pi}{2}|\lambda|^2}, \quad z\in\mathbb{C}, \quad \lambda\in\mathbb{Z}[\mathrm{i}]$, *where*

$$\varepsilon(\lambda) = \begin{cases} -1 & \textit{if } \lambda\in\mathbb{Z}[\mathrm{i}]\setminus 2\mathbb{Z}[\mathrm{i}] \\ 1 & \textit{if } \lambda\in 2\mathbb{Z}[\mathrm{i}]. \end{cases}$$

Proof: i) follows from

$$\prod_{\lambda\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}}\left(1-\frac{\mathrm{i}^k z}{\lambda}\right)^{\frac{\mathrm{i}^k z}{\lambda}+\frac{1}{2}\left(\frac{\mathrm{i}^k z}{\lambda}\right)^2} = \prod_{\mu=\frac{\lambda}{\mathrm{i}^k}\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}}\left(1-\frac{z}{\mu}\right)^{\frac{z}{\mu}+\frac{1}{2}\left(\frac{z}{\mu}\right)^2}.$$

ii) According to Lang's book [8]

$$\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(z+\frac{\lambda}{2}\right)}, \qquad z\in\mathbb{C}, \qquad \lambda\in\mathbb{Z}[\mathrm{i}]$$

where $\eta:\mathbb{Z}[\mathrm{i}]\to\mathbb{C}$ is the homomorphism of $\mathbb{Z}$-modules, related to **Weierstrass ζ-function** $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ by the identity $\zeta(z+\lambda) = \zeta(z)+\eta(\lambda)$. It suffices to establish that $\eta(\lambda) = -\pi\overline{\lambda}$, $\lambda\in\mathbb{Z}[\mathrm{i}]$. Recall from [8] Legendre's equality $\eta(\mathrm{i})-\mathrm{i}\eta(1) = 2\pi\mathrm{i}$, in order to derive

$$\eta(\lambda) = \frac{\lambda+\overline{\lambda}}{2}\eta(1)+\frac{\lambda-\overline{\lambda}}{2\mathrm{i}}\eta(\mathrm{i}) = (\eta(1)+\pi)\lambda-\pi\overline{\lambda}, \quad \lambda\in\mathbb{Z}[\mathrm{i}].$$

Combining with homogeneity $\eta(\mathrm{i}\lambda) = \frac{1}{\mathrm{i}}\eta(\lambda)$, $\lambda\in\mathbb{Z}[\mathrm{i}]$ (cf.[8]), one obtains

$$(\eta(1)+\pi)\mathrm{i}\lambda+\pi\mathrm{i}\overline{\lambda} = \eta(\mathrm{i}\lambda) = -\mathrm{i}\eta(\lambda) = -(\eta(1)+\pi)\mathrm{i}\lambda+\pi\mathrm{i}\overline{\lambda}, \quad \lambda\in\mathbb{Z}[\mathrm{i}].$$

Therefore $\eta(1) = -\pi$ and $\eta(\lambda) = -\pi\overline{\lambda}$, $\lambda\in\mathbb{Z}[\mathrm{i}]$. □

Corollary 1.

$$\frac{\sigma(z+\omega_m)}{\sigma(z-\omega_m)} = -\mathrm{e}^{2(-1)^m\omega_m\pi z}$$

$$\frac{\sigma(z+\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)} = (-1)^{m+1}\varepsilon \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z+2(-1)^m\omega_m\pi z}$$

$$\frac{\sigma(z-\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)} = (-1)^{m+1}\varepsilon \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z}.$$

for the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \mathrm{i}\omega_1$ *and* $\varepsilon = \pm 1$.

Corollary 2.

$$\frac{\sigma(z+2\varepsilon\omega_m)}{\sigma(z-1)} = \mathrm{e}^{-\pi z+(-1)^m 2\varepsilon\pi\omega_m z}$$

$$\frac{\sigma(z+(-1)^m\omega_m+\varepsilon(-1)^m\omega_{3-m})}{\sigma(z-(-1)^m\omega_m+(-1)^m\omega_{3-m})} = -\mathrm{i}^{(-1)^m\frac{(1+\varepsilon)}{2}}\mathrm{e}^{2\omega_m\pi z+(1-\varepsilon)\omega_{3-m}\pi z}.$$

for the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \mathrm{i}\omega_1$ *and* $\varepsilon = \pm 1$.

Corollary 1 and Corollary 2 follow from Lemma 2 ii) and $\bar{\omega}_m = (-1)^{m+1}\omega_m$, $\omega_m^2 = \frac{(-1)^{m+1}}{4}$.

In [5] the map $\Phi : \widetilde{\mathbb{B}/\Gamma_{-1}^{(6,8)}} \to \mathbb{P}([\Gamma_{-1}^{(6,8)}, 1])$ is shown to be a regular embedding. This is done by constructing a $\mathbb{C}$-basis of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$, consisting of binary parallel or triangular σ-quotients. An abelian function $f_{\alpha,\beta} \in \mathcal{L}$ is binary parallel if the pole divisor $(f_{\alpha,\beta})_\infty = T_\alpha + T_\beta$ consists of two disjoint smooth elliptic curves T_α and T_β. A σ-quotient $f_{i,\alpha,\beta} \in \mathcal{L}$ is triangular if $T_i \cap T_\alpha \cap T_\beta = \emptyset$ and any two of T_i, T_α and T_β intersect in a single point.

Theorem 4 (Kasparian and Kotzev [5]). *Let*

$$\Sigma_{12}(z) = \frac{\sigma(z-1)\sigma(z+\omega_1-\omega_2)}{\sigma(z-\omega_1)\sigma(z-\omega_2)}, \qquad \Sigma_1 = \frac{\sigma(u-\mathrm{i}v+\omega_3)}{\sigma(u-\mathrm{i}v)}$$

$$\Sigma_2 = \frac{\sigma(u+v+\omega_3)}{\sigma(u+v)}, \quad \Sigma_3 = \frac{\sigma(u+\mathrm{i}v+\omega_3)}{\sigma(u+\mathrm{i}v)}, \quad \Sigma_4 = \frac{\sigma(u-v+\omega_3)}{\sigma(u-v)}$$

$$\Sigma_5 = \frac{\sigma(u-\omega_2)}{\sigma(u-\omega_1)}, \ \Sigma_6 = \frac{\sigma(u-\omega_1)}{\sigma(u-\omega_2)}, \ \Sigma_7 = \frac{\sigma(v-\omega_2)}{\sigma(v-\omega_1)}, \ \Sigma_8 = \frac{\sigma(v-\omega_1)}{\sigma(v-\omega_2)}.$$

Then

i) *the space* $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{\sqrt{-1}}^{(6,8)}, \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\mathrm{sing}}\right)$ *contains the binary parallel* σ*-quotients* $f_{56}(u,v) = \Sigma_{12}(u)$, $f_{78}(u,v) = \Sigma_{12}(v)$ *and the triangular*

σ-quotients

$f_{157} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u}\Sigma_1\Sigma_5\Sigma_7,$ $\quad f_{168} = -\,\mathrm{e}^{-\pi-\pi\mathrm{i}u-\pi v-\pi\mathrm{i}v}\Sigma_1\Sigma_6\Sigma_8$

$f_{357} = -\,\mathrm{e}^{-\pi+\pi u+\pi v+\pi\mathrm{i}v}\Sigma_3\Sigma_5\Sigma_7,$ $\quad f_{368} = -\,\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi\mathrm{i}u}\Sigma_3\Sigma_6\Sigma_8$

$f_{258} = \mathrm{e}^{-\pi+\pi u-\pi\mathrm{i}v}\Sigma_2\Sigma_5\Sigma_8,$ $\quad f_{267} = \mathrm{e}^{-\pi-\pi\mathrm{i}u+\pi v}\Sigma_2\Sigma_6\Sigma_7$

$f_{458} = -\,\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u-\pi v}\Sigma_4\Sigma_5\Sigma_8,$ $\quad f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi\mathrm{i}u+\pi\mathrm{i}v}\Sigma_4\Sigma_6\Sigma_7$

ii) *a $\mathbb{C}$-basis of $\mathcal{L}$ is*

$$f_o := 1,\ f_1 := f_{157},\ f_2 := f_{258},\ f_3 := f_{368},\ f_4 := f_{467},\ f_5 := f_{56},\ f_6 := f_{78}.$$

4. Technical Preparation

The group $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ permutes the eight irreducible components of $T_{-1}^{(6,8)}$ and the $\Gamma_{-1}^{(6,8)}$-cusps. For any subgroup H of G_{-1}, the Γ_H-cusps are the H-orbits of $\partial_{\Gamma_{-1}^{(6,8)}}\mathbb{B}/\Gamma_{-1}^{(6,8)} = \{\kappa_i\,;\ 1 \leq i \leq 8\}$.

Lemma 3. *If $\varphi : G_{-1} \to S_8(\kappa_1, \ldots, \kappa_8)$ is the natural representation of $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ in the symmetric group of the $\Gamma_{-1}^{(6,8)}$-cusps, then*

$$\varphi(\tau_{33}) = (\kappa_5, \kappa_6)(\kappa_7, \kappa_8), \qquad \varphi(I) = (\kappa_1, \kappa_4, \kappa_3, \kappa_2)(\kappa_5, \kappa_6)$$
$$\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8), \qquad \varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8).$$

Proof: The $\Gamma_{-1}^{(6,8)}$-cusps κ_i are obtained by contraction of the proper transforms T_i' of T_i under the blow-up of A_{-1} at $\left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}$. Therefore the representations of G_{-1} in the permutation groups of $\{T_i\,;\ 1 \leq i \leq 8\}$, $\{T_i'\,;\ 1 \leq i \leq 8\}$ and $\{\kappa_i\,;\ 1 \leq i \leq 8\}$ coincide.

According to $\tau_{33}(u - \mathrm{i}^k v) = u - \mathrm{i}^k v + (1 - \mathrm{i}^k)\omega_3 = u - \mathrm{i}^k v (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i})$, the translation τ_{33} acts identically on T_1, T_2, T_3, T_4. Further, $\tau_{33}(u - \omega_m) = u + \omega_{3-m} \equiv u - \omega_{3-m} (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i})$ reveals the permutation $(T_5, T_6)(T_7, T_8)$ of the last four components of $T_{-1}^{(6,8)}$.

Due to the identity $I(u - \mathrm{i}^k v) = \mathrm{i}u - \mathrm{i}^k v = \mathrm{i}(u - \mathrm{i}^{k-1} v)$, the automorphism I induces the permutation (T_1, T_4, T_3, T_2) of the first four components of $T_{-1}^{(6,8)}$. Further, $I(u - \omega_m) = \mathrm{i}(u \pm \omega_{3-m})$ reveals that I permutes T_5 with T_6. Note that I acts identically on v and fixes T_7, T_8.

In a similar vein, $J(u - \mathrm{i}^k v) = u - \mathrm{i}^{k+1} v$, $J(v - \omega_m) = \mathrm{i}(v \pm \mathrm{i}\omega_{3-m})$ determine that $\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8)$. According to $\theta(u - \mathrm{i}^k v) = v - \mathrm{i}^k u = -\mathrm{i}^k(u - \mathrm{i}^{-k} v)$ and $\theta(u - \omega_m) = v - \omega_m$, one concludes that $\varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8)$. $\square$

The following lemma incorporates several arguments, which will be applied repeatedly towards determination of the target $\mathbb{P}([\Gamma_H, 1])$ and the rank of the logarithmic canonical map Φ^H.

Lemma 4. *In the notations from* Theorem 4, *for an arbitrary subgroup* H *of* $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ *and any* $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$, *let* $R_H(f) = \sum_{h \in H} h(f)$ *be the value of* ***Reynolds operator*** R_H *of* H *on* f.

i) *The space* $\mathcal{L}^H$ *of the* H*-invariants of* $\mathcal{L}$ *is spanned by* $\{R_H(f_i)\,;\ 0 \leq i \leq 6\}$.

ii) *Let* $T_i \subset (R_H(f_{i,\alpha_1,\beta_1}))_\infty, (R_H(f_{i,\alpha_2,\beta_2}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum_{\alpha=5}^{8} T_\alpha$ *for some* $1 \leq i \leq 4$, $5 \leq \alpha_j \leq 6$, $7 \leq \beta_j \leq 8$. *Then*

$$R_H(f_{i,\alpha_2,\beta_2}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}), R_H(f_{i,\alpha_1,\beta_1})).$$

iii) *Let* $\bar{\kappa}_{i_1}, \ldots, \bar{\kappa}_{i_p}$ *with* $1 \leq i_1 < \ldots < i_p \leq 4$ *be different* Γ_H*-cusps*

$$T_{i_j} \subset (R_H(f_{i_j}))_\infty \subseteq \mathrm{Orb}_H(T_{i_j}) + \sum_{\alpha=5}^{8} T_\alpha \quad \textit{for all} \quad 1 \leq j \leq p$$

and B *be a* $\mathbb{C}$*-basis of* $\mathcal{L}_2^H = \mathcal{L}_{A_{-1}}\left(\sum_{\alpha=5}^{8} T_\alpha\right)^H$. *Then the set*

$$\{R_H(f_{i_j,\alpha_j,\beta_j})\,;\ 1 \leq j \leq p\} \cup B$$

consists of linearly independent invariants over $\mathbb{C}$.

iv) *If* $R_j = R_H(f_{j,\alpha_j,\beta_j}) \not\equiv \mathrm{const}$, $R_j|_{T_j} = \infty$ *and* $R_i = R_H(f_{i,\alpha_i,\beta_i})$ *has* $R_i|_{T_j} \not\equiv \mathrm{const}$ *then for any subgroup* H_o *of* H *the projective maps*

$$\Psi^{H_o} : X/H_o \dashrightarrow \mathbb{P}(\mathcal{L}^{H_o}), \qquad \Phi^{H_o} : \widehat{\mathbb{B}/\Gamma_{H_o}} \dashrightarrow \mathbb{P}(j_1[\Gamma_{H_o}, 1])$$

are of rank $\mathrm{rk}\Phi^{H_o} = \mathrm{rk}\Psi^{H_o} = 2$.

v) *If the group* H' *is obtained from the group* H *by replacing all* $\tau_{33}^n I^k J^l \theta^m \in H$ *with* $\tau_{33}^n I^l J^k \theta^m$, *then the spaces of modular forms* $j_1[\Gamma_{H'}, 1] \simeq j_1[\Gamma_H, 1]$ *are isomorphic and the logarithmic-canonical maps have equal rank* $\mathrm{rk}\Phi^H = \mathrm{rk}\Phi^{H'}$.

Proof: i) By Theorem 4 ii), $\mathcal{L} = \mathrm{Span}_{\mathbb{C}}(f_i\,;\ 0 \leq 6)$. Therefore any $f \in \mathcal{L}$ is a $\mathbb{C}$-linear combination $f = \sum_{i=0}^{6} c_i f_i$. Due to H-invariance of f and the linearity of the representation of H in $\mathrm{Aut}(\mathcal{L})$, Reynolds operator

$$|H| f = R_H(f) = \sum_{i=0}^{6} c_i R_H(f_i).$$

ii) Let $\omega_s \in j_1 \left[\Gamma_{-1}^{(6,8)}, 1\right]^H$ are the modular forms, which transfer to $R_H(f_{i,\alpha_s,\beta_s})$, $1 \leq s \leq 2$. Since $\omega_1(\kappa_i) \neq 0$, there exists $c_i \in \mathbb{C}$, such that $\omega_i' = \omega_2 - c_i\omega_1$ vanishes at κ_i. By the assumption $(R_H(f_{i,\alpha_s,\beta_s}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum_{\alpha=5}^{8} T_\alpha$, the transfer $F_i \in \mathcal{L}^H$ of ω_i' belongs to $\mathrm{Span}_{\mathbb{C}}(1, f_{56}, f_{78})^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$.
iii) As far as the transfer $\mathrm{Trans}_1^H : j_1[\Gamma_H, 1] \to \mathcal{L}$ is a $\mathbb{C}$-linear isomorphism, it suffices to establish the linear independence of the corresponding modular forms $\{\omega_{i_j};\, 1 \leq j \leq p\} \cup \{\omega_b;\, b \in B\}$. Evaluating the $\mathbb{C}$-linear combination $\sum_{j=1}^{p} c_{i_j}\omega_{i_j} + \sum_{b\in B} c_b\omega_b = 0$ at $\bar{\kappa}_{i_1}, \ldots, \bar{\kappa}_{i_p}$, one obtains $c_{i_j} = 0$, according to $\omega_{i_j}(\bar{\kappa}_{i_s}) = \delta_j^s$ and $\omega_b(\bar{\kappa}_{i_j}) = 0$, $b \in B$, $1 \leq j \leq p$. Then $\sum_{b\in B} \omega_b = 0$ requires the vanishing of all c_b, due to the linear independence of B.
iv) If H_o is a subgroup of H then $\mathcal{L}^H$ is a subspace of $\mathcal{L}^{H_o}$, $j_1[\Gamma_H, 1]$ is a subspace of $j_1[\Gamma_{H_o}, 1]$ and $\Psi^H = \mathrm{pr}^{\mathcal{L}}\Psi^{H_o}$, $\Phi^H = \mathrm{pr}^{\Gamma_H}\Phi^{H_o}$ for the projections $\mathrm{pr}^{\mathcal{L}} : \mathbb{P}(\mathcal{L}^{H_o}) \to \mathbb{P}(\mathcal{L}^H)$, $\mathrm{pr}^{\Gamma_H} : \mathbb{P}(j_1[\Gamma_{H_o}, 1]) \to \mathbb{P}(j_1[\Gamma_H, 1])$. That is why, it suffices to justify that $\mathrm{rk}\Phi^H = \mathrm{rk}\Psi^H = 2$ is maximal. Assume the opposite and consider $R_i, R_j : X/H \dashrightarrow \mathbb{P}^1$. The commutative diagram

$$\begin{array}{ccc} X/H & \xrightarrow{(R_i,R_j)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle R_j}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ \mathbb{P}^1 & & \end{array}$$

has surjective R_j, as far as $R_j \not\equiv \mathrm{const}$. If the image $C = (R_i, R_j)(X/H)$ is a curve, then the projection $\mathrm{pr}_2 : C \to \mathbb{P}^1$ has only finite fibers. In particular, $\mathrm{pr}_2^{-1}(\infty) = R_i((R_j)_\infty) \times \infty \supseteq R_i(T_j) \times \infty$ consists of finitely many points. However, $R_i(T_j) = \mathbb{P}^1$ as an image of the non-constant elliptic function R_i: $T_j \dashrightarrow \mathbb{P}^1$. The contradiction implies that $\dim_{\mathbb{C}} C = 2$ and $\mathrm{rk}\Psi^H = 2$.
v) The transposition of the holomorphic coordinates $(u, v) \in \mathbb{C}^2$ affects non-trivially the constructed σ-quotients. However, one can replace the equations $u - \mathrm{i}^k v = 0$ of T_k, $1 \leq k \leq 4$ by $v - \mathrm{i}^{-k} u = 0$ and repeat the above considerations with interchanged u, v. The dimension of $j_1[\Gamma_H, 1]$ and the rank of Φ^H are invariant under the transposition of the global holomorphic coordinates on $\widetilde{A_{-1}} = \mathbb{C}^2$. $\square$

With a slight abuse of notation, we write $g(f)$ instead of $g^*(f)$, for $g \in G_{-1}$, $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$.

Lemma 5. *The generators* τ_{33}, I, J, θ *of* G_{-1} *act on the binary parallel and triangular* σ*-quotients from* Corollary 4 *as follows*

$$3\tau_{33}(f_{56}) = -\, f_{56}, \qquad \tau_{33}(f_{78}) = -\, f_{78}$$
$$\tau_{33}(f_{157}) = -\, \mathrm{i}e^{\frac{\pi}{2}} f_{168}, \qquad \tau_{33}(f_{168}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{157}, \qquad \tau_{33}(f_{357}) = -\, \mathrm{i}e^{-\frac{\pi}{2}} f_{368}$$
$$\tau_{33}(f_{368}) = \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \qquad \tau_{33}(f_{258}) = f_{267}, \qquad \tau_{33}(f_{267}) = f_{258}$$
$$\tau_{33}(f_{458}) = -\, f_{467}, \qquad \tau_{33}(f_{467}) = -\, f_{458}$$
$$I(f_{56}) = -\, \mathrm{i}f_{56}, \qquad I(f_{78}) = f_{78}$$
$$I(f_{157}) = -\, \mathrm{i}f_{467}, \qquad I(f_{168}) = -\, e^{-\frac{\pi}{2}} f_{458}, \qquad I(f_{357}) = \mathrm{i}f_{267}$$
$$I(f_{368}) = -\, e^{\frac{\pi}{2}} f_{258}, \qquad I(f_{258}) = \mathrm{i}f_{168}, \qquad I(f_{267}) = -\, e^{-\frac{\pi}{2}} f_{157}$$
$$I(f_{458}) = -\, \mathrm{i}f_{368}, \qquad I(f_{467}) = -\, e^{\frac{\pi}{2}} f_{357}$$
$$J(f_{56}) = f_{56}, \qquad J(f_{78}) = -\, \mathrm{i}f_{78}$$
$$J(f_{157}) = -\, \mathrm{i}e^{\frac{\pi}{2}} f_{258}, \qquad J(f_{168}) = f_{267}, \qquad J(f_{357}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{458}$$
$$J(f_{368}) = f_{467}, \qquad J(f_{258}) = f_{357}, \qquad J(f_{267}) = -\, \mathrm{i}e^{-\frac{\pi}{2}} f_{368}$$
$$J(f_{458}) = f_{157}, \qquad J(f_{467}) = \mathrm{i}e^{\frac{\pi}{2}} f_{168}$$
$$\theta(f_{56}) = f_{78}, \qquad \theta(f_{78}) = f_{56}$$
$$\theta(f_{157}) = -\, e^{\frac{\pi}{2}} f_{357}, \qquad \theta(f_{168}) = -\, e^{-\frac{\pi}{2}} f_{368}, \qquad \theta(f_{357}) = -\, e^{-\frac{\pi}{2}} f_{157}$$
$$\theta(f_{368}) = -\, e^{\frac{\pi}{2}} f_{168}, \qquad \theta(f_{258}) = f_{267}, \qquad \theta(f_{267}) = f_{258}$$
$$\theta(f_{458}) = f_{467}, \qquad \theta(f_{467}) = f_{458}.$$

Proof: Making use of Lemma 2 and Corollary 2, one computes that

$$\tau_{33}\sigma(u-1) = -e^{\pi u + \pi \mathrm{i} u}\sigma(u+\omega_1-\omega_2), \quad \tau_{33}\sigma(u+\omega_1-\omega_2) = e^{-2\pi u}\sigma(u-1)$$
$$\tau_{33}\sigma(u-\omega_1) = -e^{\pi \mathrm{i} u}\sigma(u-\omega_2), \quad \tau_{33}\sigma(u-\omega_2) = -e^{-\pi u}\sigma(u-\omega_1)$$
$$\tau_{33}(\Sigma_1) = -\mathrm{i}e^{-\frac{\pi}{2}}\Sigma_1, \quad \tau_{33}(\Sigma_2) = e^{-\pi}\Sigma_2, \quad \tau_{33}(\Sigma_3) = \mathrm{i}e^{-\frac{\pi}{2}}\Sigma_3, \quad \tau_{33}(\Sigma_4) = \Sigma_4$$
$$\tau_{33}(\Sigma_5) = e^{-\pi u - \pi \mathrm{i} u}\Sigma_6, \quad \tau_{33}(\Sigma_6) = e^{\pi u + \pi \mathrm{i} u}\Sigma_5$$
$$\tau_{33}(\Sigma_7) = e^{-\pi v - \pi \mathrm{i} v}\Sigma_8, \quad \tau_{33}(\Sigma_8) = e^{\pi v + \pi \mathrm{i} v}\Sigma_7$$
$$I\sigma(u-1) = \mathrm{i}e^{-\pi u + \pi \mathrm{i} u}\sigma(u-1), \quad I\sigma(u+\omega_1-\omega_2) = -e^{\pi u}\sigma(u+\omega_1-\omega_2)$$
$$I\sigma(u-\omega_1) = -\mathrm{i}e^{\pi \mathrm{i} u}\sigma(u-\omega_2), \quad I\sigma(u-\omega_2) = \mathrm{i}\sigma(u-\omega_1)$$
$$I(\Sigma_1) = \mathrm{i}e^{-\pi \mathrm{i} u + \pi \mathrm{i} v}\Sigma_4, \quad I(\Sigma_2) = \mathrm{i}e^{-\pi \mathrm{i} u - \pi v}\Sigma_1$$
$$I(\Sigma_3) = \mathrm{i}e^{-\pi \mathrm{i} u - \pi \mathrm{i} v}\Sigma_2, \quad I(\Sigma_4) = \mathrm{i}e^{-\pi \mathrm{i} u + \pi v}\Sigma_3$$
$$I(\Sigma_5) = -e^{-\pi \mathrm{i} u}\Sigma_6, \quad I(\Sigma_6) = -e^{\pi \mathrm{i} u}\Sigma_5, \quad I(\Sigma_7) = \Sigma_7, \quad I(\Sigma_8) = \Sigma_8$$
$$J\sigma(v+\mu) = I\sigma(u+\mu)|_{u=v}, \qquad \mu \in \mathbb{C}$$

$$4J(\Sigma_1) = \Sigma_2, \quad J(\Sigma_2) = \Sigma_3, \quad J(\Sigma_3) = \Sigma_4, \qquad J(\Sigma_4) = \Sigma_1$$
$$J(\Sigma_5) = \Sigma_5, \quad J(\Sigma_6) = \Sigma_6, \quad J(\Sigma_7) = -\mathrm{e}^{-\pi \mathrm{i} v}\Sigma_8, \quad J(\Sigma_8) = -\mathrm{e}^{\pi \mathrm{i} v}\Sigma_7$$

$$\theta\sigma(u+\mu) = \sigma(v+\mu), \qquad \mu \in \mathbb{C}$$

$$\theta(\Sigma_1) = -\mathrm{i}\mathrm{e}^{\pi u + \pi \mathrm{i} v}\Sigma_3, \qquad \theta(\Sigma_2) = \Sigma_2$$

$$\begin{aligned} \theta(\Sigma_3) &= \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u - \pi v}\Sigma_1, \quad \theta(\Sigma_4) = -\mathrm{e}^{\pi u - \pi \mathrm{i} u - \pi v + \pi \mathrm{i} v}\Sigma_4 \\ \theta(\Sigma_5) &= \Sigma_7, \quad \theta(\Sigma_6) = \Sigma_8, \quad \theta(\Sigma_7) = \Sigma_5, \quad \theta(\Sigma_8) = \Sigma_6. \end{aligned}$$

□

The following lemma is an immediate consequence of Lemma 2 and Corollary 1.

Lemma 6.

$$\left.\frac{f_{157}}{\Sigma_1}\right|_{T_1} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \left.\frac{f_{168}}{\Sigma_1}\right|_{T_1} = \mathrm{e}^{-\pi}, \quad \left.\frac{f_{258}}{\Sigma_2}\right|_{T_2} = \mathrm{e}^{-\pi}, \quad \left.\frac{f_{267}}{\Sigma_2}\right|_{T_2} = \mathrm{e}^{-\pi}$$

$$\left.\frac{f_{357}}{\Sigma_3}\right|_{T_3} = \mathrm{e}^{-\pi}, \quad \left.\frac{f_{368}}{\Sigma_3}\right|_{T_3} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \left.\frac{f_{458}}{\Sigma_4}\right|_{T_4} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \left.\frac{f_{467}}{\Sigma_4}\right|_{T_4} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}$$

$$\left.\frac{f_{157} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0, \quad \left.\frac{f_{258} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{458}}{\Sigma_5}\right|_{T_5} = 0.$$

Lemma 7.

$$[(f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368})]|_{T_2} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\left(1 + c\mathrm{e}^{-\frac{\pi}{2}}\right)$$

$$\frac{\sigma((1+\mathrm{i})v + \omega_3)}{\sigma((1+\mathrm{i})v)}\left[\mathrm{e}^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + \mathrm{e}^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right]$$

is non-constant for all $c \in \mathbb{C} \setminus \{-\mathrm{e}^{\frac{\pi}{2}}\}$.

Proof: Note that

$$\begin{aligned} f(v) &= [(f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368})]|_{T_2} \\ &= \left[\mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\Sigma_1(-v, v) - c\mathrm{e}^{-\pi + \pi \mathrm{i} v}\Sigma_3(-v, v)\right] \\ &\quad \times [\Sigma_5(-v)\Sigma_7(v) + \Sigma_6(-v)\Sigma_8(v)] \\ &= \mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\left(1 + c\mathrm{e}^{-\frac{\pi}{2}}\right)\frac{\sigma((1+\mathrm{i})v - \omega_3)}{\sigma((1+\mathrm{i})v)} \\ &\quad \times \left[\mathrm{e}^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + \mathrm{e}^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right] \end{aligned}$$

making use of Lemma 2 and Corollary 1. Obviously, $f(v)$ has no poles outside $\mathbb{Q}(\mathrm{i})$. It suffices to justify that $\lim_{v\to 0} f(v) = \infty$, in order to conclude that $f(v) \not\equiv$ const. To this end, use $\sigma(\omega_2) = \mathrm{i}\sigma(\omega_1)$ to observe that

$$f(v)\sigma((1+\mathrm{i})v)\Big|_{v=0} = 2\,\mathrm{i}e^{-\frac{\pi}{2}}\left(1 + ce^{-\frac{\pi}{2}}\right)\sigma(\omega_3) \neq 0$$

whenever $c \neq -e^{\frac{\pi}{2}}$, while $\sigma((1+\mathrm{i})v)|_{v=0} = 0$. □

5. Basic Results

Lemma 8. *For* $H = \langle IJ^2, \tau_{33}J^2\rangle$, $\langle I^2J, \tau_{33}I^2\rangle$ *with rational* A_{-1}/H *and any* $-\mathrm{Id} \in H \leq G_{-1}$*, the map* $\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$ *is constant.*

Proof: By Lemma 4 (iv), the assertion for $\langle I^2J, \tau_{33}I^2\rangle$ is a consequence of the one for $\langle IJ^2, \tau_{33}J^2\rangle$. In the case of $H = \langle IJ^2, \tau_{33}J^2\rangle$, the space $\mathcal{L}^H$ is spanned by Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0$$

$$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{267} - e^{\frac{\pi}{2}}f_{258} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368} + \mathrm{i}f_{467} + \mathrm{i}f_{458}.$$

The Γ_H-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157})|_{T_1} \neq \infty$. Therefore $R_H(f_{157}) \in \mathcal{L}_2^H = \mathbb{C}$ and $\mathrm{rk}\Phi^H = 0$.

It suffices to observe that $-\mathrm{Id}$ changes the signs of the $\mathbb{C}$-basis

$$f_{56},\ f_{78},\ f_{157},\ f_{258},\ f_{368},\ f_{467} \tag{1}$$

of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$. Then for $H_o = \langle -\mathrm{Id}\rangle$ the space $\mathcal{L}^{H_o}$ is generated by $R_{H_o}(1) = 1$. Any subgroup $H_o \leq H \leq G_{-1}$ decomposes into cosets $H = \cup_{i=1}^k h_iH_o$ and $R_H = \sum_{i=1}^k h_iR_{H_o}$ vanishes on (1). Thus, $\mathcal{L}^H = \mathbb{C}$ and $\mathrm{rk}\Phi^H = 0$. □

Note that $A_{-1}/\langle -\mathrm{Id}\rangle$ has 16 double points, whose minimal resolution is the Kummer surface X_{-1} of A_{-1}. Thus, $H \ni -\mathrm{Id}$ exactly when the minimal resolution Y of the singularities of A_{-1}/H is covered by a smooth model of X_{-1}. More precisely, all A_{-1}/H with $-\mathrm{Id} \in H$ have vanishing irregularity $0 \leq q(A_{-1}/H) \leq q(X_{-1}) = 0$. These are the Enriques $A_{-1}/\langle -\mathrm{Id}, \tau_{33}I^2\rangle$, all K3 quotients A_{-1}/H with $\langle \tau_{33}^n\rangle \neq H \leq K = \ker\det\mathcal{L}$, except $A_{-1}/\langle \tau_{33}(-\mathrm{Id})\rangle$ and the rational A_{-1}/H with $\tau_{33}IJ \in H$ for $0 \leq n \leq 1$ or $\langle -\mathrm{Id}, h_1\rangle \leq H$ for

$$h_1 \in \{I^{2m}J^{2-2m},\quad \tau_{33}^mI,\quad \tau_{33}^mJ,\quad \tau_{33}^mI^lJ^{-l}\theta\ ;\ 0 \leq m \leq 1,\quad 0 \leq l \leq 3\}.$$

Lemma 9. *The non-trivial subgroups* $H \not\ni -\mathrm{Id}$ *of* G_{-1} *are*

i) *cyclic of order two*

$$H_2(m,l) = \langle \tau_{33} I^{2m} J^{2l} \rangle \quad \textit{with} \quad 0 \leq m, l \leq 1$$

$$H_2^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle \quad \textit{with} \quad 0 \leq n \leq 1,\ 0 \leq k \leq 3,\ H_2' = \langle I^2 \rangle,\ H_2'' = \langle J^2 \rangle$$

ii) *cyclic of order four*

$$H_4'(n,m) = \langle \tau_{33}^n I J^{2m} \rangle \quad \textit{with} \quad 0 \leq n, m \leq 1$$

$$H_4''(n,m) = \langle \tau_{33}^n I^{2m} J \rangle \quad \textit{with} \quad 0 \leq n, m \leq 1$$

iii) *isomorphic to the* ***Klein group*** $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$H_{2\times 2}'(m) = \langle \tau_{33} J^{2m}, I^2 \rangle \quad \textit{with} \quad 0 \leq m \leq 1$$

$$H_{2\times 2}''(m) = \langle \tau_{33} I^{2m}, J^2 \rangle \quad \textit{with} \quad 0 \leq m \leq 1$$

$$H_{2\times 2}^\theta(k) = \langle I^k J^{-k} \theta, \tau_{33} \rangle \quad \textit{with} \quad 0 \leq k \leq 1$$

$$H_{2\times 2}^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^2 J^2 \rangle \quad \textit{with} \quad 0 \leq n, k \leq 1$$

iv) *isomorphic to* $\mathbb{Z}_4 \times \mathbb{Z}_2$

$$H_{4\times 2}'(m,l) = \langle I J^{2m}, \tau_{33} J^{2l} \rangle \quad \textit{with} \quad 0 \leq m, l \leq 1$$

$$H_{4\times 2}''(m,l) = \langle I^{2m} J, \tau_{33} I^{2l} \rangle \quad \textit{with} \quad 0 \leq m, l \leq 1.$$

Proof: If H is a subgroup of G_{-1}, which does not contain $-\mathrm{Id}$, then $H \subseteq S = \{g \in G_{-1}\,;\ -\mathrm{Id} \not\in \langle g \rangle\}$. Decompose $G_{-1} = G'_{-1} \cup G'_{-1}\theta$ into cosets modulo the abelian subgroup

$$G'_{-1} = \{\tau_{33}^n I^k J^l\,;\, 0 \leq n \leq 1, 0 \leq k, l \leq 3\} \leq G_{-1}.$$

The cyclic group, generated by $(\tau_{33}^n I^k J^l \theta)^2 = (IJ)^{k+l}$ does not contain $-\mathrm{Id} = (IJ)^2$ if and only if $k + l \equiv 0 (\mathrm{mod}\ 4)$. If $S^{(r)} = \{g \in S\,;\, g \text{ is of order } r\}$ then

$$S \cap G'_{-1}\theta = \{\tau_{33}^n I^k J^{-k}\theta\,;\, 0 \leq n \leq 1,\ \ 0 \leq k \leq 3\} = S^{(2)} \cap G'_{-1}\theta =: S_1^{(2)}$$

and $S \cap G'_{-1}\theta \subseteq S^{(2)}$ consists of elements of order two. Concerning $S \cap G'_{-1}$, observe that $(\tau_{33}^n I^k J^{k+2m})^2 = (IJ)^{2k} \in S$ for $0 \leq n, m \leq 1, 0 \leq k \leq 3$ requires $k = 2p$ to be even. Consequently

$$\{\tau_{33}^n I^k J^l\,;\, k \equiv l (\mathrm{mod}\ 2)\} \cap S$$
$$= \{\tau_{33} I^{2m} J^{2l},\ \ I^2,\ \ J^2\,;\, 0 \leq m, l \leq 1\} = S^{(2)} \cap G'_{-1} =: S_0^{(2)}$$

$$\{\tau_{33}^n I^k J^l\,;\, k \equiv l + 1 (\mathrm{mod}\ 2)\} \cap S$$
$$= \{\tau_{33}^n I^{2m+1} J^{2l}, \tau_{33}^n I^{2m} J^{2l+1}\,;\, 0 \leq n, m, l \leq 1\} = S^{(4)}.$$

In such a way, one obtains $S = \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)} \cup S^{(4)}$ of cardinality $|S| = 31$. If a subgroup H of G_{-1} is contained in S, then $|H| \leq |S| = 31$ divides $|G_{-1}| = 64$, i.e., $|H| = 1, 2, 4, 8$ or 16. The only subgroup $H < G_{-1}$ of $|H| = 1$ is the trivial one $H = \{\mathrm{Id}\}$. The subgroups $-\mathrm{Id} \notin H < G_{-1}$ of order two are the cyclic ones, generated by $h \in S_0^{(2)} \cup S_1^{(2)}$. We denote $H_2(m,l) = \langle \tau_{33} I^{2m} J^{2l} \rangle$ for $0 \leq m, l \leq 1$, $H_2^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ for $0 \leq n \leq 1$, $0 \leq k \leq 3$ and $H_2' = \langle I^2 \rangle$, $H_2'' = \langle J^2 \rangle$.

For any $h \in S^{(4)}$ one has $\langle h \rangle = \langle h^3 \rangle$, so that the subgroups $-\mathrm{Id} \notin H \simeq \mathbb{Z}_4$ of G_{-1} are depleted by $H_4'(n,m) = \langle \tau_{33}^n I J^{2m} \rangle$, $H_4''(n,m) = \langle \tau_{33}^n I^{2m} J \rangle$ with $0 \leq n, m \leq 1$.

The subgroups $-\mathrm{Id} \notin H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ of G_{-1} are generated by commuting $g_1, g_2 \in S^{(2)} = S_0^{(2)} \cup S_1^{(2)}$. If $g_1, g_2 \in S_1^{(2)}$ then $g_1 g_2 \in G'_{-1}$, so that one can always assume that $g_2 \in S_0^{(2)}$. Any $g_1 \neq g_2$ from $S_0^{(2)} \subset G'_{-1}$ generate the Klein group of order four. Moreover, if

$$S_{0,1}^{(2)} = \{\tau_{33} I^{2m} J^{2l};\ 0 \leq m, l \leq 1\}, \qquad S_{0,0}^{(2)} = \{I^2, J^2\}$$

then for any $g_1, g_2 \in S_{0,1}^{(2)}$ with $g_1 g_2 \in S$ there follows $g_1 g_2 \in S_{0,0}^{(2)}$. Thus, any $S_0^{(2)} \supset H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ has at least one generator $g_2 \in S_{0,0}^{(2)}$. The requirement $I^2 J^2 = -\mathrm{Id} \notin H$ specifies that $g_1 \in S_{0,1}^{(2)}$. In the case of $g_2 = I^2$ there is no loss of generality to choose $g_1 = \tau_{33} J^{2m}$, in order to form $H'_{2\times 2}(m)$. Similarly, for $g_2 = J^2$ it suffices to take $g_1 = \tau_{33} I^{2m}$, while constructing $H''_{2\times 2}(m)$. In order to determine the subgroups $-\mathrm{Id} \notin H = \langle g_1 \rangle \times \langle g_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ with $g_1 \in S_1^{(2)}$, $g_2 \in S_0^{(2)}$, note that $g_1 = \tau_{33}^n I^k J^{-k} \theta$ does not commute with I^2, J^2 and commutes with $g_2 = \tau_{33} I^{2m} J^{2l}$ if and only if $2m \equiv 2l \pmod 4$, i.e., $0 \leq m = l \leq 1$. Bearing in mind that $\langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^{2m} J^{2m} \rangle = \langle \tau_{33}^{n+1} I^{k+2m} J^{-k+2m} \theta, \tau_{33} I^{2m} J^{2m} \rangle$, one restricts the values of k to $0 \leq k \leq 1$. For $m = 0$ denote $H_{2\times 2}^{\theta}(k) = \langle I^k J^{-k} \theta, \tau_{33} \rangle$. For $m = 1$ put $H_{2\times 2}^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^2 J^2 \rangle$.

Let $-\mathrm{Id} \notin H \subset S$ be a subgroup of order 8. The non-abelian such H are isomorphic to quaternionic group $\mathbb{Q}_8 = \langle s, t;\ s^4 = \mathrm{Id},\ \ s^2 = t^2,\ \ sts = t \rangle$ or to dihedral group $\mathbb{D}_4 = \langle s, t;\ s^4 = \mathrm{Id},\ \ t^2 = \mathrm{Id},\ \ sts = t \rangle$. Note that $s \in S^{(4)}$ and $sts = t$ require $st \neq ts$. As far as $S^{(4)} \cup S_0^{(2)} \subset G'_{-1}$ for the abelian group $G'_{-1} = \langle \tau_{33}, I, J \rangle$, it suffices to consider $t = \tau_{33}^n I^k J^{-k} \theta \in S_1^{(2)}$ and $s = \tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ with $0 \leq n, m, l \leq 1$, $0 \leq p, k \leq 3$. However, $sts = \tau_{33}^n I^{k+2l+1} J^{k+2l+1} \theta \neq t$ reveals the non-existence of a non-abelian group $-\mathrm{Id} \notin H \leq G_{-1}$ of order eight.

The abelian groups $H \subset S = \{\mathrm{Id}\} \cup S^{(2)} \cup S^{(4)}$ of order eight are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq H \subset S$ is generated by $s =$

$\tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ and $t \in S_0^{(2)}$, as far as $t' = \tau_{33}^n I^k J^{-k}\theta \in S_1^{(2)}$ has

$$st' = \tau_{33}^{m+n} I^{p+k} J^{2l+1-(p+k)}\theta \neq \tau_{33}^{m+n} I^{2l+1-(p-k)} J^{p-k}\theta = t's.$$

For $s = \tau_{33}^n I^{2m+1} J^{2l} \in S^{(4)}$ there holds $\langle s, t\rangle = \langle s^3, t\rangle$ and it suffices to consider $s = \tau_{33}^n I J^{2l}$. Further, $t \not\in \langle s^2\rangle = \langle I^2\rangle$ and $s^2 t \neq -\mathrm{Id}$ specify that $t = \tau_{33} I^{2p} J^{2q}$ for some $0 \leq p, q \leq 1$. Replacing eventually t by $ts^2 = tI^2$, one attains $t = \tau_{33} J^{2q}$. On the other hand, the generator $s = \tau_{33} I J^{2l} \in S^{(4)}$ of $H = \langle s, t\rangle$ can be restored by $st = IJ^{2(l+q)}$, so that $H = H'_{4\times 2}(l, q) = \langle IJ^{2l}, \tau_{33} J^{2q}\rangle$ for some $0 \leq l, q \leq 1$. Exchanging I with J, one obtains the remaining groups $H''_{4\times 2}(l, q) = \langle I^{2l} J, \tau_{33} I^{2q}\rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$, contained in S.

If $-\mathrm{Id} \not\in H \subset S$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then arbitrary different elements $s, t, r \in H$ of order two commute and generate H. For any $x \in S$ and $M \subseteq S$, consider the centralizer $C_M(x) = \{y \in M;\, xy = yx\}$ of x in M. Looking for $s \in S^{(2)}$, $t \in C_{S^{(2)}}(s)$ and $r \in C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)$, one computes that

$$C_{S^{(2)}}(\tau_{33}^n I^2) = C_{S^{(2)}}(\tau_{33}^n J^2) = S_0^{(2)}$$

$$C_{S^{(2)}}(\tau_{33} I^{2m} J^{2m}) = S^{(2)} = S_0^{(2)} \cup S_1^{(2)}$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m} J^{-2m}\theta) = \{\tau_{33}^p I^{2q} J^{-2q}\theta,\ \ \tau_{33} I^{2p} J^{2p};\, 0 \leq p, q \leq 1\}$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m+1} J^{-2m-1}\theta) = \{\tau_{33}^p I^{2q+1} J^{-2q-1}\theta,\ \ \tau_{33} I^{2p} J^{2p};\, 0 \leq p, q \leq 1\}.$$

Any subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \subset \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)}$ intersects $S_1^{(2)}$, due to $|S_0^{(2)}| = 6$. That allows to assume that $s \in S_1^{(2)}$ and observe that

$$C_{S^{(2)}}(s) = \{s,\ \ (-\mathrm{Id})s,\ \ \tau_{33}s,\ \ \tau_{33}(-\mathrm{Id})s,\ \ \tau_{33},\ \ \tau_{33}(-\mathrm{Id})\}.$$

If $t = \tau_{33} I^{2p} J^{2p} \in C_{S^{(2)}}(s)$ then $C_{S^{(2)}}(t) = S^{(2)}$, so that

$$H \setminus \{\mathrm{Id}, s, t\} \subseteq [C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)] \setminus \{s, t\} = C_{S^{(2)}} \setminus \{s, t\} \tag{2}$$

with $5 = |H \setminus \{\mathrm{Id}, s, t\}| \leq |C_{S^{(2)}}(s) \setminus \{s, t\}| = 4$ is an absurd. For $t \in C_{S^{(2)}}(s) \setminus \{\tau_{33} I^{2p} J^{2p};\, 0 \leq p \leq 1\}$ one has $C_{S^{(2)}}(t) = C_{S^{(2)}}(s)$, which again leads to (2). Therefore, there is no subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$ of G_{-1}.

Concerning the non-existence of subgroups $-\mathrm{Id} \not\in H \subset S$ of order 16, the abelian $-\mathrm{Id} \not\in H \subset S$ of order 16 may be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ is generated by $s, t \in S^{(4)}$ with $s^2 \neq t^2$. Replacing, eventually, s by s^3 and t by t^3, one has $s = \tau_{33}^n I J^{2m}$, $t = \tau_{33}^p I^{2q} J$ with $0 \leq n, m, p, q \leq 1$. Then $s^2 t^2 = I^2 J^2 = -\mathrm{Id} \in H$ is an absurd. The groups $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are generated by $s \in S^{(4)}$ and $t, rinC_{S^{(2)}}(s)$ with $r \in C_{S^{(2)}}(t)$. In the case of $s = \tau_{33}^n I J^{2m}$, the centralizer $C_{S^{(2)}}(s) = S_0^{(2)}$. Bearing in mind that $s^2 = I^2$, one observes that $\langle t, r\rangle \cap \{I^2, J^2\} = \emptyset$. Therefore $t, r \in \{\tau_{33} I^{2p} J^{2q};\, 0 \leq p, q \leq 1\}$, whereas $tr \in \{\mathrm{Id}, I^2, J^2, -\mathrm{Id}\}$. That reveals the non-existence of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$. The groups $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

contain 15 elements of order two, while $|S^{(2)}| = 14$. Therefore there is no abelian group $-\operatorname{Id} \notin H \leq G_{-1}$ of order 16.

There are three non-abelian groups of order 16, which do not contain a non-abelian subgroup of order 8 and consist of elements of order 1, 2 or 4. If

$$\langle s, t;\, s^4 = e,\ \ t^4 = e,\ \ st = ts^3 \rangle \simeq H \subset S$$

then $s, t \in S^{(4)} \subset G'_{-1} = \langle \tau_{33}, I, J \rangle$ commute and imply that s is of order two. The assumption

$$\langle a, b, c;\, a^4 = e,\ \ b^2 = e,\ \ c^2 = e,\ \ cbca^2b = e,\ \ ba = ab,\ \ ca = ac \rangle \simeq H \subset S$$

requires $b, c \in C_{S^{(2)}}(a) = S_0^{(2)} = \{\tau_{33} I^{2m} J^{2l},\ \ I^2,\ \ J^2;\, 0 \leq m, l \leq 1\}$. Then b and c commute and imply that $cbca^2b = e = a^2 = e$. Finally, for

$$G_{4,4} = \langle s, t;\, s^4 = e,\ \ t^4 = e,\ \ stst = e,\ \ ts^3 = st^3 \rangle$$

there follows $s, t \in S^{(4)} \subset G'_{-1}$, whereas $st = ts$. Consequently, $s^2 = t^2$ and $G_{4,4} = \{s^i t^j;\, 0 \leq i \leq 3,\ \ 0 \leq j \leq 1\}$ is of order ≤ 8, contrary to $|G_{4,4}| = 16$. Thus, there is no subgroup $-\operatorname{Id} \notin H \leq G_{-1}$ of order 16. □

Throughout, we use the notations $H_\alpha^\beta(\gamma)$ from Lemma 9 and denote by $\Gamma_\alpha^\beta(\gamma)$ the corresponding lattices with $\Gamma_\alpha^\beta(\gamma)/\Gamma_{-1}^{(6,8)} = H_\alpha^\beta(\gamma)$.

Theorem 5. *For the groups* $H = H'_{4\times 2}(p,q) = \langle IJ^{2p}, \tau_{33}J^{2q}\rangle$, $H''_{4\times 2}(p,q) = \langle I^{2p}J, \tau_{33}I^{2q}\rangle$, $H'_4(1-m,m) = \langle \tau_{33}^{1-m} IJ^{2m}\rangle$, $H''_4(1-m,m) = \langle \tau_{33}^{1-m} I^{2m} J\rangle$, $H'_{2\times 2}(1) = \langle \tau_{33}J^2, I^2\rangle$, $H''_{2\times 2}(1) = \langle \tau_{33}I^2, J^2\rangle$, $H^\theta_{2\times 2}(n,m) = \langle \tau_{33}^n I^m J^{-m}\theta$, $\tau_{33}I^2J^2\rangle$ *with* $0 \leq p, q \leq 1$, $(p,q) \neq (1,1)$ *and* $0 \leq n, m \leq 1$ *the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^1$$

is dominant and not globally defined. The Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_H}$ *are birational to ruled surfaces with elliptic bases whenever* $H = H'_{4\times 2}(0,0)$, $H''_{4\times 2}(0,0)$, $H'_4(1,0)$ *or* $H''_4(1,0)$. *The remaining ones are rational surfaces.*

Proof: According to Lemma 4(v), it suffices to prove the theorem for $H'_{4\times 2}(p,q)$ with $(p,q) \neq (1,1)$, $H'_4(1-m,m)$ $H'_{2\times 2}(1)$ and $H^\theta_{2\times 2}(n,m)$.

If $H = H'_4(1,0) = \langle \tau_{33} I\rangle$, then $\mathcal{L}^H$ is generated by $1 \in \mathbb{C}$ and Reynolds operators

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 0,\ \ R_H(f_{157}) = f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{258} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{458}$$

$$R_H(f_{168}) = f_{168} - \mathrm{i} f_{267} + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368} + \mathrm{e}^{-\frac{\pi}{2}} f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} R_H(f_{368}).$$

There are four $\Gamma'_4(1,0)$-cusps : $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5$, $\bar{\kappa}_6$, $\bar{\kappa}_7 = \bar{\kappa}_8$. Applying Lemma 4 ii) to $T_1 \subset (R_H(f_{157}))_\infty, R_H(f_{168})_\infty \subseteq \sum_{i=1}^{8} T_i$, one concludes that $R_H(f_{168}) \in \operatorname{Span}_{\mathbb{C}}(1, R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^2$ and $\Phi^{H'_4(1,0)}$ is a dominant map to $\mathbb{P}(\mathcal{L}^H) \simeq \mathbb{P}^1$. Since $R_H(f_{157})|_{T_6} \neq \infty$, the entire $[\Gamma'_4(1,0),1]$ vanishes at $\bar{\kappa}_6$ and $\Phi^{H'_4(1,0)}$ is not defined at $\bar{\kappa}_6$.

The group $H = H'_{4\times 2}(0,0) = \langle I, \tau_{33}\rangle$ contains $F = H'_4(1,0)$ as a subgroup of index two with non-trivial coset representative I. Therefore $R_H(f_{56}) = R_F(f_{56}) + IR_F(f_{56}) = 0$, $R_H(f_{78}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,0)} \leq 1$. Due to

$$R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} - e^{\frac{\pi}{2}} f_{258} - e^{\frac{\pi}{2}} f_{267} + f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{458} - \mathrm{i}f_{467}$$

$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$. Lemma 6 provides $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Therefore $\dim_{\mathbb{C}} \mathcal{L}^H = 2$ and $\Phi^{H'_{4\times 2}(0,0)}$ is a dominant map to $\mathbb{P}^1$. The $\Gamma_{4\times 2}(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \overline{\kappa_3} = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Again from Lemma 6, $\left.\frac{f_{157} - e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{458}}{\Sigma_5}\right|_{T_5} = 0$, so that $R_H(f_{157})$ is regular over $T_5 + T_6$. As a result, $\Phi^{H'_{4\times 2}(0,0)}$ is not defined at $\bar{\kappa}_5 = \bar{\kappa}_6$.

For $H = H'_4(0,1) = \langle IJ^2\rangle$, the space $\mathcal{L}^H$ is spanned by 1 and Reynolds operators

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{267} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{467}$$

$$R_H(f_{168}) = f_{168} + \mathrm{i}f_{258} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} + e^{-\frac{\pi}{2}} f_{458} = \mathrm{i}R_H(f_{258}).$$

The $\Gamma'_4(0,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$, $\bar{\kappa}_7$ and $\bar{\kappa}_8$. Note that $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$, in order to conclude that $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$ by Lemma 4 ii). Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_4(0,1)}$ is a dominant map to $\mathbb{P}^1$. Lemma 6 supplies $\left.\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$ and justifies that $\Phi^{H'_4(0,1)}$ is not defined at $\bar{\kappa}_5$.

For $H = H'_{4\times 2}(1,0) = \langle IJ^2, \tau_{33}\rangle$ note that $R_H(f_{56}) = 0$, $R_H(f_{78}) = 0$, as far as $H'_4(1,0)$ is a subgroup of $H'_{4\times 2}(1,0)$. Further,

$$R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} + e^{\frac{\pi}{2}} f_{267} + e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + f_{368} + \mathrm{i}f_{467} - \mathrm{i}f_{458}$$

has a pole over $\sum_{i=1}^{4} T_i$, according to $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ by Lemma 6 and the transitiveness of the $H'_4(1,0)$-action on $\{\kappa_i;\ 1 \leq i \leq 4\}$. Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_{4\times 2}(1,0)}$ is a dominant map to $\mathbb{P}^1$. One computes immediately that the $\Gamma'_{4\times 2}(1,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Again from Lemma 6, $\left.\frac{f_{157} + e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i}f_{458}}{\Sigma_5}\right|_{T_5} = 0$, $R_H(f_{157})$ has no pole at $T_5 + T_6$ and $\Phi^{H'_{4\times 2}(1,0)}$ is not defined at $\bar{\kappa}_5 = \bar{\kappa}_6$.

If $H = H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ then

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 4f_{78}, \quad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - f_{368}$$

$$R_H(f_{258}) = f_{258} - f_{267} - \mathrm{i}e^{-\frac{\pi}{2}} f_{467} - \mathrm{i}e^{-\frac{\pi}{2}} f_{458} \quad \text{and} \quad 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_{2\times 2}(1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 reveals that $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}e^{\frac{\pi}{2}}f_{357}-f_{368}}{\Sigma_3}\Big|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = \frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, so that $R_H(f_{157})$, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78})$ and $\mathcal{L}^H \simeq \mathbb{C}^2$. As a result, $\Phi^{H'_{2\times 2}(1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$ and $\bar{\kappa}_2$.
For the group $H = H'_{4\times 2}(0,1) = \langle I, \tau_{33}J^2\rangle$, containing $H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ there follows $R_H(f_{56}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,1)} \leq 1$. Therefore $R_H(f_{78}) = 8f_{78}$,

$$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{258} - e^{\frac{\pi}{2}}f_{267} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368} - \mathrm{i}f_{458} - \mathrm{i}f_{467}$$

and $1 \in \mathbb{C}$ span $\mathcal{L}^H$. The $\Gamma'_{4\times 2}(0,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78}) \simeq \mathbb{C}^2$. Thus, $\Phi^{H'_{4\times 2}(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$.
If $H = H^{\theta}_{2\times 2}(0,0) = \langle \theta, \tau_{33}I^2J^2\rangle$ then $\mathcal{L}^H$ is spanned by $1 \in \mathbb{C}$,

$$R_H(f_{56}) = 2(f_{56} + f_{78}), \quad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} - e^{\frac{\pi}{2}}f_{357} - \mathrm{i}f_{368}$$

and $R_H(f_{467}) = 2(f_{467} + f_{458})$, due to $R_H(f_{258}) = 0$. The $\Gamma^{\theta}_2(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 provides $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, $\frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{157}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. Therefore $\Phi^{H^{\theta}_2(0,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_4$.
For $H = H^{\theta}_{2\times 2}(0,1) = \langle IJ^{-1}\theta, \tau_{33}I^2J^2\rangle$ one has

$$R_H(f_{56}) = 2(f_{56} + \mathrm{i}f_{78}), \qquad R_H(f_{157}) = 0, \qquad R_H(f_{168}) = 0$$

$R_H(f_{368}) = 2(f_{368} - \mathrm{i}e^{\frac{\pi}{2}}f_{357})$, $R_H(f_{258}) = f_{258} - f_{267} - e^{-\frac{\pi}{2}}f_{458} - e^{-\frac{\pi}{2}}f_{467}$.
The $\Gamma^{\theta}_{2\times 2}(0,1)$-cusps are $\bar{\kappa}_1$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 implies that $\frac{f_{368}-\mathrm{i}e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, $\frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$, $\frac{f_{458}+f_{467}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{368}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}$. Consequently, $\Phi^{H^{\theta}_{2\times 2}(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_4$.
In the case of $H = H^{\theta}_{2\times 2}(1,0) = \langle \tau_{33}\theta, \tau_{33}I^2J^2\rangle$, the Reynolds operators are

$$R_H(f_{56}) = 2(f_{56} - f_{78}), \qquad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + \mathrm{i}f_{368} + e^{\frac{\pi}{2}}f_{357}$$

$$R_H(f_{258}) = 2(f_{258} - f_{267}), \qquad R_H(f_{458}) = 0, \quad R_H(f_{467}) = 0.$$

The $\Gamma^{\theta}_{2\times 2}(1,0)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 yields $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}f_{368}+e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$. Consequently, $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Bearing in mind that $R_H(f_{56})|_{T_5} =$

∞, one concludes that $\Phi^{H^\theta_{2\times 2}(1,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$.

Finally, for $H = H^\theta_{2\times 2}(1,1) = \langle \tau_{33} I J^{-1}\theta, \tau_{33} I^2 J^2\rangle$ one has

$$R_H(f_{56}) = 2(f_{56} - \mathrm{i} f_{78}), \quad R_H(f_{157}) = 2(f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168}), \quad R_H(f_{357}) = 0$$

$$R_H(f_{368}) = 0 \quad \text{and} \quad R_H(f_{258}) = f_{258} - f_{267} + e^{-\frac{\pi}{2}} f_{467} + e^{-\frac{\pi}{2}} f_{458}.$$

The $\Gamma^\theta_{2\times 2}(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \overline{\kappa_8}$. Lemma 6 implies that $\left.\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = \left.\frac{f_{258}-f_{267}}{\Sigma_2}\right|_{T_2} = 0$, so that $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. As a result, $\Phi^{H^\theta_{2\times 2}(1,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_3$ and $\bar{\kappa}_2$. □

Theorem 6. *If $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2\rangle$, $H''_{2\times 2}(0) = \langle \tau_{33}, J^2\rangle$, $H^\theta_{2\times 2}(n) = \langle I^n J^{-n}\theta, \tau_{33}\rangle$ with $0 \leq n \leq 1$, $H'_4(n,n) = \langle \tau_{33}^n I J^{2n}\rangle$, $H''_4(n,n) = \langle \tau_{33}^n I^{2n} J\rangle$ with $0 \leq n \leq 1$ or $H_2(1,1) = \langle \tau_{33} I^2 J^2\rangle$ then the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^2$$

is dominant and not globally defined. The surface $\widehat{\mathbb{B}/\Gamma_H}$ is K3 *for $H = H_2(1,1)$, rational for $H = H'_4(1,1)$, $H''_4(1,1)$ and ruled with an elliptic base for all the other aforementioned H.*

Proof: By Lemma 4 v), it suffices to consider $H'_{2\times 2}(0)$, $H^\theta_{2\times 2}(n)$, $H'_4(n,n)$ and $H_2(1,1)$.

In the case of $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2\rangle$, $\mathcal{L}^H$ is spanned by

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + f_{368}$$

$$R_H(f_{258}) = f_{258} + f_{267} - \mathrm{i}e^{-\frac{\pi}{2}} f_{458} + \mathrm{i}e^{-\frac{\pi}{2}} f_{467} \quad \text{and} \quad 1 \in \mathbb{C}.$$

The $\Gamma'_{2\times 2}(0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 provides $\left.\frac{f_{157}-\mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Similarly, $\left.\frac{f_{258}+f_{267}}{\Sigma_2}\right|_{T_2} = 2e^{-\pi} \neq 0$ suffices for $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent, according to Lemma 4 iii) and constitute a $\mathbb{C}$-basis for $\mathcal{L}^H$. In order to assert that $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, we use that $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$ by Lemma 7 with $c = \mathrm{i}e^{\frac{\pi}{2}}$. Lemma 6 provides $\left.\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$, in order to conclude that $R_H(f_{157})|_{T_5} \neq \infty$ and the entire $[\Gamma'_{2\times 2}(0), 1]$ vanishes at $\bar{\kappa}_5$. Therefore $\Phi^{H'_{2\times 2}(0)}$ is a dominant map to $\mathbb{P}([\Gamma'_{2\times 2}(0), 1]) = \mathbb{P}^2$, which is not defined at $\bar{\kappa}_5$.

For $H = H^\theta_{2\times 2}(0) = \langle \theta, \tau_{33}\rangle$, the Reynolds operators are

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} - e^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{368}$$

$$R_H(f_{258}) = 2(f_{258} + f_{267}), \qquad R_H(f_{467}) = 0$$

generate $\mathcal{L}^H$. The $\Gamma^{\theta}_{2\times 2}(0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. According to Lemma 6, $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, so that $R_H(f_{157})|_{T_1} = \infty$. Further, $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$ and the lemma provides $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent and $\mathcal{L}^H \simeq \mathbb{C}^3$ by Lemma 4 iii). We claim that

$$R_H(f_{258})|_{T_1} = -2e^{-\pi \mathrm{i} v} \frac{\sigma((1+\mathrm{i})v + \omega_3)}{\sigma((1+\mathrm{i})v)} \left[\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2} + e^{2\pi(1+\mathrm{i})v} \frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} \right]$$

is non-constant. On one hand, $R_H(f_{258})|_{T_1}$ has no poles on $\mathbb{C} \setminus \mathbb{Q}(\mathrm{i})$. On the other hand, $\left[\frac{1}{2} R_H(f_{258})\Big|_{T_1} \right] \sigma((1+\mathrm{i})v)\Big|_{v=0} = -\sigma(\omega_3) \left[\frac{1}{\mathrm{i}^2} + \mathrm{i}^2 \right] \neq 0$, so that $\lim_{v \to 0} [R_H(f_{258})|_{T_1}] = \infty$. According to Lemma 4 iv), $R_H(f_{157})|_{T_1} = \infty$ and $R_H(f_{258})|_{T_1} \not\equiv \mathrm{const}$ suffice for $\Phi^{H^{\theta}_{2\times 2}(0)}$ to be a dominant map to $\mathbb{P}^2$. The entire $\mathcal{L}^H$ takes finite values on T_4, so that $\Phi^{H^{\theta}_{2\times 2}(0)}$ is not defined at $\bar{\kappa}_4$.

Concerning $H = H^{\theta}_{2\times 2}(1) = \langle IJ^{-1}\theta, \tau_{33} \rangle$, one computes that

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0, \qquad R_H(f_{157}) = 2(f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168})$$

$$R_H(f_{368}) = 0, \quad R_H(f_{258}) = f_{258} + f_{267} - e^{-\frac{\pi}{2}} f_{458} + e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma^{\theta}_{2\times 2}(1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6 we have $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ and $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$. Therefore $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{258})|_{T_2} = \infty$ and 1, $R_H(f_{157})$, $R_H(f_{258})$ constitute a $\mathbb{C}$-basis of $\mathcal{L}^H$, according to Lemma 4 iii). Applying Lemma 7 with $c = 0$, one concludes that $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$. Then Lemma 4 iv) implies that $\Phi^{H^{\theta}_{2\times 2}(1)}$ is a dominant map to $\mathbb{P}^2$. The lack of $f \in \mathcal{L}^H$ with $f|_{T_3} = \infty$ reveals that $\Phi^{H^{\theta}_{2\times 2}(1)}$ is not defined at $\bar{\kappa}_3$.

If $H = H'_4(0,0) = \langle I \rangle$ then the Reynolds operators are

$$R_H(f_{56}) = 0, \ \ R_H(f_{78}) = 4f_{78}, \ \ R_H(f_{157}) = f_{157} - e^{\frac{\pi}{2}} f_{267} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i} f_{467}$$

$$R_H(f_{168}) = f_{168} - \mathrm{i} f_{258} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{458} \ \text{ and } \ R_H(1) = 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_4(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$, $\bar{\kappa}_7$ and $\bar{\kappa}_8$. According to Lemma 4 ii), the inclusions $T_1 \subset (R_H(f_{157}))_{\infty}, (R_H(f_{168}))_{\infty} \subseteq \sum_{i=1}^{8} T_i$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^3$. Observe that $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$, in order to apply Lemma 4 iv) and assert that $\Phi^{H'_4(0,0)}$ is a dominant map to $\mathbb{P}^2$. As far as $\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0$ by

Lemma 6, the abelian function $R_H(f_{157})$ has no pole on T_5. Therefore $\Phi^{H'_4(0,0)}$ is not defined at $\bar{\kappa}_5$.
For $H'_4(1,1) = \langle \tau_{33} I J^2 \rangle$ the Reynolds operators are

$$R_h(f_{56}) = 0, \quad R_H(f_{78}) = 4f_{78}, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{258} + ie^{\frac{\pi}{2}} f_{357} - i f_{458}$$

$$R_H(f_{168}) = f_{168} + i f_{267} + ie^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma'_4(1,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5$, $\bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Due to $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$, Lemma 4 ii) applies to provide $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Thus, $\mathcal{L}^H \simeq \mathbb{C}^3$. According to Lemma 4 iv), $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$ suffices for $\Phi^{H'_4(1,1)}$ to be a dominant rational map to $\mathbb{P}^2$. Further, $\left.\frac{f_{157} + ie^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$ by Lemma 6 implies that $R_H(f_{157})$ has no pole over T_5 and $\Phi^{H'_4(1,1)}$ is not defined at $\bar{\kappa}_5$.
If $H = H_2(1,1) = \langle \tau_{33} I^2 J^2 \rangle$ then $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \quad R_H(f_{56}) = 2f_{56}, \quad R_H(f_{78}) = 2f_{78}, \quad R_H(f_{157}) = f_{157} + ie^{\frac{\pi}{2}} f_{168}$$

$$R_H(f_{368}) = f_{368} - ie^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{258}) = f_{258} - f_{267}, \quad R_H(f_{467}) = f_{467} + f_{458}.$$

The $\Gamma_2(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\kappa}_3$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6 one has $\left.\frac{f_{157} + ie^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = \left.\frac{f_{368} - ie^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\right|_{T_3} = \left.\frac{f_{258} - f_{267}}{\Sigma_2}\right|_{T_2} = \left.\frac{f_{467} + f_{458}}{\Sigma_4}\right|_{T_4} = 0$. Thus, $R_H(f_{157}), R_H(f_{368}), R_H(f_{258}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$ and $\mathcal{L}^H \simeq \mathbb{C}^3$. Bearing in mind that $R_H(f_{56})|_{T_5} = \infty$, $R_H(f_{78})|_{T_5} \not\equiv \mathrm{const}$, one applies Lemma 4 iv) and concludes that $\Phi^{H_2(1,1)}$ is a dominant map to $\mathbb{P}^2$. Since $\mathcal{L}^H$ has no pole over $\sum_{i=1}^{4} T_i$, the map $\Phi^{H_2(1,1)}$ is not defined at $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\kappa}_4$. $\square$

Let us recall from Hacon and Pardini's [1] that the geometric genus $p_g(X) = \dim_{\mathbb{C}} H^0(X, \Omega^2_X)$ of a smooth minimal surface X of general type is at most 4. The next theorem provides a smooth toroidal compactification $Y = \left(\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}\right)'$ with abelian minimal model $A_{-1}/\langle \tau_{33} \rangle$ and $\dim_{\mathbb{C}} H^0(Y, \Omega^2_Y(T')) = 5$.

Theorem 7. i) *For $H = H'_2 = \langle I^2 \rangle$, $H''_2 = \langle J^2 \rangle$, $H_2(n, 1-n) = \langle \tau_{33} I^{2n} J^{2-2n} \rangle$ or $H^\theta_2(n,k) = \langle \tau^n_{33} I^k J^{-k} \theta \rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 3$ the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^3$$

has maximal $\mathrm{rk}\Phi^H = 2$. *For $H \neq H_2(n, 1-n)$ the rational map Φ^H is not globally defined and $\widehat{\mathbb{B}/\Gamma_H}$ are ruled surfaces with elliptic bases. In the case of $H = H_2(n, 1-n)$ the surface $\widehat{\mathbb{B}/\Gamma_H}$ is hyperelliptic.*

ii) *For* $H = H_2(0,0) = \langle \tau_{33} \rangle$ *the smooth surface* $\left(\mathbb{B}/\Gamma_{\langle \tau_{33}\rangle}\right)'$ *has abelian minimal model* $A_{-1}/\langle \tau_{33}\rangle$ *and the logarithmic-canonical map*

$$\Phi^{\langle \tau_{33}\rangle} : \widehat{\mathbb{B}/\Gamma_{\langle \tau_{33}\rangle}} \dashrightarrow \mathbb{P}([\Gamma_{\langle \tau_{33}\rangle}, 1]) = \mathbb{P}^4$$

is of maximal $\mathrm{rk}\Phi^{\langle \tau_{33}\rangle} = 2$.

Proof: i) By Lemma 4 v), it suffices to prove the statement for H_2', $H_2(1,0)$ and $H_2^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k}\theta\rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 2$.
Note that H_2', $H_2(1,0)$ are subgroups of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2\rangle$ and $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$. By Lemma 4 iv) that suffices for $\mathrm{rk}\Phi^{H_2'} = \mathrm{rk}\Phi^{H_2(1,0)} = 2$.
In the case of $H = H_2' = \langle I^2\rangle$, the Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 2f_{78}$$

$$R_H(f_{157}) = f_{157} + \mathrm{i}e\frac{\pi}{2}f_{357}, \qquad R_H(f_{168}) = f_{168} + \mathrm{i}e^{-\frac{\pi}{2}}f_{368}$$

$$R_H(f_{258}) = f_{258} - \mathrm{i}e^{-\frac{\pi}{2}}f_{458}, \qquad R_H(f_{267}) = f_{267} + \mathrm{i}e^{-\frac{\pi}{2}}f_{467}.$$

The Γ_2'-cusps are $\bar\kappa_1 = \bar\kappa_3$, $\bar\kappa_2 = \bar\kappa_4$, $\bar\kappa_5$, $\bar\kappa_6$, $\bar\kappa_7$ and $\bar\kappa_8$. According to Lemma 4 ii), the inclusions $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Similarly, from $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ there follows $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{258}))$. As a result, one concludes that the space of the invariants $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over T_6, the rational map $\Phi^{H_2'}$ is not defined at $\bar\kappa_6$.
If $H = H_2(1,0) = \langle \tau_{33} I^2\rangle$, then $\mathcal{L}^H$ is spanned by

$$1 \in \mathbb{C}, \qquad R_H(f_{56}) = 2f_{56}, \qquad R_H(f_{78}) = 0$$

$$R_H(f_{157}) = f_{157} + f_{368}, \qquad R_H(f_{258}) = f_{258} + \mathrm{i}e^{-\frac{\pi}{2}}f_{467}.$$

The $\Gamma_2(1,0)$-cusps are $\bar\kappa_1 = \bar\kappa_3$, $\bar\kappa_2 = \bar\kappa_4$, $\bar\kappa_5 = \bar\kappa_6$, $\bar\kappa_7 = \bar\kappa_8$. According to Lemma 4 iii), the inclusions $T_1 + T_3 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 + T_4 \subset (R_H(f_{258}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for the linear independence of 1, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{258})$.
Further, observe that $H_2^{\theta}(n,0) = \langle \tau_{33}^n\theta\rangle$ are subgroups of $H^{\theta}_{2\times 2}(0) = \langle \tau_{33}, \theta\rangle$ with $\mathrm{rk}\Phi^{H^{\theta}_{2\times 2}(0)} = 2$. Therefore $\mathrm{rk}\Phi^{H_2^{\theta}(n,0)} = 2$ by Lemma 4 iv).
If $H = H_2^{\theta}(0,0) = \langle\theta\rangle$ then

$$R_H(f_{56}) = f_{56} + f_{78}, \quad R_H(f_{157}) = f_{157} - e^{\frac{\pi}{2}}f_{357}, \quad R_H(f_{368}) = f_{368} - e^{\frac{\pi}{2}}f_{168}$$

$$R_H(f_{258}) = f_{258} + f_{267}, \qquad R_H(f_{467}) = f_{467} + f_{458}.$$

The $\Gamma_2^\theta(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$ and $\bar{\kappa}_6 = \bar{\kappa}_8$. According to Lemma 4 ii), $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ implies $R(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R(f_{157}))$. Lemma 6 supplies $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2\mathrm{e}^{-\pi} \neq 0$ and $\frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$. Therefore $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{467}) \subset \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Thus, $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. The entire $[\Gamma_2^\theta(0,0), 1]$ vanishes at $\bar{\kappa}_4$ and $\Phi^{H_2^\theta(0,0)}$ is not globally defined.
For $H = H_2^\theta(1,0) = \langle \tau_{33}\theta \rangle$ the space $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \qquad R_H(f_{56}) = f_{56} - f_{78}$$

$$R_H(f_{157}) = f_{157} + \mathrm{i} f_{368}, \quad R_H(f_{258}) = 2f_{258}, \quad R_H(f_{467}) = 0.$$

The $\Gamma_2^\theta(1,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$ and $\bar{\kappa}_6 = \bar{\kappa}_7$. Making use of $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 \subset (R_H(f_{258}))_\infty \subset T_2 + \sum_{\alpha=5}^{8} T_\alpha$, one applies Lemma 4 iii), in order to conclude that

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

The abelian functions from $\mathcal{L}^H$ have no poles along T_4, so that $\Phi^{H_2^\theta(1,0)}$ is not defined at $\bar{\kappa}_4$.
Observe that $H_2^\theta(n,1) = \langle \tau_{33}^n I J^{-1}\theta \rangle$ are subgroups of $H_{2\times 2}^\theta(1) = \langle \tau_{33}, IJ^{-1}\theta \rangle$ with $\mathrm{rk}\Phi^{H_{2\times 2}^\theta(1)} = 2$, so that $\mathrm{rk}\Phi^{H_2^\theta(n,1)} = 2$ as well.
More precisely, Reynolds operators for $H = H_2^\theta(0,1) = \langle IJ^{-1}\theta \rangle$ are

$$R_H(f_{56}) = f_{56} + \mathrm{i} f_{78}, \quad R_H(f_{157}) = f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}, \quad R_H(f_{368}) = f_{368} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}$$

$$R_H(f_{258}) = f_{258} - \mathrm{e}^{-\frac{\pi}{2}} f_{458}, \qquad R_H(f_{267}) = f_{267} + \mathrm{e}^{-\frac{\pi}{2}} f_{467}.$$

The Γ_2^θ-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$, $\bar{\kappa}_6 = \bar{\kappa}_7$. By Lemma 6 one has $\frac{f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{368} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, whereas $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{368}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Applying Lemma 4 ii) to the inclusions $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$, one concludes that $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{258}))$. Altogether

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

Since $\mathcal{L}^H$ has no pole over T_3, the rational map $\Phi^{H_2^\theta(0,1)}$ is not defined at $\bar{\kappa}_3$.
If $H = H_2^\theta(1,1) = \langle \tau_{33} I J^{-1}\theta \rangle$ then

$$R_H(f_{56}) = f_{56} - \mathrm{i} f_{78}, \qquad R_H(f_{157}) = 2f_{157}$$

$$R_H(f_{368}) = 0, \qquad R_H(f_{258}) = f_{258} + e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2^\theta(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$ and $\bar{\kappa}_6 = \bar{\kappa}_8$. Making use of $R_H(f_{157})|_{T_1} = \infty$, $T_H(f_{258})|_{T_2} = \infty$, one applies Lemma 4 iii), in order to conclude that $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over T_3, the rational map $\Phi^{H_2^\theta(1,1)}$ is not defined at $\bar{\kappa}_3$.

Reynolds operators for $H = H_2^\theta(0,2) = \langle I^2 J^2 \theta \rangle$ are

$$R_H(f_{56}) = f_{56} - f_{78}, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{168}) = f_{168} + e^{-\frac{\pi}{2}} f_{368}$$

$$R_H(f_{258}) = f_{258} - f_{267}, \qquad R_H(f_{467}) = f_{467} - f_{458}.$$

The $\Gamma_2^\theta(0,2)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$, $\overline{\kappa_6} = \overline{\kappa_8}$. Lemma 4 ii) applies to $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ to provide $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}))$. By Lemma 6 one has $\left.\frac{f_{258} - f_{267}}{\Sigma_2}\right|_{T_2} = 0$ and $\left.\frac{f_{467} - f_{458}}{\Sigma_4}\right|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. As a result, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$ and $R_H(f_{467})|_{T_4} = \infty$. Lemma 4 iii) reveals that $1 \in \mathbb{C}$, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$ form a $\mathbb{C}$-basis of $\mathcal{L}^H$. Since $\mathcal{L}^H$ has no pole over T_2, the rational map $\Phi^{H_2^\theta(0,2)}$ is not defined over $\bar{\kappa}_2$.

In the case of $H = H_2^\theta(1,2) = \langle \tau_{33} I^2 J^2 \theta \rangle$ one has

$$R_H(f_{56}) = f_{56} + f_{78}, \qquad R_H(f_{157}) = f_{157} - \mathrm{i} f_{368}$$

$$R_H(f_{258}) = 0, \qquad R_H(f_{467}) = 2 f_{467}.$$

The $\Gamma_2^\theta(1,2)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$ and $\bar{\kappa}_6 = \bar{\kappa}_7$. Lemma 4 iii) applies to $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$, $T_4 \subset (R_H(f_{467}))_\infty \subseteq T_4 + T_6 + T_7$, in order to justify the linear independence of 1, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$. Since $\mathcal{L}^H \simeq \mathbb{C}^4$ has no pole over T_2, the rational map $\Phi^{H_2^\theta(1,2)}$ is not defined at $\bar{\kappa}_2$.

ii) For $H = H_2(0,0) = \langle \tau_{33} \rangle$ one has the following Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0, \qquad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}$$

$$R_H(f_{258}) = f_{258} + f_{267}, \quad R_H(f_{368}) = f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{467}) = f_{467} - f_{458}.$$

There are six $\Gamma_{\langle \tau_{33} \rangle}$-cusps: $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\kappa}_3$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By the means of Lemma 6 one observes that $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\left.\frac{f_{258} + f_{267}}{\Sigma_2}\right|_{T_2} = 2e^{-\pi} \neq 0$, $\left.\frac{f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\right|_{T_3} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\left.\frac{f_{467} - f_{458}}{\Sigma_4}\right|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. Therefore $T_i \subset (R_H(f_{i,\alpha_i,\beta_i}))_\infty \subseteq T_i + \sum_{\delta=5}^{8} T_\delta$ for $1 \leq i \leq 4$, $(\alpha_1, \beta_1) = (5,7)$, $(\alpha_2, \beta_2) = (5,8)$, $(\alpha_3, \beta_3) = (6,8)$, $(\alpha_4, \beta_4) = (6,7)$. According to Lemma 4 iii), that suffices for 1, $R_H(f_{157})$, $R_H(f_{258})$, $R_H(f_{368})$, $R_H(f_{467})$ to be a $\mathbb{C}$-basis of $\mathcal{L}^H$. Bearing in mind that $H_2(0,0) = \langle \tau_{33} \rangle$ is a subgroup of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$ with $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, one concludes that $\mathrm{rk}\Phi^{\langle \tau_{33} \rangle} = 2$. □

References

[1] Hacon Ch. and Pardini R., *Surfaces with $p_g = q = 3$*, Trans. Amer. Math. Soc. **354** (2002) 2631–1638.

[2] Hemperly J., *The Parabolic Contribution to the Number of Independent Automorphic Forms on a Certain Bounded Domain*, Amer. J. Math. **94** (1972) 1078–1100.

[3] Holzapfel R.-P., *Jacobi Theta Embedding of a Hyperbolic 4-space with Cusps*, In: Geometry, Integrability and Quantization IV, I. Mladenov and G. Naber (Eds), Coral Press, Sofia 2002, pp 11–63.

[4] Holzapfel R.-P., *Complex Hyperbolic Surfaces of Abelian Type*, Serdica Math. J. **30** (2004) 207–238.

[5] Kasparian A. and Kotzev B., *Normally Generated Subspaces of Logarithmic Canonical Sections*, to appear in Ann. Univ. Sofia.

[6] Kasparian A. and Kotzev B., *Weak Form of Holzapfel's Conjecture*, J. Geom. Symm. Phys. **19** (2010) 29–42.

[7] Kasparian A. and Nikolova L., *Ball Quotients of Non-Positive Kodaira Dimension*, submitted to CRAS (Sofia).

[8] Lang S., *Elliptic Functions*, Addison-Wesley, London 1973, pp 233–237.

[9] Momot A., *Irregular Ball-Quotient Surfaces with Non-Positive Kodaira Dimension*, Math. Res. Lett. **15** (2008) 1187–1195.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 290–304

MODIFIED COSMOLOGICAL EQUATIONS AND THE EINSTEIN STATIC UNIVERSE

LUCA PARISI and ROSANGELA CANONICO

Dipartimento di Fisica "E. R. Caianiello", Università di Salerno, I-84084 Fisciano, INFN Sezione di Napoli, GC di Salerno, Via Ponte Don Melillo, Italy

Abstract. The stability properties of the Einstein Static solution of General Relativity is altered when corrective terms arising from modifications of the underlying gravitational theory appear in the cosmological equations. Employing dynamical system techniques and numerical integrations, we discuss the stability of static cosmological solutions in the framework of two recently proposed quantum gravity models, namely Loop Quantum Cosmology and Horava-Lifshitz gravity.

1. Introduction

The **Einstein Static** (ES) Universe is an exact solution of Einstein's equations describing a closed Friedmann-Robertson-Walker model sourced by a perfect fluid and a cosmological constant (see, for example [23]). This solution is unstable to homogeneous perturbations as shown by Eddington [15], furthermore it is always neutrally stable against small inhomogeneous vector and tensor perturbations and neutrally stable against adiabatic scalar density inhomogeneities with high enough sound speed [2].

In recent years there has been renewed interest in the ES Universe because of its relevance for the Emergent Universe scenario [16, 17, 31] in which the ES solution plays a crucial role, being an initial state for a past-eternal inflationary cosmological model. In the Emergent Universe scenario the horizon problem is solved before inflation begins, there is no singularity, no exotic physics is involved, and the quantum gravity regime can even be avoided. This model, relying on the choice of a particular initial state, suffers from a fine-tuning problem which is ameliorated

*Reprinted from J. Geom. Symm. Phys. **22** (2011) 51–65.

when modifications to the cosmological equations arise but then a mechanism is needed to trigger the expanding phase of the Universe (see [27, 28]).

The existence of ES solutions along with their stability properties has been widely investigated in the framework of **General Relativity** (GR)for several kinds of matter fields sources (see [3] and references therein). ES solutions also exist in several modified gravity models [8] ranging from the Randall-Sundrum and DGP braneworld scenarios [12, 18, 22, 37, 42] to Gauss-Bonnet modified gravity and $f(R)$ theories [4–6, 13, 20, 21, 36]. The issue of the existence and stability of ES solutions has also been considered in the semiclassical regime of **Loop Quantum Cosmology** (LQC), in either the case of correction to the matter sector [32] or the case of correction to the gravitational sector [34]. Recently the same issue has been also considered in the framework of **Hořava-Lifshitz** (HL) gravity [41] and IR modified Hořava gravity [7, 19].

When dealing with higher order modified cosmological equations, the existence of many new ES solutions is possible, whose stability properties, depending on the details of the single theory or family of theories taken into account, are substantially modified with respect to the classical ES solution of General Relativity (GR).

Often in such analysis the case of closed $(k = 1)$ cosmological models is the only one considered, neglecting the intriguing possibility of static solutions in open $(k = -1)$ cosmological models. It is interesting that, due to the aforementioned corrections to the cosmological equations, open ES models may be found even in the case of a vanishing cosmological constant or when the perfect fluid has vanishing energy density.

In this paper we systematically review the stability properties of static cosmological solutions arising in the framework of two recently proposed quantum gravity models, namely Loop Quantum Cosmology and Horava-Lifshitz gravity, both providing modified cosmological equations. To this aim, we employ dynamical system techniques and numerical integrations. This work is based on the results presented in [10, 33, 34].

This paper is structured as follows. In Section 2, we consider static solutions in the framework of LQC, following and enlarging the analysis already performed in [34]. It is shown that, beside the ES solution of GR, a LQC solution arises also in the case of open cosmological models which stability is also completely characterized. Following the same approach, in Section 3 we consider static cosmological solutions in the context of HL gravity with detailed balance and projectability condition. Two solutions are found along with their stability properties. In Section 4, some conclusions are eventually drawn.

2. Loop Quantum Cosmology

In Loop Quantum Cosmology the quantization techniques borrowed by **Loop Quantum Gravity**, a background-independent nonperturbative quantum theory of gravity, are applied to symmetry reduced models (see [9] and references therein).

For the sake of simplicity, in this section we consider the modified Friedmann equations arising in the semiclassical regime of LQC [1,39]. We consider gravitational modifications only, neglecting the inverse volume correction to the matter sector. The motivation is twofold: the analysis of this system allows a more transparent comparison with the case of GR and moreover it allows us to follow the notations introduced in [34] which will also be easily used in the analysis of the HL gravity presented in the next section.

The model considered is sourced by a perfect fluid with linear equation of state $p = w\rho$ plus a cosmological constant Λ. The classical energy conservation equation still holds

$$\dot{\rho} = -3\rho H(1+w) \tag{1}$$

while the loop quantum effects lead to a modification to the classical **Friedmann equation**

$$H^2 = \left(\frac{\kappa}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} + \frac{3k}{\kappa\rho_c a^2}\right) \tag{2}$$

and to the **Raychaudhuri equation**

$$\begin{aligned} \dot{H} &= -\frac{\kappa}{2}\rho\,(1+w)\left(1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c}\right) \\ &\quad + \left[1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c} - \frac{3\rho(1+w)}{\rho_c}\right]\frac{k}{a^2} + \frac{6k^2}{\kappa\rho_c a^4}. \end{aligned} \tag{3}$$

Notice that we are considering at once the $k = 0$ case and the $k = \pm 1$ cases [1,39]. Here $\kappa = 8\pi G = 8\pi/M_P^2$, and the critical LQC energy density is $\rho_c \approx 0.82 M_P^4$.

2.1. Static Solutions

The system of equations (1)-(4) admits two static solutions, i.e. solutions characterized by $\dot{a} = \dot{H} = \dot{\rho} = 0$. The first solution corresponds to the standard ES Universe in GR while the second solution arises from the LQC corrective terms

$$\rho_{GR} = \frac{2\Lambda}{\kappa(1+3w)}, \qquad a^2_{GR} = \frac{2k}{\kappa\rho_{GR}(1+w)} \tag{4}$$

$$\rho_{LQ} = \frac{2(\Lambda - \kappa\rho_c)}{\kappa(1+3w)}, \qquad a^2_{LQ} = \frac{2k}{\kappa\rho_{LQ}(1+w)}. \tag{5}$$

The conditions under which these static solutions exist are summarized in Table 1 which follow from $a^2 > 0$ and $\rho > 0$. The presence of the curvature index k is worth stressing, indeed the previous analysis [34] can be enlarged to enclose the $k = -1$ case where the two solutions still exist.

2.2. Stability Analysis

The stability of the solutions equations (4) and (5) can be characterized using dynamical system theory and performing a linearized stability analysis. To this aim, we first have to rewrite the system of equations (1)-(4) in the form of a genuine dynamical system. Indeed, in these equations the three variables a, H and ρ appear but the actual dynamics is constrained on a two-dimensional surface described by the modified Friedmann equation (see Fig.1). Thus, following [34], we solve

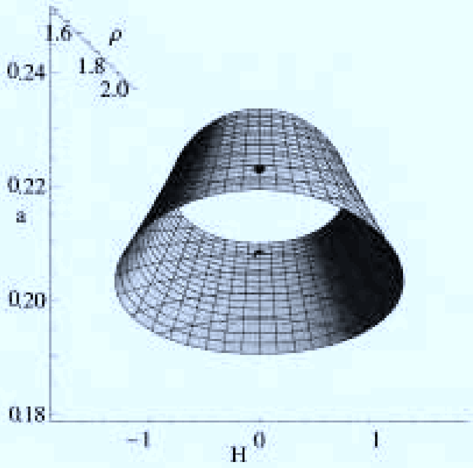

Figure 1. Friedmann constraint as hypersurface in the a, H, ρ space the for the case $k = -1$, $\Lambda < 0$, $w < -1$ with $\Lambda = -100$, $w = -2$, $\kappa = 25.13274123$. The ES and LQ solutions are depicted as black dots on top and underneath the surface respectively.

equation (2) for $1/a^2$. Two solutions are found

$$\frac{1}{a^2} = g_{\pm}(\rho, H) \tag{6}$$

where

$$g_{\pm} = \frac{2(\kappa\rho + \Lambda) + \kappa\rho_c\left(1 \pm \sqrt{1 - 12H^2/\kappa\rho_c}\right)}{6k}. \tag{7}$$

Substituting the solutions (6) into equation (4), we find two branches for the time derivative of the Hubble parameter, thus the original system splits in a pair of two-dimensional nonlinear dynamical systems in the variables ρ and H (see Fig. 2)

$$\text{GR}: \dot{\rho} = -3H\rho(1+w) \quad \text{and} \quad \dot{H} = F_{-}(\rho, H) \tag{8}$$

$$\text{LQ}: \dot{\rho} = -3H\rho(1+w) \quad \text{and} \quad \dot{H} = F_{+}(\rho, H) \tag{9}$$

where

$$F_{\pm} = -\frac{\kappa}{2}(1+w)\rho\left(1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}\right)+\frac{6k^2g_{\pm}^2}{\kappa\rho_c}$$
$$+g_{\pm}k\left[1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}-3(1+w)\frac{\rho}{\rho_c}\right]. \qquad (10)$$

Each one of the systems (8) and (9) admits a fixed point representing a static solution, that is

$$\text{GR}: H=0 \quad \text{and} \quad \rho_o = \frac{2\Lambda}{\kappa(1+3w)} \qquad (11)$$

$$\text{LQ}: H=0 \quad \text{and} \quad \rho_o = \frac{2(\Lambda-\kappa\rho_c)}{\kappa(1+3w)} \qquad (12)$$

respectively. Substituting these values of ρ_o in equation (2) one gets exactly the values of the constant scale factor in terms of the parameters as in equations (4) and (5).

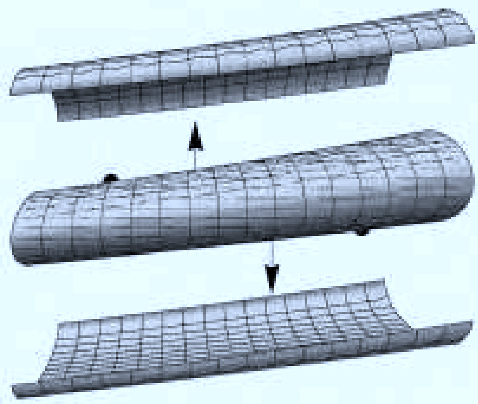

Figure 2. Splitting of the Friedmann constraint in two local charts around the fixed points.

Finally, to characterize the stability of the solutions equations (4) and (5) we evaluate the eigenvalues of the Jacobian matrix for the two systems equations (8) and (9) at the fixed points equations (11) and (12) respectively.

For the system in equation (8), we recover the usual properties of the ES solution in GR. The eigenvalues of the linearized system at the fixed point are

$$\lambda_{GR} = \pm\sqrt{\Lambda(1+w)}. \qquad (13)$$

In the case of positive curvature index $k=1$, these are either real with opposite signs for $\Lambda > 0$ and $w > -1/3$ - thus the fixed point is unstable (of the saddle type) - or purely imaginary for $\Lambda < 0$ and $-1 < w < -1/3$, so the fixed point is a center. In the case of negative spatial curvature index $k=-1$, these are again real with opposite signs for $\Lambda < 0$ and $w < -1$, so the fixed point is unstable (of the saddle type). In Fig. 3 an example of the latter case is depicted.

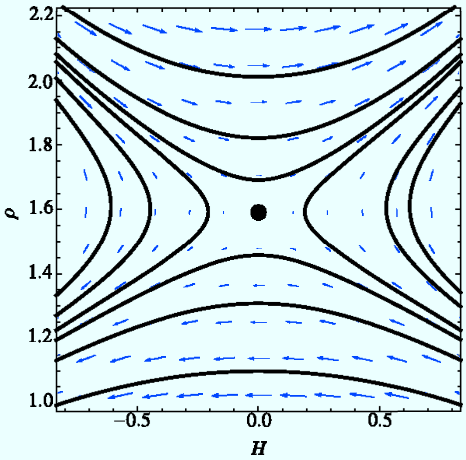

Figure 3. Dynamical behavior of the system around the GR fixed point for the case $k = -1$, $\Lambda < 0$, $w < -1$ with $\Lambda = -100$, $w = -2$, $\kappa = 25.13274123$.

For the system equation (9) the eigenvalues at the fixed point are

$$\lambda_{LQ} = \pm\sqrt{(\kappa\rho_c - \Lambda)(1 + w)}. \tag{14}$$

In the case of positive curvature index $k = 1$, the LQ fixed point is either unstable (of the saddle kind), when $\kappa\rho_c > \Lambda$ and $-1 < w < -1/3$, or a center for the linearized system, i.e. a neutrally stable fixed point, when $\kappa\rho_c < \Lambda$ and $w > -1/3$. In the case of negative spatial curvature index $k = -1$, the eigenvalues are purely imaginary for $\kappa\rho_c > \Lambda$ and $w < -1$, so we have a center for the linearized system again. In the latter case the fixed point is nonhyperbolic thus the linearization theorem does not apply. Nevertheless a numerical integration of the fully nonlinear system equation (9) for initial conditions near the fixed point confirms the result of the linearized stability analysis (see Fig. 4). It's worth stressing that in open LQC models a stable ES solution exists in the case of positive values of the cosmological constant as long as $\Lambda < \kappa\rho_c$.

The results of the linearized stability analysis are summarized in Table 1.

3. Hořava-Lifshitz Gravity

The Hořava-Lifshitz gravity [24, 25] is a power-counting renormalizable theory of (3+1)-dimensional quantum gravity. In the ultraviolet limit, the theory has a Lifshitz-like anisotropic scaling between space and time characterized by the dynamical critical exponent $z = 3$. In the IR limit the theory flows to the relativistic value $z = 1$.

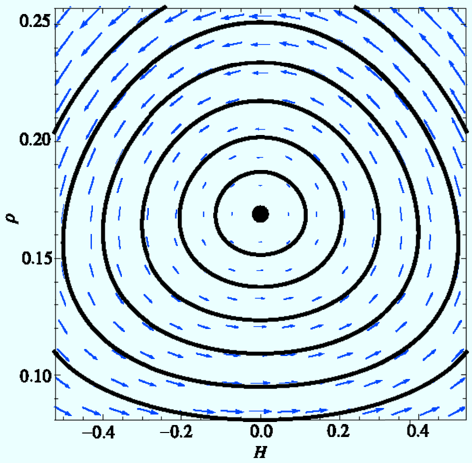

Figure 4. Dynamical behavior of the system around the LQ fixed point for the case $k = -1$, $\Lambda < \kappa\rho_c$, $w < -1$ with $\Lambda = 10$, $w = -2$, $\kappa = 25.13274123$.

Table 1. Existence conditions and stability conditions for the static solutions in equations (4) and (5).

	k	Λ	w	Stability
GR	1	> 0	$w > -1/3$	saddle
		< 0	$-1 < w < -1/3$	center
	-1	< 0	$w < -1$	saddle
LQ	1	$< \kappa\rho_c$	$-1 < w < -1/3$	center
		$> \kappa\rho_c$	$w > -1/3$	saddle
	-1	$< \kappa\rho_c$	$w < -1$	center

The effective speed of light c, the effective **Newton constant** G and the effective **cosmological constant** Λ of the low-energy theory, emerge from the relevant deformations of the deeply nonrelativistic $z = 3$ theory which dominates at short distances [24, 25]

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}, \qquad G = \frac{\kappa^2}{32\pi c}, \qquad \Lambda = \frac{3}{2}\Lambda_W. \tag{15}$$

The first of the equations in(15) imposes a relation among the parameters c, Λ_W and λ. Thus, in order to have a real emergent speed of light c, for $\lambda > 1/3$ the cosmological constant has to be negative Λ_W. However, after an analytic continuation of the parameters (see [29]), a real speed of light for $\lambda > 1/3$ implies a positive cosmological constant Λ_W. Thus, mimicking the notation introduced in [30], we introduce a two-valued parameter $\epsilon = \pm 1$, in order to examine both the aforementioned cases at once.

The HL cosmology has been systematically studied using dynamical systems theory in [11, 14, 26, 38], it has also been investigated in [40] using conservation laws of mechanics. Here we consider static solutions of the cosmological equations for the HL gravity when both the detailed balance condition and projectability condition hold.

First we recast the modified Friedmann equations of [29] in a form which allows an easy comparison with the formerly considered case of LQC[1].

The **modified Friedmann equation** reads

$$H^2 = \frac{2}{3\lambda - 1}\left[\frac{\kappa}{3}\rho + \epsilon\left(\frac{\Lambda}{3} - \frac{k}{a^2} + \frac{3k^2}{4\Lambda a^4}\right)\right] \tag{16}$$

and the **modified Raychaudhuri equation** reads

$$\dot{H} = \frac{2}{3\lambda - 1}\left[-\frac{\kappa}{2}\rho(1+w) + \epsilon\left(\frac{k}{a^2} - \frac{3k^2}{2\Lambda a^4}\right)\right]. \tag{17}$$

The conservation equation for the energy density of the perfect fluid still holds unchanged:

$$\dot{\rho} = -3\rho H(1+w). \tag{18}$$

Besides the overall factor $\frac{2}{3\lambda-1}$ on the right hand side of equations (16) and (17), the modifications to the cosmological equations of GR consist of the higher order terms $\propto k^2/\Lambda a^4$ which become dominant at short distance scales and do not affect the classical cosmological equations in the case of flat models.

3.1. Static Solutions

It can be readily found, imposing the conditions $\dot{a} = \dot{H} = \dot{\rho} = 0$, that the system of equations (18)-(17) admits the following two static solutions

$$\rho_{HL1} = 0, \qquad a^2_{HL1} = \frac{3k}{2\Lambda} \tag{19}$$

$$\rho_{HL2} = \frac{-16\epsilon\Lambda}{(3w-1)^2\kappa}, \qquad a^2_{HL2} = \frac{(3w-1)k}{2\Lambda(1+w)}. \tag{20}$$

The conditions under which these static solutions exist are summarized in Table 2 and Table 3.

The presence of the curvature index k and the parameter ϵ in equations (19) and (20) is worth being stressed; indeed the analysis presented in [41] can be enlarged to enclose the $k = -1$ case where new interesting possibilities arise. For instance a physically meaningful ES solution is present even in the case of vanishing energy density of the perfect fluid, i.e., equation (19).

---

[1] According to the definitions given in Section II, $c = 1$ and $\kappa = 8\pi G$, equation (16) and equation (17) have been written accordingly.

3.2. Stability Analysis

The stability analysis can be easily performed reducing the original system to an actual two-dimensional autonomous dynamical system by making use of the Friedmann constraint (see Fig.5). In this case the simplest and most straightforward

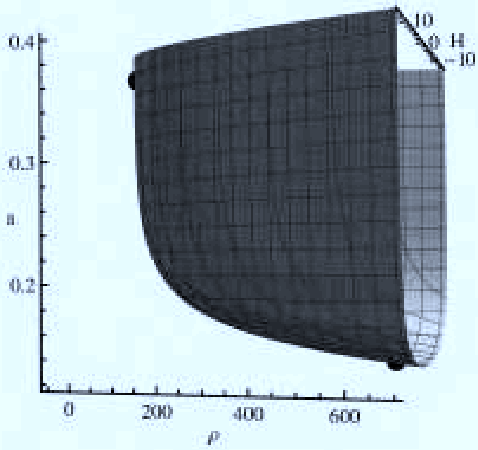

Figure 5. Friedmann constraint as hypersurface in the a, H, ρ space the for the case $k = -1$ with $\epsilon = 1$, $\lambda > 1/3$, $\Lambda < 0$, $w > 1/3$. The two black dots represent the HL1 (upper) and HL2 (lower) static solutions.

choice is to eliminate the dependence on ρ from the other equations, being equation (16) linear in ρ, that is, to consider the projection on the (H, a)-plane (see Fig.6). This allows us to describe the dynamics with just one set of equations. Indeed, solving equation (16) for ρ

$$\rho = \frac{3}{2\kappa}\left(3\lambda - 1\right)H^2 - \frac{\epsilon}{\kappa}\left(\Lambda - \frac{3k}{a^2} + -\frac{3k^2}{4\Lambda a^4}\right) \tag{21}$$

and substituting into equation (17) one gets a first order nonlinear differential equation

$$\dot{H} = \frac{\epsilon}{3\lambda - 1}\left[(1+w)\Lambda - \frac{(3w+1)k}{a^2} + \frac{3k^2(3w-1)}{4\Lambda a^4}\right] - \frac{3}{2}\left(1+w\right)H^2 \tag{22}$$

which, together with the definition of the Hubble parameter

$$\dot{a} = aH \tag{23}$$

provides a genuine two-dimensional autonomous dynamical system in the variables a and H. The system admits two fixed points with energy densities as in equations (19) and (20) and in order to characterize the stability of these solutions, we evaluate the eigenvalues of the Jacobian matrix for the system equations (22) and (23) at the fixed points corresponding to equations (19) and (20) respectively.

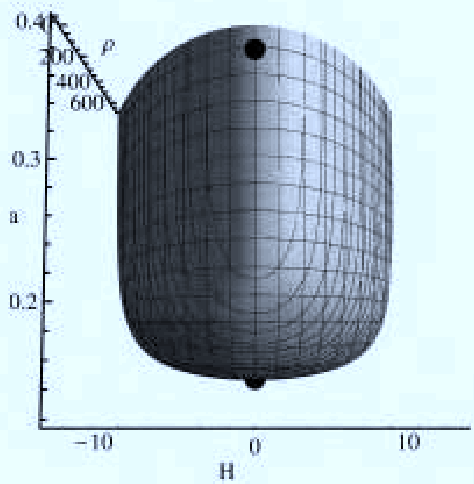

Figure 6. Friedmann constraint as seen from the (H, a)-plane.

The eigenvalues at the fixed point $HL1$ read

$$\lambda_{HL1} = \pm\frac{2\sqrt{6(3\lambda-1)\epsilon\Lambda}}{3(3\lambda-1)}. \tag{24}$$

For all the admitted values of the parameters this is a pair of purely imaginary eigenvalues thus the fixed point is a center for the linearized system. The point is nonhyperbolic so the linearized analysis may fail to be predictive at nonlinear order, nevertheless a numerical integration proves that this fixed point is actually a center (see Fig. 7).

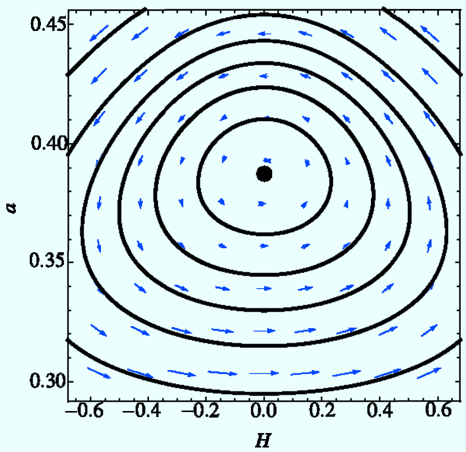

Figure 7. Dynamical behavior of the system around the $HL1$ fixed point for the case $k = -1$ with $\epsilon = 1$, $\lambda > 1/3$, $\Lambda < 0$, $w > 1/3$.

The results of the stability analysis for the fixed point $HL1$ are summarized in Table 2.

Table 2. Existence conditions and stability conditions for the static solution $HL1$.

ϵ	λ	k	Λ	Stability
-1	$< 1/3$	-1	< 0	center
	$> 1/3$	1	> 0	
1	$< 1/3$	1	> 0	
	$> 1/3$	-1	< 0	

The eigenvalues at the fixed point $HL2$ read

$$\lambda_{HL2} = \pm\frac{2\sqrt{-2(3w-1)(3\lambda-1)(1+w)\epsilon\Lambda}}{(3\lambda-1)(3w-1)}. \tag{25}$$

According to the admitted values of the parameters this is either a pair of purely imaginary eigenvalues, so the fixed point is a center for the linearized system, or a pair of real eigenvalues with opposite signs, so the fixed point is unstable (of the saddle type). In particular, the solution is a center for $-1 < w < 1/3$ and is a saddle for $w < -1$ or $w > 1/3$ (for an example of the latter case see Fig. 8).

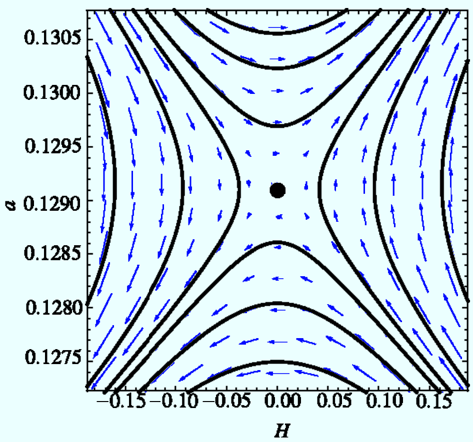

Figure 8. Dynamical behavior of the system around the $HL2$ fixed point for the case $k = -1$ with $\epsilon = 1$, $\lambda > 0$, $\Lambda < 0$, $w > 1/3$.

The results of the stability analysis for the fixed point $HL2$ are summarized in Table 3.

4. Conclusions

Here we have considered the existence of static solutions in the framework of two recently proposed quantum gravity models, namely, LQC and HL gravity and eventually we have shown that the inclusion of a negative curvature index $k = -1$ enlarges the ranges of existence of the solutions affecting their stability properties

Table 3. Existence conditions and stability conditions for the static solution $HL2$.

ϵ	λ	k	Λ	w	Stability
-1	$> 1/3$	-1	> 0	$-1 < w < 1/3$	center
		1	> 0	$w < -1$	saddle
				$w > 1/3$	
1	$> 1/3$	-1	< 0	$w < -1$	saddle
				$w > 1/3$	
		1	< 0	$-1 < w < 1/3$	centre

thus providing new interesting results. The solutions found display stability conditions rather different from those of the corresponding solutions in closed models and from the stability properties of the standard ES solution of GR.

In the case of LQC gravitational modifications to the Friedmann equations, a negative curvature index allows a neutrally stable static solution with $\Lambda < \kappa\rho_c$ and $w < -1$, in contrast to the GR case. In particular the LQC static solution exists and is stable in the case of positive values of the cosmological constant as long as $\Lambda < \kappa\rho_c$.

In the case of HL gravity two static solutions are found. The inclusion of the negative curvature index leads to a static solution ($HL1$) with negative cosmological constant and vanishing energy density which is neutrally stable against homogeneous perturbations. Furthermore, a negative curvature index allows a static solution ($HL2$) which can be either a saddle, for $w < -1$ and $w > 1/3$, or a center for $-1 < w < 1/3$.

As already observed in the frameworks of different modified models [27, 28, 32, 34], the regime of infinite cycles about the center fixed points must be eventually broken in order to enter the current expanding universe phase. To this aim a further mechanism is needed, whose analysis is beyond the scope of this paper.

Acknowledgements

The first named author would like to thank to C. Böhmer, F. Lobo, R. Maartens and K. Vandersloot for useful discussions. The authors are grateful to G. Vilasi for his thoughtfull advices and encouragement. This work is partially supported by Agenzia Spaziale Italiana (ASI) through the WP5000 project, by the Italian Ministero Istruzione Università e Ricerca (MIUR) through the PRIN 2008 grant and by the INFN-MICINN collaboration.

References

[1] Ashtekar A., Pawlowski T., Singh P. and Vandersloot K., *Loop Quantum Cosmology of k=1 FRW Models*, Phys. Rev. D **75** (2007) 024035.

[2] Barrow J., Ellis G., Maartens R. and Tsagas C., *On the Stability of the Einstein Static Universe*, Class. Quant. Grav. **20** (2003) L155-L164.

[3] Barrow J. and Tsagas C., *On the Stability of Static Ghost Cosmologies*, Class. Quant. Grav. **26** (2009) 195003.

[4] Böhmer C., *The Einstein Static Universe with Torsion and the Sign Problem of the Cosmological Constant*, Class. Quant. Grav. **21** (2004) 1119-1124.

[5] Böhmer C., Hollenstein L. and Lobo F., *Stability of the Einstein Static Universe in f(R) Gravity*, Phys. Rev. D **76** (2007) 084005.

[6] Böhmer C. and Lobo F., *Stability of the Einstein Static Universe in Modified Gauss-Bonnet Gravity*, Phys. Rev. D **79** (2009) 067504.

[7] Böhmer C. and Lobo F., *Stability of the Einstein Static Universe in IR Modified Hořava Gravity*, arXiv:0909.3986 [gr-qc].

[8] Böhmer C., Hollenstein L., Lobo F. and Seahra S., *Stability of the Einstein Static Universe in Modified Theories of Gravity*, arXiv:1001.1266 [gr-qc].

[9] Bojowald M., *Loop Quantum Cosmology*, Living Rev. Rel. **11** (2008) 4.

[10] Canonico R., Parisi L., *Stability of the Einstein Static Universe in Open Cosmological Models*, Phys. Rev. D **82** (2010) 064005.

[11] Carloni S., Elizalde E. and Silva P., *An Analysis of the Phase Space of Horava-Lifshitz Cosmologies,* Class. Quant. Grav. **27** (2010) 045004.

[12] Clarkson C. and Seahra S., *Braneworld Resonances*, Class. Quant. Grav. **22** (2005) 3653-3688.

[13] Clifton T. and Barrow J., *The Existence of Gödel, Einstein and de Sitter Universes*, Phys. Rev. D **72** (2005) 123003.

[14] Czuchry E., *The Phase Portrait of a Matter Bounce in Horava-Lifshitz Cosmology*, arXiv:0911.3891 [hep-th].

[15] Eddington A., *On the Instability of Einstein's Spherical World*, Mon. Not. Roy. Astron. Soc. **90** (1930) 668-688.

[16] Ellis G. and Maartens R., *The Emergent Universe: Inflationary Cosmology with no Singularity*, Class. Quant. Grav. **21** (2004) 223-232.

[17] Ellis G., Murugan J. and Tsagas C., *The Emergent Universe: An Explicit Construction*, Class. Quant. Grav. **21** (2004) 233-249.

[18] Gergely L. and Maartens R., *Brane-world Generalizations of the Einstein Static Universe*, Class. Quant. Grav. **19** (2002) 213-222.

[19] Ghodsi A and Hatefi E., *Extremal Rotating Solutions in Horava Gravity*, Phys. Rev. D **81** (2010) 044016.

[20] Goheer N., Goswami R. and Dunsby P., *Dynamics of f(R)-cosmologies Containing Einstein Static Models*, Class. Quant. Grav. **26** (2009) 105003.

[21] Goswami R., Goheer N. and Dunsby P., *Existence of Einstein Static Universes and their Stability in Fourth-Order Theories of Gravity*, Phys. Rev. D. **78** (2008) 044011.

[22] Gruppuso A., Roessl E. and Shaposhnikov M., *Einstein Static Universe as a Brane in Extra Dimensions*, JHEP **011** (2004) 0408.

[23] Hawking S. and Ellis G., *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge, 1973.

[24] Horava P., *Quantum Gravity at a Lifshitz Point*, Phys. Rev. D **79** (2009) 084008.

[25] Horava P., *Membranes at Quantum Criticality*, JHEP **0903** (2009) 020.

[26] Leon G. and Saridakis E., *Phase-Space Analysis of Horava-Lifshitz Cosmology*, JCAP **0911** (2009) 006.

[27] Lidsey J. and Mulryne D., *A Graceful Entrance to Braneworld Inflation,* Phys. Rev. D **73** (2006) 083508.

[28] Lidsey J., Mulryne D., Nunes N. and Tavakol R., *Oscillatory Universes in Loop Quantum Cosmology and Initial Conditions for Inflation,* Phys. Rev. D **70** (2004) 063521.

[29] Lu H., Mei J. and Pope C., *Solutions to Horava Gravity*, Phys. Rev. Lett. **103** (2009) 091301.

[30] Minamitsuji M., *Classification of Cosmology with Arbitrary Matter in the Hořava-Lifshitz Theory*, Phys. Lett. B **684** (2010) 194-198.

[31] Mukherjee S., Paul B., Dadhich N., Maharaj S. and Beesham A., *Emergent Universe with Exotic Matter*, Class. Quant. Grav. **23** (2006) 6927-6933.

[32] Mulryne D., Tavakol R., Lidsey J. and Ellis G., *An Emergent Universe From a Loop*, Phys. Rev. D **71** (2005) 123512.

[33] Parisi L., *Dynamical Systems Techniques in Cosmology. An Example: LQC and the Einstein Static Universe*, J. Geom. Symm. Phys. **14** (2009) 67-83, (see also Geometry, Integrability and Quantization X, Avangard Press, Sofia 2009, pp 211-226).

[34] Parisi L., Bruni M., Maartens R. and Vandersloot K., *The Einstein Static Universe in Loop Quantum Cosmology*, Class. Quant. Grav. **24** (2007) 6243-6254.

[35] Park M., *The Black Hole and Cosmological Solutions in IR Modified Horava Gravity*, JHEP **0909** (2009) 123.

[36] Seahra S. and Böhmer C., *Einstein Static Universes are Unstable in Generic f(R) Models*, Phys. Rev. D **79** (2009) 064009.

[37] Seahra S., Clarkson C. and Maartens R., *Delocalization of Brane Gravity by a Bulk Black Hole*, Class. Quant. Grav. **22** (2005) L91-L102.

[38] Son E. and Kim W., *Smooth Cosmological Phase Transition in the Horava-Lifshitz Gravity,* arXiv:1003.3055 [gr-qc].

[39] Vandersloot K., *Loop Quantum Cosmology and the k = -1 RW Model*, Phys. Rev. D **75** (2007) 023523.

[40] Wang A. and Wu Y., *Thermodynamics and Classification of Cosmological Models in the Horava-Lifshitz Theory of Gravity,* JCAP **0907** (2009) 012.

[41] Wu P. and Yu H., *Emergent Universe from the Hovava-Lifshitz Gravity*, Phys. Rev. D **81** (2010) 103522.

[42] Zhang K., Wu P. and Yu H., *The Stability of Einstein Static Universe in the DGP Braneworld*, Phys. Lett. B **690** (2010) 229-232.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 305–319

CONSTANT MEAN CURVATURE SURFACES AT THE INTERSECTION OF INTEGRABLE GEOMETRIES

ÁUREA QUINTINO

Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa
1649-003 Lisboa, Portugal

Abstract. The constant mean curvature surfaces in three-dimensional space-forms are examples of isothermic constrained Willmore surfaces, characterized as the constrained Willmore surfaces in three-space admitting a conserved quantity. Both constrained Willmore spectral deformation and constrained Willmore Bäcklund transformation preserve the existence of a conserved quantity. The class of constant mean curvature surfaces in three-dimensional space-forms lies, in this way, at the intersection of several integrable geometries, with classical transformations of its own, as well as constrained Willmore transformations and transformations as a class of isothermic surfaces. Constrained Willmore transformation is expected to be unifying to this rich transformation theory.

1. Introduction

Minimal surfaces appear as the area-minimizing surfaces amongst all those spanning a given boundary. The Euler-Lagrange equation of the underlying variational problem turns out to be the zero mean curvature equation. A physical model of a minimal surface can be obtained by dipping a wire frame into a soap solution. The resulting soap film is minimal, in the sense that it always tries to organize itself so that its surface area is as small as possible whilst spanning the wire contour. This minimal surface area is reached for the flat position, which is also the position in which the membrane is the most relaxed, i.e., where the elastic energy is minimal - these surfaces are elastic energy extremals and, in this way, examples of Willmore surfaces. In fact, a classical result by Thomsen [23] characterizes isothermic Willmore surfaces in three-space as minimal surfaces in some three-dimensional space-form.

Unlike flat soap films, soap bubbles do not extremize the elastic energy - they exist under a certain surface tension, in an equilibrium where slightly greater pressure inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are examples of area-minimizing surfaces under the constraint of volume enclosed - these are surfaces of (non-zero) constant mean curvature and, therefore, examples of constrained Willmore surfaces (which are not Willmore surfaces), elastic energy extremals with respect to infinitesimally conformal variations (rather than with respect to all variations). Indeed, as established by Richter [18], **constant mean curvature** (CMC) surfaces in three-dimensional space-forms are, in particular, isothermic constrained Willmore surfaces.

In [16], a spectral deformation and a Bäcklund transformation of constrained Willmore surfaces are defined and a permutability between the two is established. It is shown that all these transformations corresponding to the zero multiplier preserve the class of Willmore surfaces. The class of CMC surfaces in three-dimensional space-forms is characterized as the class of constrained Willmore surfaces in three-space admitting a conserved quantity. It is shown that, for special choices of parameters, both spectral deformation and Bäcklund transformation preserve the class of constrained Willmore surfaces admitting a conserved quantity, and, in particular, the class of CMC surfaces in three-dimensional space-forms.

The class of constant mean curvature surfaces in three-space lies, in this way, at the intersection of several integrable geometries, with constrained Willmore spectral deformation and Bäcklund transformations, an isothermic spectral deformation (classically defined by Bianchi [2] and, independently, Calapso [10]), as well as a classical spectral deformation of its own (the Bonnet transformation [4]), and, in the Euclidean case, isothermic Darboux transformations (classically discovered by Darboux [12]) or, equivalently [15], Bianchi-Bäcklund transformation [1]. The isothermic spectral deformation is known to preserve the constancy of the mean curvature of a surface in some space-form, cf. [9]. In [16], it is shown that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation. These spectral deformations of CMC surfaces in three-dimensional space-forms are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation. In [14] it is shown that, for special choices of parameters, the Darboux transformation of isothermic surfaces in Euclidean three-space preserves the constancy of the mean curvature in $\mathbb{R}^3$, as well as the mean curvature itself. Isothermic Darboux transformation of a CMC surface in Euclidean three-space is expected to be obtained as a particular case of constrained Willmore Bäcklund transformation. Constrained Willmore transformation is in this way expected to be unifying to this rich transformation theory.

Our theory is local and, throughout this text, with no need for further reference, restriction to a suitable non-empty open set shall be underlying.

2. Constrained Willmore Surfaces

In modern literature on the elasticity of membranes, a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature is considered the elastic energy of a membrane. By neglecting the total mean curvature (by physical considerations) and having in consideration that the total Gaussian curvature of compact orientable Riemannian surfaces without boundary is a topological invariant, Willmore [25] defined the **Willmore (elastic) energy** of a compact oriented Riemannian surface, without boundary, isometrically immersed in $\mathbb{R}^3$, to be

$$\mathcal{W} = \int H^2 \mathrm{d}A$$

i.e., the total squared mean curvature. The Willmore functional "extends" (for more details, see [16]) to isometric immersions of compact oriented Riemannian surfaces in Riemannian manifolds by means of half of the total squared norm of Π^0, the trace-free part of the second fundamental form, which, in fact, amongst surfaces in $\mathbb{R}^3$, differs from $\mathcal{W}$ by the total Gaussian curvature, but still shares then the critical points with $\mathcal{W}$. And so does

$$\mathcal{W} = \int_M |\Pi^0|^2 \mathrm{d}A$$

which is what we consider as the **Willmore energy functional**.

By definition the **Willmore surfaces** are the extremals of the Willmore energy. The class of **constrained Willmore** (CW) **surfaces** appears as the generalization of the class of Willmore surfaces that arises when we consider extremals of the Willmore functional with respect to infinitesimally conformal variations - those satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(X^{1,0}, X^{1,0})_t = 0$$

fixing $X^{1,0}$ a $(1,0)$-vector field - rather than with respect to all variations (Note that conformal variations are characterized by $(X^{1,0}, X^{1,0})_t = 0$, fixing $X^{1,0}$ a $(1,0)$-vector field). Under a conformal change of the metric, the squared norm of the trace-free part of the second fundamental form and the area element change in an inverse way, leaving the Willmore energy unchanged. In particular, this establishes the class of (constrained) Willmore surfaces as a Möbius invariant class.

Our study is one of (constrained) Willmore surfaces in n-dimensional space-forms[1] with $n \geq 3$, which, in view of the conformal invariance mentioned above, we

[1] Throughout this text, we will, alternatively, use n-space to refer to n-dimensional space-form (Euclidean, spherical or hyperbolic).

approach as immersions

$$\Lambda : (M, \mathcal{C}_\Lambda) \to \mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$$

of an oriented compact[2] surface M into the conformal n-sphere, which we model on the projective space of the light-cone $\mathcal{L} \subset \mathbb{R}^{n+1,1}$, following Darboux [11], (for a modern account, see [5]) providing M with the conformal structure $\mathcal{C}_\Lambda$ induced from the one on $\mathbb{P}(\mathcal{L})$ (and with the canonical complex structure).

A fundamental construction in conformal geometry of surfaces is the mean curvature sphere congruence, the bundle of two-spheres tangent to the surface and sharing with it the mean curvature vector at each point (although the mean curvature vector is not conformally invariant, under a conformal change of the metric it changes in the same way for the surface and the osculating two-sphere). Let

$$S : M \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$$

be the mean curvature sphere congruence of Λ (the k-spheres of $\mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$ are exactly the manifolds $\mathbb{P}(\mathcal{L} \cap V)$ with V a $(k+1,1)$-plane of $\mathbb{R}^{n+1,1}$ (see [5]).). We have a decomposition $\underline{\mathbb{R}}^{n+1,1} = S \oplus S^\perp$ and then a decomposition of the trivial flat connection d on $\underline{\mathbb{R}}^{n+1,1}$ as

$$\mathrm{d} = \mathcal{D} \oplus \mathcal{N}$$

for $\mathcal{D} = \nabla^S + \nabla^{S^\perp}$, with ∇^S and $\nabla^{S^\perp}$ the connections induced on S and $S^\perp$, respectively, by d. Set

$$\Lambda^{1,0} := \Lambda \oplus \mathrm{d}\sigma(T^{1,0}M), \qquad \Lambda^{0,1} := \Lambda \oplus \mathrm{d}\sigma(T^{0,1}M)$$

two subbundles of $S^{\mathbb{C}}$, defined independently of the choice of $\sigma \in \Gamma(\Lambda)$ never-zero, and then $\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}$.

In generalization of what is presented in [7] for the particular case of $n = 4$, we have (see [16])

$$\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (\mathrm{d}S \wedge *\mathrm{d}S)$$

a manifestly conformally invariant formulation of the Willmore energy. This formulation makes it clear that

$$\mathcal{W}(\Lambda) = E(S)$$

the Willmore energy of Λ coincides with the Dirichlet energy of S (with respect to any of the metrics in the conformal class on M). N. Ejiri [13] and independently, M. Rigoli [19] proved, furthermore, that

$$\Lambda \text{ Willmore} \Leftrightarrow S \text{ harmonic}$$

so that Λ is a Willmore surface if and only if $S : (M, \mathcal{C}_\Lambda) \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ is a harmonic map. According to Uhlenbeck [24], it follows that Λ is a Willmore

[2] A natural extension to surfaces that are not necessarily compact will take place at some point below.

surface if and only if $\mathrm{d}^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1}$ is a flat connection, for all λ in $\mathbb{S}^1$. More generally (for more details, see [16]) we have

Theorem 1 ([6]). *Λ is a constrained Willmore surface if and only if there exists a real one-form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that the connection*

$$\mathrm{d}^\lambda_q := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1} \tag{1}$$

*is flat, for all $\lambda \in \mathbb{S}^1$. In that case, q is said to be a multiplier to Λ and Λ is said to be a q-*CW *surface.*

In Theorem 1 and throughout this text, we consider the identification

$$\wedge^2\mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$$

of the exterior power $\wedge^2\mathbb{R}^{n+1,1}$ with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$ via

$$\omega \mapsto v_1 \wedge v_2(\omega) := (v_1, \omega)v_2 - (v_2, \omega)v_1$$

for $v_1, v_2, \omega \in \mathbb{R}^{n+1,1}$.

The characterization of constrained Willmore surfaces in space-forms presented in Theorem 1 provides a natural extension of the concept to surfaces that are not necessarily compact.

Willmore surfaces are the 0-CW surfaces. The zero multiplier is not necessarily the only multiplier to a CW surface with no constraint on the conformal structure, though. In fact, the uniqueness of multiplier characterizes non-isothermic constrained Willmore surfaces

Proposition 1 ([16]). A constrained Willmore surface has a unique multiplier if and only if it is not an isothermic surface.

A classical result by Thomsen [23] characterizes isothermic Willmore surfaces in three-space as minimal surfaces in some three-dimensional space-form. Constant mean curvature surfaces in three-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by Richter [18]. However, isothermic constrained Willmore surfaces in three-space are not necessarily CMC surfaces in some space-form, as established by an example due to Burstall [3], of a constrained Willmore cylinder that does not have constant mean curvature in any space-form.

For later reference, it is convenient to denote, alternatively, d^λ_q by $\mathrm{d}^{\lambda,q}_S$ and to use $\hat{\mathrm{d}}^{\lambda,q}_V$ for the analogue defined for a general non-degenerate subbundle V of $(\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}} = \underline{\mathbb{C}}^{n+2}$, provided with the complex bilinear extension of the metric on $\underline{\mathbb{R}}^{n+1,1}$, a general one-form q with values in $\wedge^2\underline{\mathbb{C}}^{n+2}$ and $\hat{\mathrm{d}}$ a general flat metric connection on $\underline{\mathbb{C}}^{n+2}$. The characterization of *q-constrained harmonicity* of the bundle S consisting of the flatness of $\mathrm{d}^{\lambda,q}_S$, for all λ in $\mathbb{S}^1$, extends naturally to a notion respecting a general non-degenerate subbundle V of $\underline{\mathbb{C}}^{n+2}$ and a general $q \in \Omega^1(\wedge^2\underline{\mathbb{C}}^{n+2})$, by means of the flatness of the connection $\mathrm{d}_V^{\lambda,q}$, for all $\lambda \in \mathbb{S}^1$. For more details, see [16].

3. Transformations of Constrained Willmore Surfaces

Constrained Willmore surfaces in space-forms form a Möbius invariant class of surfaces with strong links to the theory of integrable systems, admitting, amongst others, a spectral deformation, defined by the action of a loop of flat metric connections, and Bäcklund transformations, defined by applying a dressing action [16]. All these transformations are closely related and all those corresponding to the zero multiplier preserve the class of Willmore surfaces.

3.1. Spectral Deformation

For each λ in $\mathbb{S}^1$, the flatness of the metric connection d_q^λ establishes the existence of an isometry

$$\phi_q^\lambda : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$$

of bundles, defined on a simply connected component of M, preserving connections, unique up to a Möbius transformation. We use an interpretation of loop group theory by Burstall and Calderbank and define a spectral deformation of Λ which is supposed to be a $q-$CW surface into new constrained Willmore surfaces by setting, for each λ in $\mathbb{S}^1$

$$\Lambda_q^\lambda := \phi_q^\lambda \Lambda.$$

This comes as an immediate consequence of the fact that $(\mathrm{d}_q^\lambda)_{q_\lambda}^\mu = \mathrm{d}_q^{\lambda\mu}$, for $q_\lambda = \lambda^{-2}q^{1,0} + \lambda^2 q^{0,1}$ and for all $\lambda, \mu \in \mathbb{S}^1$, which readily establishes Λ_q^λ as a $\mathrm{Ad}_{\phi_q^\lambda}(q_\lambda)$-CW surface. In particular, spectral deformation corresponding to the zero multiplier preserves the class of Willmore surfaces. For each λ, we refer to Λ_q^λ as the transformation of Λ defined (in the ambit of Möbius geometry) by the flat metric connection d_q^λ.

3.2. Bäcklund Transformation

We use a version of the dressing action theory of Terng-Uhlenbeck [22] to define a transformation of Λ into new constrained Willmore surfaces. We start by defining a transformation on the level of constrained harmonic bundles. For that, we give conditions on a **dressing** $r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ such that the gauging

$$\hat{\mathrm{d}}_S^{\lambda,\tilde{q}} := r(\lambda) \circ \mathrm{d}_S^{\lambda,q} \circ r(\lambda)^{-1}$$

of d^λ_q by $r(\lambda)$, for each λ, establishes the constrained harmonicity of some bundle $\hat{S}$ from the constrained harmonicity of S, as follows. Define $\tilde{q} \in \Omega^1(\wedge^2\underline{\mathbb{C}}^{n+2})$ by setting

$$\tilde{q}^{1,0} := \mathrm{Ad}_{r(0)}q^{1,0}, \qquad \tilde{q}^{0,1} := \mathrm{Ad}_{r(\infty)}q^{0,1}.$$

Set, furthermore

$$\hat{q} = \mathrm{Ad}_{r(1)^{-1}}\tilde{q}$$

and

$$\hat{S} = r(1)^{-1}S.$$

Lemma 1 ([16]). *Let $\rho \in \Gamma(\underline{\mathbb{C}}^{n+2})$ be reflection across S, $\rho = \pi_S - \pi_{S^\perp}$, for π_S and $\pi_{S^\perp}$ the orthogonal projections of $\underline{\mathbb{C}}^{n+2}$ onto $S^{\mathbb{C}}$ and $(S^\perp)^{\mathbb{C}}$, respectively. Suppose $r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ is such that*

i) *$\lambda \mapsto r(\lambda)$ is holomorphic and invertible at $\lambda = 0$ and $\lambda = \infty$*
ii) *$\rho r(\lambda)\rho^{-1} = r(-\lambda)$, for all λ*
iii) *$\lambda \mapsto \hat{\mathrm{d}}^{\lambda,\tilde{q}}_S$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$.*

Then, for $\hat{\mathrm{d}} := \hat{\mathrm{d}}^{1,\tilde{q}}_S$, the notation $\hat{\mathrm{d}}^{\lambda,\tilde{q}}_S$ is not merely formal, that is, the connection denoted by $\hat{\mathrm{d}}^{\lambda,\tilde{q}}_S$ is of the form (1).

Suppose that 1 is in the domain of r. In that case, and under the hypotheses of Lemma 1, it follows immediately, in view of the specific form of $\hat{\mathrm{d}}^{\lambda,\tilde{q}}_S$, that

$$r(1)^{-1} \circ \hat{\mathrm{d}}^{\lambda,\tilde{q}}_S \circ r(1) = \mathrm{d}^{\lambda,\hat{q}}_{\hat{S}}$$

which establishes the $\hat{q}$-constrained harmonicity of $\hat{S}$ from the q-constrained harmonicity of S.

Now set

$$\hat{\Lambda}^{1,0} := r(1)^{-1}r(\infty)\Lambda^{1,0}, \qquad \hat{\Lambda}^{0,1} := r(1)^{-1}r(0)\Lambda^{0,1}$$

and

$$\hat{\Lambda} := \hat{\Lambda}^{1,0} \cap \hat{\Lambda}^{0,1}.$$

The isotropy of both $\Lambda^{1,0}$ and $\Lambda^{0,1}$ establishes that for both $\hat{\Lambda}^{1,0}$ and $\hat{\Lambda}^{0,1}$ and therefore, the nullity of the bundle $\hat{\Lambda}$. On the other hand, an extra condition on r, namely, $\det r(0)|_S = \det r(\infty)|_S$, establishes $\hat{\Lambda}$ as a line bundle. Actually, condition ii) in Lemma 1 establishes, in particular, that $r(0)|_S, r(\infty)|_S \in \Gamma(O(S))$. One verifies, furthermore, that, if $\hat{\Lambda} \subset (\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$ is a real bundle, then $\hat{S}$ is the mean curvature sphere congruence of the surface $\hat{\Lambda}$ and, ultimately, that if $\hat{q}$ is real, then $\hat{\Lambda}$ is a $\hat{q}$-CW surface.

Following the philosophy of Terng-Uhlenbeck [22], we then construct $r = r(\lambda)$ satisfying the conditions above, as well as establishing the reality of $\hat{\Lambda}$ and $\hat{q}$ from

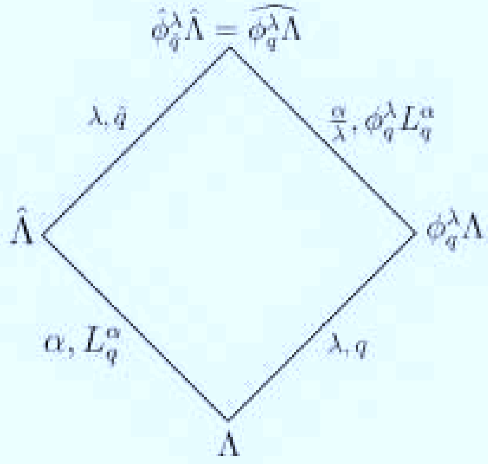

Figure 1. A permutability of spectral deformation and Bäcklund transformation of constrained Willmore surfaces.

the reality of Λ and q, respectively. We consider a two-step process of transformations of the type

$$r^{(-)}_{\alpha,L}(\lambda) := (-)\frac{\alpha-\lambda}{\alpha+\lambda}\,\pi_L + \pi_{(L\oplus\rho L)^\perp} + (-)\,\frac{\alpha+\lambda}{\alpha-\lambda}\,\pi_{\rho L}$$

(respectively), parametrized by $\alpha \in \mathbb{C}\backslash\mathbb{S}^1$ non-zero and $L = L^\alpha_q \subset \underline{\mathbb{C}}^{n+2}$ a $\mathrm{d}^{\alpha,q}_S$-parallel null line bundle such that $\rho L \cap L^\perp = \backslash\{0\}$. Namely, we consider

$$r = r_{\alpha,L^\alpha} r^-_{\beta,L^\beta}$$

for $\beta = \overline{\alpha}^{-1}$, $L^\beta = \overline{L}$ and $L^\alpha = r^-_{\beta,L^\beta}(\alpha)L$, with α and L as above. We refer to $\hat{\Lambda}$ as the Bäcklund transform of Λ of parameters α, L^α_q. Note that Bäcklund transformation corresponding to the zero multiplier preserves the class of Willmore surfaces.

3.3. Spectral Deformation vs Bäcklund Transformation

Spectral deformation and Bäcklund transformation of constrained Willmore surfaces permute, as follows

Theorem 2 ([16]). *Let α, L^α_q be Bäcklund transformation parameters to Λ corresponding to the multiplier q, let λ be in $\mathbb{S}^1$ and $\phi^\lambda_q : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}^{\lambda,q}_S) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ and $\hat{\phi}^\lambda_{\hat{q}} : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}^{\lambda,\hat{q}}_{\hat{S}}) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ be isometries preserving connections. Then the Bäcklund transform of parameters $\frac{\alpha}{\lambda}$, $\phi^\lambda_q L^\alpha_q$ of the spectral deformation $\phi^\lambda_q\Lambda$ of Λ, of parameter λ, corresponding to the multiplier q, coincides with the spectral deformation of parameter λ corresponding to the multiplier $\hat{q}$ of the Bäcklund transform of parameters α, L^α_q of Λ, i.e., the diagram in* Fig.1 *commutes.*

4. Conserved Quantities under CW Transformation

Suppose Λ is a q-CW surface. Let $p(\lambda) = \lambda^{-1}v + v_0 + \lambda\bar{v}$ be a Laurent polynomial with $v_0 \in \Gamma(S^{\mathbb{C}})$ real, $v \in \Gamma((S^{\mathbb{C}})^{\perp})$ and $v_\infty := p(1) \neq 0$. We say that $p(\lambda)$ is a **q-conserved quantity** of Λ if $\mathrm{d}_q^{\lambda} p(\lambda) = 0$, for all $\lambda \in \mathbb{C}\backslash\{0\}$ and then following the idea by Burstall and Calderbank we have

Lemma 2 ([16]). *$p(\lambda)$ is a q-conserved quantity of Λ if and only if*

$$\mathrm{d}v_\infty = 0, \quad \mathcal{D}^{0,1}v = 0, \qquad \mathcal{N}^{1,0}v + q^{1,0}v_0 = 0.$$

The characterization above, of a q-conserved quantity $p(\lambda)$ of Λ, shows, in particular, that $p(\lambda)$ determines q (for details, see [16]). There is then no ambiguity on referring to $p(\lambda)$ simply as a conserved quantity of Λ.

For special choices of parameters, both spectral deformation and Bäcklund transformation of constrained Willmore surfaces preserve the existence of a conserved quantity, as follows

Theorem 3 ([16]). *Let μ be in $\mathbb{S}^1$ and $\phi_q^\mu : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_S^{\mu,q}) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ be an isometry preserving connections. Suppose that either v_0 is non-zero or $\overline{\mu}v + \mu\overline{v}$ is non-zero. In that case, if $p(\lambda)$ is a q-conserved quantity of Λ, then $\phi_q^\mu p(\mu\lambda)$ is a $\mathrm{Ad}_{\phi_q^\mu}(q_\mu)$-conserved quantity of the spectral deformation $\phi_q^\mu \Lambda$ generated by the parameter μ of Λ.*

We have also

Theorem 4 ([16]). *Suppose $p(\lambda)$ is a q-conserved quantity of Λ. Let α, L_q^α be Bäcklund transformation parameters to Λ corresponding to the multiplier q and let r be the corresponding dressing. If $p(\alpha) \perp L_q^\alpha$, then*

$$\hat{p}(\lambda) := r(1)^{-1}\, r(\lambda)\, p(\lambda)$$

is a $\hat{q}$-conserved quantity of the Bäcklund transform $\hat{\Lambda}$ of Λ of parameters α, L_q^α.

5. Example: Constant Mean Curvature Surfaces in Three-dimensional Space-forms

The class of constant mean curvature surfaces in three-dimensional space-forms is characterized as the class of constrained Willmore surfaces in three-space admitting a conserved quantity. It follows that, for special choices of parameters, both spectral deformation and Bäcklund transformation of constrained Willmore surfaces preserve the class of CMC surfaces in three-dimensional space-forms. The class of CMC surfaces in three-dimensional space-forms lies, in this way, at the intersection of several integrable geometries, with classical transformations of its own, as well as constrained Willmore transformations and transformations as a class of isothermic surfaces. Constrained Willmore transformation is expected to be unifying to this rich transformation theory.

In contrast to constrained Willmore surfaces, constant mean curvature surfaces are not conformally-invariant objects, which requires carrying a distinguished space-form. Following [5] we start by realizing all space-forms as submanifolds of the light-cone, given $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero

$$S_{v_\infty} := \{v \in \mathcal{L}\,;\,(v, v_\infty) = -1\}$$

inherits from $\mathbb{R}^{n+1,1}$ a positive definite metric of (constant) sectional curvature $-(v_\infty, v_\infty)$. For each v_∞, the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defines a diffeomorphism

$$\pi_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L} \cap \langle v_\infty\rangle^\perp).$$

Let us consider the particular case $n = 3$. Let T and $\perp$ denote the orthogonal projections of $\underline{\mathbb{R}}^{4,1}$ onto S and $S^\perp$, respectively. Suppose the surface Λ is not contained in any two-sphere. This condition ensures (see [16]) that, given $v_\infty \in \mathbb{R}^{4,1}$ non-zero, Λ is (locally) a surface in $\mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L} \cap \langle v_\infty\rangle^\perp) \cong S_{v_\infty}$

$$\Lambda \cong (\pi_{S_{v_\infty}})^{-1} \circ \Lambda : M \to S_{v_\infty}$$

with mean curvature given, up to sign, by

$$H_\infty = (v_\infty^\perp, v_\infty^\perp)^{\frac{1}{2}}.$$

In particular, Λ is a minimal surface in the space-form S_{v_∞} (i.e., $H_\infty = 0$) if and only if $v_\infty \in \Gamma(S)$.

5.1. CMC Surfaces and Conserved Quantities

According to Lemma 2, the existence of a conserved quantity $p(\lambda)$ of Λ establishes, in particular, the constancy of $v_\infty := p(1)$. Furthermore we have

Theorem 5 ([16]). *If Λ is a* CW *surface and $p(\lambda)$ is a conserved quantity of Λ, then Λ has constant mean curvature in the space-form S_{v_∞}, for $v_\infty = p(1)$. Reciprocally, if H_∞ is constant, for some non-zero $v_\infty \in \mathbb{R}^{4,1}$, then*

$$p_\infty(\lambda) := \lambda^{-1}\frac{1}{2}v_\infty^\perp + v_\infty^T + \lambda\frac{1}{2}v_\infty^\perp$$

is a conserved quantity of the CW *surface Λ. Constant mean curvature surfaces in three-dimensional space-forms are the constrained Willmore surfaces in three-space admitting a conserved quantity.*

Next we establish a conserved quantity with respect to a general multiplier to a surface with constant mean curvature in some three-space. The conclusion that these surfaces allow CW spectral deformation and CW Bäcklund transformation into new ones will then follow from Theorems 3 and 4.

As suggested by Proposition 1, the characterization of the set of multipliers to a constrained Willmore surface is closely related to the isothermic condition. Isothermic surfaces are classically defined by the existence of conformal curvature line cooordinates, i.e., conformal coordinates with respect to which the second fundamental form is diagonal. This is a conformally-invariant condition, although the second fundamental form is not conformally-invariant, and it can be reformulated in a manifestly invariant way, as follows (This formulation is also discussed in [6] and [21].)

Theorem 6 ([8]). *Λ is isothermic if and only if there is a non-zero real closed form $\eta \in \Omega^1(\Lambda \wedge \Lambda^1)$. In that case, we say that (Λ, η) is isothermic.*

In the conditions of Theorem 6, the form η is unique up to non-zero constant real scale, cf. [21].

Following Proposition 1, we have, furthermore

Proposition 2 ([16]). Suppose (Λ, η) is an isothermic q-CW surface. Then the set of multipliers to Λ is the one-dimensional affine space $q + \langle * \eta \rangle_{\mathbb{R}}$.

Fix $v_\infty \in \mathbb{R}^{4,1}$ non-zero. Suppose Λ has constant mean curvature in S_{v_∞}. Define $N \in \Gamma(S^\perp)$ unit by setting $v_\infty^\perp = H_\infty N$ (in the particular case Λ is minimal in S_{v_∞}, N is defined only up to sign). Write σ_∞ for $(\pi_{S_{v_\infty}})^{-1} \circ \Lambda$. Set $\eta_\infty := \frac{1}{2}\sigma_\infty \wedge \mathrm{d}N$, a form derived by F. Burstall and D. Calderbank which establishes Λ as an isothermic surface and for which scaling by the mean curvature in S_{v_∞} setting

$$q_\infty := H_\infty \eta_\infty$$

provides a multiplier to Λ (see [16])

Proposition 3. (Λ, η_∞) is an isothermic q_∞-CW surface.

Proposition 3 makes it clear, in particular, that minimal surfaces in three-dimensional space-forms are examples of Willmore surfaces.

For each $t \in \mathbb{R}$, set

$$q_\infty^t := q_\infty + t * \eta_\infty .$$

Proposition 4 ([17]). For each $t \in \mathbb{R}$

$$p_\infty^t(\lambda) := \lambda^{-1}\frac{1}{2}(H_\infty - \mathrm{i}t)N + v_\infty^T + \lambda\frac{1}{2}(H_\infty + \mathrm{i}t)N$$

is a q_∞^t-conserved quantity of Λ.

5.2. CMC Surfaces at the Intersection of Integrable Geometries

The results in Section 5.1 combine to establish the following

Theorem 7 ([17]). *The class of* CMC *surfaces in three-dimensional space-forms is preserved by both* CW *spectral deformation and* CW *Bäcklund transformation, for special choices of parameters, with preservation of both the space-form and the mean curvature in the latter case.*

Fix $v_\infty \in \mathbb{R}^{4,1}$ non-zero and suppose Λ has constant mean curvature in S_{v_∞}. The fact that a Bäcklund transform of Λ still is a surface of constant mean curvature H_∞ in S_{v_∞}, as stated above, is not immediate from Theorem 4. In contrast, it is immediate from Theorem 3 that, given λ in $\mathbb{S}^1$ and $\phi^\lambda_{t,\infty} : (\underline{\mathbb{R}}^{4,1}, \mathrm{d}^\lambda_{q^t_\infty}) \to (\underline{\mathbb{R}}^{4,1}, \mathrm{d})$ an isometry preserving connections, the spectral deformation $\phi^\lambda_{t,\infty}\Lambda$ of Λ, of parameter λ, corresponding to the multiplier q^t_∞ has constant mean curvature

$$H^\lambda_{t,\infty} = \mid \mathrm{Re}\,(\lambda H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))\mid$$

in the space-form $S_{v^\lambda_{t,\infty}}$ for

$$v^\lambda_{t,\infty} := \phi^\lambda_{t,\infty}(v^T_\infty + ((\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))N).$$

Zero curvature representation provides a context in which Bonnet transformation [4] of CMC surfaces in $\mathbb{R}^3$ can be generalized to CMC surfaces in general three-space, as follows (*the classical* CMC *spectral deformation*). For each $\lambda \in \mathbb{S}^1$, set

$$\mathrm{d}^\lambda_\infty := \mathcal{D} + \lambda\mathcal{N}^{1,0} + \lambda^{-1}\mathcal{N}^{0,1} + 2(\lambda - 1)q^{1,0}_\infty + 2(\lambda^{-1} - 1)q^{0,1}_\infty.$$

Theorem 8 ([16]). *The connection* $\mathrm{d}^\lambda_\infty$ *is flat, for all* $\lambda \in \mathbb{S}^1$. *Besides, if for each* $\lambda \in \mathbb{S}^1$, $\phi^\lambda_\infty : (\underline{\mathbb{R}}^{4,1}, \mathrm{d}^\lambda_\infty) \to (\underline{\mathbb{R}}^{4,1}, \mathrm{d})$ *is an isometry preserving connections, then*

i) $v^\lambda_\infty := \phi^\lambda_\infty v_\infty$ *is a non-zero constant section of* $\underline{\mathbb{R}}^{4,1}$

ii) *the transformation* $\Lambda^\lambda_\infty := \phi^\lambda_\infty\Lambda$ *of* Λ, *defined by the flat metric connection* d^λ_∞, *has constant mean curvature* H_∞ *in the space-form* $S_{v^\lambda_\infty}$

iii) *the family* $\phi^\lambda_\infty\sigma_\infty$, *with* $\lambda \in \mathbb{S}^1$, *is a family of isometrical deformations of* σ_∞ *in a fixed space-form, preserving the mean curvature.*

In [8], the spectral deformation of isothermic surfaces in $\mathbb{R}^3$ (or, equivalently, in general three-space) classically discovered by Bianchi [2] and, independently, Calapso [10] is generalized to n-space, for general n, by means of zero curvature representation, as follows (*the isothermic spectral deformation*). Let η be a non-zero real one-form with values in $\Lambda \wedge \Lambda^{(1)}$. For each $t \in \mathbb{R}$, set

$$\mathrm{d}^\eta_t := \mathrm{d} + t\eta.$$

Theorem 9 ([8]). *(Λ, η) is isothermic if and only if d_t^η is a flat connection, for each $t \in \mathbb{R}$. In that case, the transformation Λ_t^η of Λ defined by the flat metric connection d_t^η is still isothermic, for each $t \in \mathbb{R}$.*

The isothermic spectral deformation is known [9] to preserve the constancy of the mean curvature in some three-dimensional space-form, defining then a transformation of CMC surfaces into new ones. In fact [16], given $t \in \mathbb{R}$ and $\phi_t^\infty : (\mathbb{R}^{4,1}, \mathrm{d}_t^{\eta_\infty}) \to (\mathbb{R}^{4,1}, \mathrm{d})$ an isometry preserving connections, the deformation $\phi_t^\infty \Lambda$ of Λ has constant mean curvature H_t^∞ in the space-form $S_{v_t^\infty}$, for

$$v_t^\infty := \phi_t^\infty (v_\infty + \frac{t}{2} N)$$

with

$$(H_t^\infty)^2 = (H_\infty + \frac{t}{2})^2.$$

Proposition 5 ([17]). The classical CMC spectral deformation of parameter other than -1 can be obtained as constrained Willmore spectral deformation

$$\mathrm{d}_\infty^\lambda = \mathrm{d}_{q_\infty^{t_\lambda}}^{\lambda^{-1}}$$

for $\lambda \neq -1$ in $\mathbb{S}^1$ and

$$t_\lambda := \mathrm{i} H_\infty \frac{1-\lambda}{1+\lambda} \in \mathbb{R}.$$

Furthermore: for all $\lambda \in \mathbb{S}^1$

$$\mathrm{d}_\infty^{\lambda^{-1}} = \mathrm{d}_{q_\infty}^\lambda + 2H_\infty (1 - \operatorname{Re} \lambda)\, \eta_\infty^\lambda$$

for $\eta_\infty^\lambda = \lambda^{-1} \eta_\infty^{1,0} + \lambda \eta_\infty^{0,1}$. Hence the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation and, in the particular case of a minimal surface, the classical CMC spectral deformation coincides, up to reparametrization, with constrained Willmore spectral deformation corresponding to the zero multiplier.

These spectral deformations of CMC surfaces in three-dimensional space-forms are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation, cf. Theorem 2.

CMC surfaces in Euclidean three-space enjoy, furthermore, Darboux transformation as isothermic surfaces or, equivalently [15], Bianchi-Bäcklund transformation, as discussed in [20] (cf. [1]). In fact, in [14], it is shown that, for special choices of parameters, the transformation of isothermic surfaces in $\mathbb{R}^3$ classically discovered by Darboux [12] preserves the constancy of the mean curvature in $\mathbb{R}^3$, as well as the mean curvature itself. This is also the case for constrained Willmore Bäcklund transformation, cf. Theorem 7. In [14], a description of Darboux transformation of constant mean curvature surfaces in Euclidean three-space is presented in the quaternionic setting. It is based on the solution of a Riccati equation and it displays a striking similarity with the Darboux transformation of constrained Willmore surfaces in four-space defined in [16]. Non-trivial Darboux transformation of constrained Willmore surfaces can be obtained as a particular case of constrained Willmore Bäcklund transformation, as established in [16]. We believe that isothermic Darboux transformation of a CMC surface in Euclidean three-space can be obtained as a particular case of constrained Willmore Bäcklund transformation.

Acknowledgements

This work was partially supported by Fundação para a Ciência e a Tecnologia, Portugal, e Fundação da Universidade de Lisboa, with the Grant with reference CMAF-BPD-01/09.

References

[1] Bianchi L., *Vorlesungen über Differentialgeometrie*, (Anhang zu Kapitel XVII: Zur Transformationstheorie der Flächen mit constantem positiven Krümmungsmass) Teubner, Leipzig 1899, pp 641-648.

[2] Bianchi L., *Ricerche Sulle Superficie Isoterme e Sulla Deformazione delle Quadriche*, Annali di Matematica **11** (1905) 93-157.

[3] Bohle C., Peters G. and Pinkall U., *Constrained Willmore Surfaces*, Calc. Var. Partial Diff. Eqs. **32** (2008) 263-277.

[4] Bonnet O., *Mémoire sur la Théorie des Surfaces Applicables*, Journal de l'École Polytechnique **42** (1867) 72-92.

[5] Burstall F., *Isothermic Surfaces: Conformal Geometry, Clifford Algebras and Integrable Systems*, In: Integrable Systems, Geometry and Topology, C.-L. Terng (Ed), International Press Studies in Mathematics vol. **36** (2006) 1-82.

[6] Burstall F. and Calderbank D., *Conformal Submanifold Geometry I-III*, arXiv:1006.5700v1 (2010).

[7] Burstall F., Ferus D., Leschke K., Pedit F. and Pinkall U., *Conformal Geometry of Surfaces in* $\mathbb{S}^4$ *and Quaternions*, LNM vol. 1772, Springer, Heidelberg, 2002.

[8] Burstall F., Donaldson N., Pedit F. and Pinkall U., *Isothermic Submanifolds of Symmetric R-Spaces*, arXiv:0906.1692v2 (2002).

[9] Burstall F., Pedit F. and Pinkall U., *Schwarzian Derivatives and Flows of Surfaces*, Contemp. Math. **308** (2002) 39-61.

[10] Calapso P., *Sulle Superficie a Linee di Curvatura Isoterme*, Rendiconti Circolo Matematico di Palermo **17** (1903) 275-286.

[11] Darboux J., *Leçons sur la Théorie Générale des Surfaces et les Applications Géometriques du Calcul Infinitésimal, Parts 1 and 2*, Gauthier-Villars, Paris, 1887.

[12] Darboux J., *Sur les surfaces isothermiques*, C. R. Acad. Sci. Paris **128** (1899) 1299-1305.

[13] Ejiri N., *Willmore Surfaces with a Duality in* $S^n(1)$, Proc. Lond. Math. Soc. **57** (1988) 383-416.

[14] Hertrich-Jeromin U. and Pedit F., *Remarks on Darboux Transforms of Isothermic Surfaces*, Documenta Mathematica **2** (1997) 313-333.

[15] Kobayashi S. and Inoguchi J., *Characterizations of Bianchi-Bäcklund Transformations of Constant Mean Curvature Surfaces*, Int. J. Math. **16** (2005) 101-110.

[16] Quintino A., *Constrained Willmore Surfaces: Symmetries of a Möbius Invariant Integrable System*, PhD Thesis, University of Bath, 2008.

[17] Quintino A., *Constrained Willmore Surfaces: Symmetries of a Möbius Invariant Integrable System* - Based on the author's PhD Thesis, arXiv:0912.5402v1 (2009).

[18] Richter J., *Conformal Maps of a Riemann Surface into the Space of Quaternions*, PhD thesis, Technische Universität Berlin, 1997.

[19] Rigoli M., *The Conformal Gauss Map of Submanifolds of the Möbius Space*, Annals of Global Analysis and Geometry **5** (1987) 97-116.

[20] Sterling I. and Wente H., *Existence and Classification of Constant Mean Curvature Multibubbletons of Finite and Infinite Type*, Indiana Univ. Math. J. **42** (1993) 1239-1266.

[21] Santos S., *Special Isothermic Surfaces*, PhD thesis, University of Bath, 2008.

[22] Terng C.-L. and Uhlenbeck K., *Bäcklund Transformations and Loop Group Actions*, C.P.A.M. **53** (2000) 1-75.

[23] Thomsen G., *Über konforme Geometrie I, Grundlagen der Konformen Flaechentheorie*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **3** (1923) 31-56.

[24] Uhlenbeck K., *Harmonic Maps into Lie Groups (Classical Solutons of the Chiral Model)*, J. Diff. Geom. **30** (1989) 1-50.

[25] Willmore T., *Note on Embedded Surfaces*, Analele Stiintifice ale Universitatii "Alexandru Ioan Cuza" Iasi, Sect. Ia (N.S.) **11** (1965) 493-496.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 320–328

MONODROMY AND THE BOHR-SOMMERFELD GEOMETRIC QUANTIZATION*

NICOLA SANSONETTO

Department of Computer Science, Università di Verona, 37134 Verona, Italy

Abstract. We study the linear part of the monodromy of completely integrable Hamiltonian systems via Bohr-Sommerfeld Geometric Quantization. We relate monodromy to the ambiguity in the choice of the pre-quantum connection and to the action of the (connected component of the) gauge group.

1. Introduction

In the framework of Bohr-Sommerfeld geometric quantization, we study (quantum) monodromy from different viewpoints. Monodromy, together with the so-called Chern-Duistermaat class and the Lagrangian class, provides an obstruction to the global definition of action-angle variables for completely integrable Hamiltonian systems [7, 9]. Our specific contributions relate monodromy to the freedom of choice of a pre-quantum connection and to $\mathcal{G}_0$-equivalence of connections ($\mathcal{G}_0$ is the connected component of the identity of the gauge group $\mathcal{G}$ of a pre-quantum line bundle).

The present work is organized as follows. In Section 2 we first review Liouville-Arnold theorem and the obstructions to existence of global action-angles coordinates and then we quickly review the geometric quantization method. In Section 3 we state and prove the main results of the paper. A short section with conclusions and perspectives follows.

2. Liouville-Arnold Theorem and Geometric Quantization

In this section we review some basic facts about completely integrable Hamiltonian systems and geometric quantization. We will also introduce the notation that will be used throughout the paper.

---

*Reprinted from J. Geom. Symm. Phys. **20** (2010) 97–106.

2.1. Completely Integrable Hamiltonian Systems

Let (M, ω) be a $2n$-dimensional symplectic manifold, and fix $h : M \longrightarrow \mathbb{R}$, a smooth function on M (the Hamiltonian), with its associated vector field X_h, fulfilling $i_{X_h} \omega = -\mathrm{d}h$. The triple (M, ω, h) is called a **Hamiltonian system** on M, and it is said to be **completely integrable** in a subset $\widetilde{M}$ of M if it admits n mutually Poisson-commuting first integrals, which are functionally independent almost everywhere in M, and, restricting the latter, if necessary, the joint level sets of the first integrals are compact and connected. The Liouville-Arnold Theorem (see e.g. [1, 7]) gives sufficient conditions for the complete integrability of a Hamiltonian system.

Theorem 1 (Liouville-Arnold). *Let (M, ω) be a $2n$-dimensional symplectic manifold. Let $f = (f_1, \ldots, f_n) : M \longrightarrow \mathbb{R}^n$*

- *be a surjective submersion (i.e., the* energy-momentum mapping)
- *have compact and connected fibers $f^{-1}(x)$*
- *its components pairwise Poisson-commute, i.e., $\{f_i, f_j\} = 0$ for every $i, j = 1, \ldots, n$.*

Let A be the set of regular values of f. Then for each $x \in A$

1. *the fibers $f^{-1}(x)$ of f are diffeomorphic to $\mathbb{T}^n$*
2. *there exists an open neighborhood U_x of x in A and a diffeomorphism $(\boldsymbol{a}, \boldsymbol{\alpha}) : f^{-1}(U_x) \longrightarrow V \times \mathbb{T}^n$ with V an open subset of $\mathbb{R}^n$ such that $\boldsymbol{a} = (a_1, \cdots, a_n) = \kappa \circ f$ for some diffeomorphism $\kappa : f(U_x) \longrightarrow V$*
3. *the coordinates $(\boldsymbol{a}, \boldsymbol{\alpha})$ on M are Darboux coordinates, the so-called action-angles coordinates, that is $\omega = \mathrm{d}\boldsymbol{a} \wedge \mathrm{d}\boldsymbol{\alpha}$.*

From the geometric point of view the Liouville-Arnold Theorem ensures that M has a $\mathbb{T}^n$-bundle structure with Lagrangian fibers and moreover, $f^{-1}(A)$ is a local toric principal bundle with structure group $\mathbb{T}^n$ with Lagrangian fibers, whose structure group $\mathbb{T}^n$ acts in a Hamiltonian way, with momentum map given by the projection bundle map. Besides the action-angles coordinates are bundle coordinates. We want to stress that the construction of the toric principal bundle or, equivalently, the existence of global action-angle coordinates is only local. Liouville-Arnold Theorem also implies that the base manifold A of the $\mathbb{T}^n$-bundle is an integral affine manifold. Since also the fibers of the bundle carry an affine structure, the transition functions between two intersecting domain of action-angle coordinates $(\boldsymbol{a}, \boldsymbol{\alpha})$ and $(\boldsymbol{a}', \boldsymbol{\alpha}')$ are

$$\boldsymbol{a}' = Z^{-T}\,\boldsymbol{a} + \boldsymbol{z}, \qquad \boldsymbol{\alpha}' = Z\,\boldsymbol{\alpha} + \mathcal{F}(\boldsymbol{a}) \qquad \bmod 2\pi \tag{1}$$

where $Z \in \mathrm{SL}_n(\mathbb{Z})$, $\boldsymbol{z} \in \mathbb{R}^n$ and $\mathcal{F} : a(\pi^{-1}(V) \cap \pi^{-1}(V')) \longrightarrow \mathbb{R}^n$ such that $\partial_{a_i}(Z\mathcal{F})_j = \partial_{a_j}(Z\mathcal{F})_i$ with $i, j = 1, \ldots, n$. From (1) clearly follows the non-uniqueness of the action-angle coordinates which is due to the three arbitrary choices that must be made in the proof of Liouville-Arnold Theorem - first, the choice of a basis of the period lattice, second a choice of a local Lagrangian section (that is an origin to count the angles) and finally a constant of the integration in the derivation of the actions. This freedom affects the globalization of the construction of Liouville-Arnold Theorem. A first answer was given in terms of cocycles by Nekhoroshev [9] in 1976 and exhaustively and independently by Duistermaat [7] in 1980, in terms of the sheaf theory.

Theorem 2 ([7]). *With the notation of the previous theorem, the torus bundle $\pi : f^{-1}(A) \longrightarrow \mathbb{R}^n$ is topologically trivial if and only if the monodromy and the Chern-Duistermaat class of the $\mathbb{T}^n$-bundle are trivial. Moreover if the symplectic form is exact then the existence of global action-angle coordinates is equivalent to the triviality of the Lagrangian toric fibration.*

Remarks 1.
- The Chern-Duistermaat class is the Chern class of the bundle and it describes the obstruction to the existence of a global section of the bundle (we referee to [2, 12] for a detailed discussion on the Chern class in the case of completely integrable Hamiltonian systems). In local terms it means that even if the action variables are globally defined, the function $\mathcal{F}$ in (1) is not.
- In the case of a system with two degrees of freedom possessing an isolated critical value (of focus-focus type) of the energy-momentum map f, the Chern-Duistermaat class is trivial since the base manifold A admits a Leray cover with empty triple intersections. Therefore the only obstruction to the triviality of the fibration is the monodromy.

The coarsest and most known obstruction to the global existence of the action-angles coordinates is the monodromy (actually its linear part), which is the one we are interested in this paper. From the geometric point of view the monodromy is the obstruction to the global "principality" of the toric bundle. Equivalently, taken a point x in A and a basis for the first homology group of the fiber $\pi^{-1}(x)$ over x, if we carry it over loops in A, when we arrive again back at x, we have a map that describes the change of the basis of the first homology of the fibers and it depends only on the homotopy type of A. Therefore the monodromy is given by the representation (Duistermaat's idea)

$$\mathcal{M} : \pi_1(A) \longrightarrow \mathrm{Aut}(\mathrm{H}_1(\mathbb{T}^n, \mathbb{Z})) \cong \mathrm{SL}_n(\mathbb{Z}). \tag{2}$$

From a local point of view, (Nekhoroshev's idea) the monodromy is the obstruction to patch together charts on the base manifold A around the singularities. Indeed the product of $r > 1$ matrices in $\mathrm{SL}_n(\mathbb{Z})$ need not to be the identity. Another characterization of monodromy has been suggested by Weinstein [7], but see [2] for a detailed discussion in terms of the holonomy of a suitable flat connection. Since every Lagrangian fibration admits an affine, flat and torsion free connection induced by the standard connection of $\mathbb{R}^n$ (see [13]), it turns out that the holonomy of this connection is the monodromy of the bundle. Let us discuss in more detail this aspect since it will be crucial for the sequel. It will be convenient for us to study Hamiltonian monodromy from a differential geometric point of view (see [5, 7, 13]). Indeed, it is well-known (see [13]) that a Lagrangian fibration admits an affine, flat, torsion free connection $\nabla^{\mathrm{Ehr}} : TM \longrightarrow VM$ (the vertical bundle over M) on the Lagrangian leaves, which is an Ehresmann good connection for the fibration (i.e., that is every smooth curve on the base has a horizontal lift). The $\mathrm{GL}(n, \mathbb{Z})$-holonomy representation $\mathrm{hol}(\nabla^{\mathrm{Ehr}})$ of ∇^{Ehr} is the monodromy representation $\mathcal{M}(\pi_1(A))$ of the $\mathbb{T}^n$-bundle over A. Moreover the monodromy representation actually takes values in $\mathrm{SL}(n, \mathbb{Z})$ upon choosing suitable bases of the tangent spaces of the base space.

Remarks 2.

- In [8] (and independently in [15]) is given a sufficient condition for the non-triviality of monodromy near isolated focus-focus singularities: more precisely, the (local) monodromy near a topologically stable focus-focus point (in the interior of the energy-momentum range) is non-trivial.
- There are various examples of completely integrable Hamiltonian systems that present monodromy: the spherical pendulum [5,7], the champagne bottle [2], the Lagrange top (see [5] and reference therein).
- Upon quantization of a completely integrable Hamiltonian system, one has a natural notion of quantum monodromy μ_q which is equal to $(\mu_c)^{-T}$, where μ_c denotes the classical monodromy. (See [10] for a rigorous introduction to quantum monodromy).

2.2. Geometric Quantization

Let us now briefly review the basics of geometric quantization. For a complete account we refer to [3, 14]. Recall that if (M, ω) is a real symplectic manifold of even dimension such that $\left[\frac{1}{2\pi}\omega\right] \in \mathrm{H}^2(M, \mathbb{Z})$, then the Weil-Kostant Theorem states that there exists a complex line bundle (L, ∇, h) over M equipped with a hermitian metric h and a compatible connection ∇ with curvature $F_\nabla = \omega$. Hence $[\omega] = c_1(L)$, the first Chern class of $L \to M$. The connection ∇ is called a **pre-quantum connection** and $L \to M$ the **pre-quantum line bundle**. The different choices of $L \to M$ and ∇ are parametrized by the first cohomology group $\mathrm{H}^1(M, \mathbb{S}^1)$ (see e.g. [14]). In more detail given any complex line bundle $L \to M$, the connections thereon are classified, up to gauge equivalence, by their **curvature** (fixing the topological type of the line bundle, via the first Chern class) and by their **holonomy**, specified, in turn, on a basis of (real) homology one-cycles $[\gamma_i]$ for $\mathrm{H}_1(M,\mathbb{R})$, of dimension b_1, the first Betti number of M - represented, for instance, by smooth curves passing through a given point. The holonomy is trivial if M is simply connected. The gauge group $\mathcal{G}$ consists, in this case, of all smooth maps $g : M \to \mathbb{S}^1$ - explicitly, $g : x \mapsto \mathrm{e}^{\mathrm{i}\varphi(x)}$, obvious notation - and it is not connected in general, its connected components being parametrized by the degree of the maps $g : M \to \mathbb{S}^1$. The connected component (of the identity) of $\mathcal{G}$ will be denoted by $\mathcal{G}_0$, as usual, and will play an important role in what follows.

Given a connection ∇_0, any other connection is of the form $\nabla = \nabla_0 + a$, with $a \in \Lambda^1(M)$, (i.e., they build up an affine space modelled on the space of one-forms $\Lambda^1(M)$) and the relation between their respective curvatures is $F_\nabla = F_{\nabla_0} + \mathrm{d}\eta$ Therefore, the curvatures are the same if and only if η is closed. This being the case, a determines a de Rham cohomology class $[a] \in \mathrm{H}^1(M,\mathbb{R})$, fully recovered via the **period map**

$$\mathrm{H}^1(M,\mathbb{R}) \ni [a] \mapsto \left(\int_{\gamma_1} a \,, \ldots , \int_{\gamma_{b_1}} a \right) \in \mathbb{R}^n. \tag{3}$$

The **gauge group** $\mathcal{G}$ acts on connections via $\nabla \mapsto \nabla + g \cdot \mathrm{d}g^{-1} = \nabla - \mathrm{i}\,\mathrm{d}\varphi$. Therefore, the set of all gauge inequivalent connections (possessing the same curvature) is clearly given by $\mathrm{H}^1(M,\mathbb{R})/\mathrm{H}^1(M,\mathbb{Z})$ and, if M is a *torus*, then the above set is again a torus, the **Jacobian** of M. If the initial connection has zero curvature, then the above space parametrises **flat connections** up to gauge equivalence.

Coming back to the specific geometric quantization setting, given a Lagrangian submanifold Λ of the symplectic manifold M, the symplectic two-form ω vanishes upon restriction to Λ by definition, and any (semi-local) symplectic potential θ becomes a closed form thereon, defining a (semi-local) connection form pertaining to the restriction of the pre-quantum connection ∇, denoted by the same symbol. The latter is a flat connection and a global covariantly constant section of the restriction of the pre-quantum line bundle exists if and only if it has trivial holonomy, that is the **Bohr-Sommerfeld condition** is fulfilled

$$\left[\frac{1}{2\pi}\theta\right] \in \mathrm{H}^1(M,\mathbb{Z}) \qquad \text{i.e.,} \qquad \int_\gamma \theta \in 2\pi\mathbb{Z}$$

for any closed loop γ in Λ.

A covariantly constant section (which we call WKB-, or BS-wave function) takes the form

$$s(m) := \mathrm{hol}_\gamma(\nabla) \cdot s(m_0) = \mathrm{e}^{\mathrm{i}\int_\gamma \theta}\, s(m_0) \tag{4}$$

with γ denoting any path connecting a chosen point m_0 in Λ with a generic point $m \in \Lambda$, $\mathrm{hol}_\gamma(\nabla)$ being the holonomy along γ of the restriction to Λ of the pre-quantum connection ∇. The r.h.s. of (4) tacitly assumes the choice of a trivialization of $L\mid_\Lambda \longrightarrow \Lambda$ around m_0 and m in a corresponding local chart.

Remarks 3.
- We stress the fact that the Bohr-Sommerfeld condition forces us to deal with $\mathcal{G}_0$-equivalence classes (i.e., the degree of the gauge maps must be zero) in order to avoid trivialities.
- Our definition of WKB-wave function is slightly different from the conventional one (see e.g. [14]). Indeed we do not require square-integrability and we do not twist the prequantization bundle with Δ_∇ (whose smooth sections consist of the complex n-forms on Λ), thus neglecting the "amplitude-squared".
- There is a version of the Bohr-Sommerfeld condition incorporating the Maslov class, but we shall not need this refinement in what follows.

We also recall that the pre-quantum connection ∇ allows the construction of the (Hermitian) pre-quantum observables $Q(\cdot)$ via the formula

$$Q(f) = -\mathrm{i}\nabla_{X_f} + f = -\mathrm{i}\,(X_f - i_{X_f}\theta) + f$$

with f a smooth function on M. The connection is determined up to a closed one-form, yielding a corresponding ambiguity in the definition of the quantum observable $Q(f)$ attached to f. This fact will be exploited in the sequel.

3. Monodromy via Bohr-Sommerfeld Geometric Quantization

In this section we will detect (quantum) monodromy via Bohr-Sommerfeld geometric quantization and analyse how the monodromy itself affects geometric quantization as well in different ways.

Let us consider the geometric quantization of a completely integrable Hamiltonian system on a symplectic manifold of dimension $2n$ with vanishing Chern class and with vanishing affine monodromy, i.e. the vector z and the function $\mathcal{F}$ in (1) must vanish. (This two assumptions are not necessary for the result of our work but will simplify the exposition and improve the clarity of the results).

3.1. The Pre-quantum Connection

In this section we show that monodromy can be detected exploiting the freedom in the choice of the pre-quantum connection.

Let us perform geometric quantization in a neighborhood U of a Lagrangian torus $\mathbb{T}^n$. Let L denote the pre-quantum line bundle and ∇ the pre-quantum connection. Moreover let θ be the (local) connection form.

Theorem 3 ([11]). A. *The monodromy is the holonomy of the so-called* BS-*adapted connection induced by the Liouville one-form.*

B. *The monodromy is the holonomy of the so-called monodromy connection induced by the vertical one-form, which in coordinates reads $\theta' = -\boldsymbol{\alpha}\,\mathrm{d}\boldsymbol{a}$.*

Proof: A. Consider the standard connection ∇ given by the Liouville one-form $\theta = \boldsymbol{a}\,\mathrm{d}\boldsymbol{\alpha}$. Since this connection is vertical we call it a **BS-adapted connection**. It fulfills $\nabla_{X_b} = X_b$ with X_a any vector field tangent to a Lagrangian section. Moreover it is obviously flat along the fibers, since the restriction of the symplectic form on the fibers vanishes, being the fibers Lagrangian submanifolds. Given a BS-adapted connection, the action variables may be recover as follow $a_k = \frac{1}{2\pi i} \log \mathrm{hol}(\nabla|_{\mathbb{T}^n}, \gamma_k)$ where γ_k's yield a basis of one-cycles in $\mathbb{T}^n$, thus making clear local definition of the actions. Hence, monodromy may be view as the obstruction to patch together geometric quantization bundles equipped with a local BS-adapted connection. Note, however, that there is non global obstruction to prequantization by the Weil-Kostant Theorem.

B. Consider now the connection ∇' defined by the form $\theta' = -\boldsymbol{\alpha}\,\mathrm{d}\boldsymbol{a}$. We call this connection a *monodromy connection*, since parallel transport along a non trivial loop contained in a local Lagrangian section $\alpha = c$, whereupon it is flat, produces a holonomy $\mathrm{hol}(\nabla') = \mathrm{e}^{-\mathrm{i}\Delta \boldsymbol{a}}$ due to the possible non-globality of the action variables.

□

3.2. The Gauge Approach

In this section show how to detect monodromy using a gauge-equivalence theoretic interpretation of the pre-quantum connections.

Theorem 4 ([11]). 1. *The monodromy representation* (2) *can be viewed as a map $\widetilde{M} : \pi_1(A) \longrightarrow \mathcal{G}/\mathcal{G}_0 \cong \mathrm{SL}_n(\mathbb{Z})$, which acts transitively on $\mathcal{BS}$, as expression* (5) *below shows, and can be read both on wave functions and observables.*

2. *Take a* BS-*adapted connection and perform a change of coordinates according with* (1)*, then, remaining in the same Hilbert space, monodromy eventually induces a change in the quantum action operator.*

Proof: 1. Upon enforcing Bohr-Sommerfeld condition take the integral de Rham class of $\nabla := \nabla|_{\mathbb{T}^n}$. i.e., $[\theta]$ via the period map (3), and denote by $\mathcal{BS}$ the set of all classes $[\theta_\nabla]$.Then

$$\mathcal{BS} \cong \mathrm{H}^1(\mathbb{T}^n, \mathbb{Z}) = \mathcal{G} \cdot [\nabla_0] \tag{5}$$

with ∇_0 a fixed flat connection. Thus $\mathcal{BS}$ is a $\mathcal{G}$-homogeneous space isomorphic to $\mathbb{Z}^n$, whereupon the connected component $\mathcal{G}_0$ of the gauge group $\mathcal{G}$ acts trivially. Hence $\mathcal{G}/\mathcal{G}_0 \cong \mathrm{SL}_n(\mathbb{Z})$ acts freely on $\mathcal{BS}$ and provides a natural (quantum) monodromy representation: $\widetilde{\mathcal{M}} : \pi_1(A) \longrightarrow \mathcal{G}/\mathcal{G}_0 \cong \mathrm{SL}_n(\mathbb{Z}^n)$.

2. Let us perform a Darboux change of coordinates on a fixed BS-torus according to (1). Then extend the change of coordinates to a canonical transformation in a neighborhood of the fixed torus: $\boldsymbol{a}' = Z^{-T}\boldsymbol{a}$ and $\boldsymbol{\alpha}' = Z\boldsymbol{\alpha}$ with $Z \in \mathrm{SL}_n(\mathbb{Z})$. Then the Hamiltonian vector fields $X_{a_k} = \partial_{\alpha_k}$ for $k = 1, \ldots, n$ of the action variables change consistently $\partial_{\alpha_k} = Z^{-T}\partial_{\alpha_k}$. The quantum operator associated with the action a_k is $\hat{a}_k = \mathrm{i}X_{a_k} = -\mathrm{i}\partial_{a_k}$, as can be easily checked $\hat{a}_k = -\mathrm{i}(X_{a_k} - i_{X_{a_k}}\theta) + a_k = -\mathrm{i}X_{a_k}$, since $i_{\partial_{a_k}} \sum_{j=1}^{n} a_j \,\mathrm{d}\alpha_j = a_k$. Therefore after the change of coordinates the quantum operator takes the form $\boldsymbol{a}' = -\mathrm{i}Z^{-T}\partial_{\boldsymbol{\alpha}}$. Then, using Nekhoroshev's idea, if we glue together $r > 1$ charts surrounding an isolated singularity of focus-focus type we obtain a non-trivial product $Z = \Pi_{\nu=1}^{r} Z_\nu$ of matrices in $\mathrm{SL}_n(\mathbb{Z})$. This is equivalent to say that monodromy manifests itself via a non-trivial $\mathrm{SL}_n(\mathbb{Z})$-representation $[\gamma] \longmapsto Z = Z([\gamma])$ of the fundamental group $\pi_1(A)$ of the base manifold A.

□

Remark 1. We stress the fact that in spectroscopy, monodromy manifests itself precisely through a shift of the energy levels, see e.g. [4] and references therein.

4. Conclusions and Perspectives

We have reviewed some general methods to compute the (quantum) monodromy of completely integrable Hamiltonian systems using the geometric quantization procedure. In particular, recovering Weinstein idea, we have detected the monodromy via a choice of the pre-quantum connection and using Nekhoroshev original idea performing a parallel transport along a nontrivial loop around a singularity.

As future work it would be of interest to try to extend our results to fractional monodromy and to study all Duistermaat singularities in the framework of geometric quantization.

Acknowledgements

I wish to thank to Professor M. Spera for many enlightening discussions on these topics, F. Fassò for many discussion on monodromy and L. Bates for having stimulated me to work on this topic.

References

[1] Arnold V. and Givental A., *Symplectic Geometry. Dynamical Systems IV*, Encyclopaedia Math. Sci. vol. 4, Springer, Berlin 2001, pp 1–138.

[2] Bates L., *Monodromy in the Champagne Bottle*, ZAMP **42** (1991) 837–847.

[3] Brylinski J.-L., *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser, Basel, 1993.

[4] Child M., *Quantum States in Champagne Bottle*, J. Phys. A **31** (1998) 657–670.

[5] Cushman R. and Bates L., *Global Aspects of Classical Integrable Systems*, Birkhäuser, Basel, 1997.

[6] Dazord P. and Delzant T., *Le Problème Général des Variables Actions-Angles*, J. Diff. Geom. **26** (1987) 223–251.

[7] Duistermaat J., *On Global Action-Angle Coordinates*, Comm. Pure Appl. Math. **33** (1980) 687–706.

[8] Matveev V., *Integrable Hamiltonian Systems with Two Degrees of Freedom. Topological Structure of Saturated Neighborhoods of Saddle-Saddle and Focus Points*, Mat. Sb. **187** (1996) 29–58.

[9] Nekhoroshev N., *Action-Angle Variables, and their Generalizations* (in Russian), Trudy Moskov. Mat. Obšč. **26** (1972) 181–198.

[10] Vũ Ngoc S., *Quantum Monodromy in Integrable Systems*, Commun. Math. Phys. **203** (1999) 465–479.

[11] Sansonetto N. and Spera M., *Hamiltonian Monodromy Via Geometric Quantization and Theta Functions*, J. Geom. Phys. **60** (2010) 501–512.

[12] Sepe D., *Topological Classification of Lagrangian Fibrations*, J. Geom. Phys. **60** (2010) 341–351.

[13] Weinstein A., *Symplectic Manifolds and their Lagrangian Submanifolds*, Adv. Math. **16** (1971) 329–346.

[14] Woodhouse N., *Geometric Quantization*, Clarendon Press, Oxford, 1992.

[15] Zung N., *A Note on Focus-Focus Singularities*, Diff. Geom. Appl. **7** (1997) 123–130.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 329–341

EINSTEIN METRICS WITH TWO-DIMENSIONAL KILLING LEAVES AND THEIR APPLICATIONS IN PHYSICS

GAETANO VILASI

Dipartimento di Fisica "E. R. Caianiello", Università degli Studi di Salerno
INFN, Sezione di Napoli, GC Salerno, 84084 Fisciano (Salerno), Italy

Abstract. Solutions of vacuum Einstein's field equations, for the class of pseudo-Riemannian four-metrics admitting a non Abelian two dimensional Lie algebra of Killing fields, are explicitly described. When the distribution orthogonal to the orbits is completely integrable and the metric is not degenerate along the orbits, these solutions are parameterized either by solutions of a transcendental equation (the tortoise equation), or by solutions of a linear second order differential equation in two independent variables. Metrics, corresponding to solutions of the tortoise equation, are characterized as those that admit a three dimensional Lie algebra of Killing fields with two dimensional leaves. Metrics, corresponding to the case in which the commutator of the two Killing fields is isotropic, represent nonlinear gravitational waves.

1. Introduction

The aim of this paper is to illustrate some interesting and, in some sense, surprising physical properties of special solutions of Einstein field equations belonging to the larger class of Einstein metrics invariant for a non-Abelian Lie algebra of Killing vector fields generating a two dimensional distribution.

Some decades ago, by using a suitable generalization of the *Inverse Scattering Transform*, Belinsky and Sakharov [3] were able to determine four-dimensional Ricci-flat Lorentzian metrics invariant for an Abelian two dimensional Lie algebra of Killing vector fields such that the distribution $\mathcal{D}^{\perp}$ orthogonal to the one, say $\mathcal{D}$, generated by the Killing fields is transversal to $\mathcal{D}$ and Frobenius-integrable.

Thus, as a first step, it has been natural to consider [16] the problem of characterizing all gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields such that

I the distribution $\mathcal{D}$, generated by vector fields of $\mathcal{G}$, is two dimensional

II the distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

As we will see in Sections 4 and 5, the condition of transversality can be relaxed. This case, when the metric g restricted to any integral (two dimensional) submanifold (Killing leaf) of the distribution $\mathcal{D}$ is degenerate, splits naturally into two sub-cases according to whether the rank of g restricted to Killing leaves is 1 or 0. Sometimes, in order to distinguish various cases occurring in the sequel, the notation $(\mathcal{G}, r)$ will be used: metrics satisfying the conditions I and II will be called *of* $(\mathcal{G}, 2)$ *−type*, metrics satisfying conditions I and II, except the transversality condition, will be called *of* $(\mathcal{G}, 0)$ *−type* or *of* $(\mathcal{G}, 1)$ *−type* according to the rank of their restriction to Killing leaves.

According to whether the dimension of $\mathcal{G}$ is three or two, two qualitatively different cases can occur. Both of them, however, have in common the important feature that all manifolds satisfying the assumptions I and II are in a sense fibered over ζ-complex curves [18].

When $\dim \mathcal{G} = 3$, assumption II follows from I and the local structure of this class of Einstein metrics can be explicitly described. Some well known exact solutions, *e.g.* Schwarzschild, belong to this class.

A two dimensional $\mathcal{G}$, is either Abelian ($\mathcal{A}_2$) or non-Abelian ($\mathcal{G}_2$) and a metric g satisfying I and II, with $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$, will be called $\mathcal{G}$ *-integrable*. The study of $\mathcal{A}_2$-integrable Einstein metrics goes back to Einstein and Rosen [9]. Recent results can be found in [7].

The greater rigidity of $\mathcal{G}_2$-integrable metrics, for which some partial results can be found in [1, 8, 10], allows an exhaustive analysis. It will be shown that the ones of $(\mathcal{G}, 2)$ - type are parameterized by solutions of a linear second order differential equation on the plane which, in its turn, depends linearly on the choice of a ζ-*harmonic* function (see later). Thus, this class of solutions has a *bilinear structure* and, as such, admits two *superposition laws.*

All possible situations, corresponding to a two dimensional Lie algebras of isometries, are described in Table 1 where a non integrable two dimensional distribution which is part of a three dimensional integrable distribution has been called *semi-integrable* and in which the cases indicated with bold letters have been essentially solved [2, 7, 16–18].

In Section 1, four dimensional metrics of $(\mathcal{G}_2, 2)$ - *type* invariant for a non Abelian two dimensional Lie algebra are characterized from a geometric point of view. The solutions of corresponding Einstein field equations are explicitly written. The construction of global solutions is descrided in Section 2 and some examples are given in Section 3. Sections 4 and 5 are devoted to metrics of $(\mathcal{G}, 1)$ - *type* and of $(\mathcal{G}, 0)$ *−type* respectively. In Section 6 the case in which the commutator of

Table 1. Cases indicated with bold characters admit Ricci flat metrics, the remaining ones are under investigation.

	$\mathcal{D}^{\perp}$, $r = 0$	$\mathcal{D}^{\perp}$, $r = 1$	$\mathcal{D}^{\perp}$, $r = 2$
$\mathcal{G}_2$	**integrable**	**integrable**	**integrable**
$\mathcal{G}_2$	**semi-integrable**	**semi-integrable**	semi-integrable
$\mathcal{G}_2$	**non-integrable**	**non-integrable**	non-integrable
$\mathcal{A}_2$	**integrable**	**integrable**	integrable
$\mathcal{A}_2$	semi-integrable	semi-integrable	**semi-integrable**
$\mathcal{A}_2$	non-integrable	non-integrable	non-integrable

generators of the Lie algebra is of *light-type* is analyzed from a physical point of view. Harmonic coordinates are also introduced. Moreover, the wave-like character of the solutions is checked through the Zel'manov and the Pirani criteria. The canonical Landau-Lifchitz and the Bel energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. Realistic sources for such gravitational waves are also described. Eventually, the analysis of the polarization leads to the conclusion that these fields are spin-1 gravitational waves.

2. Metrics of $(\mathcal{G}_2, 2)$ - Type

In the following, we will consider four-dimensional manifolds and Greek letters take values from 1 to 4, the first Latin letters take values from 3 to 4, while i, j from 1 to 2. Moreover, $\mathcal{K}il\,(g)$ will denote the Lie algebra of all Killing fields of a metric g while *Killing algebra* will denote a sub-algebra of $\mathcal{K}il\,(g)$. Moreover, an integral (two dimensional) submanifold of $\mathcal{D}$ will be called a *Killing leaf*, and an integral (two dimensional) submanifold of $\mathcal{D}^{\perp}$ *orthogonal leaf*.

2.1. Geometric Aspects

- *Semiadapted coordinates.*
  Let g be a metric on the space-time $\mathcal{M}$ (a connected smooth manifold) and $\mathcal{G}_2$ one of its Killing algebras whose generators X, Y satisfy $[X, Y] = sY$, $s = 0, 1$
  The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is two-dimensional and in the neighborhood of a non singular point a chart (x^1, x^2, x^3, x^4) exists such that
  $$X = \frac{\partial}{\partial x^3}, \qquad Y = \exp\left(sx^3\right)\frac{\partial}{\partial x^4}\cdot$$
  From now on such a chart will be called *semiadapted* (to the Killing fields).

- *Invariant metrics*
  It can be easily verified [16, 17] that in a semiadapted chart g has the form

$$g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2\left(l_i + sm_i x^4\right)\mathrm{d}x^i\mathrm{d}x^3 - 2m_i\mathrm{d}x^i\mathrm{d}x^4$$
$$+\left(s^2\lambda\left(x^4\right)^2 - 2s\mu x^4 + \nu\right)\mathrm{d}x^3\mathrm{d}x^3 + 2\left(\mu - s\lambda x^4\right)\mathrm{d}x^3\mathrm{d}x^4$$
$$+\lambda\mathrm{d}x^4\mathrm{d}x^4, \qquad i = 1,2; j = 1,2$$

  with g_{ij}, m_i, l_i, λ, μ, ν arbitrary functions of (x^1, x^2).
- *Killing leaves.*
  Condition II allows to construct semi-adapted charts, with new coordinates (x, y, x^3, x^4), such that the fields $e_1 = \partial/\partial x$, $e_2 = \partial/\partial y$, belong to $\mathcal{D}^\perp$. In such a chart, called from now on *adapted*, the components l_i's and m_i's vanish.
  As it has already said, we will call **Killing leaf** an integral (two dimensional) submanifold of $\mathcal{D}$ and **orthogonal leaf** an integral (two dimensional) submanifold of $\mathcal{D}^\perp$. Since $\mathcal{D}^\perp$ is transversal to $\mathcal{D}$, the restriction of g to any Killing leaf, S, is non-degenerate. Thus, $(S, g|_S)$ is a homogeneous two dimensional Riemannian manifold. Then, the Gauss curvature $K(S)$ of the Killing leaves is constant (depending on the leave). In the appropriate chart $(p = x^3|_S, q = x^4|_S)$ one has

$$g|_S = \left(s^2\widetilde{\lambda}q^2 - 2s\widetilde{\mu}q + \widetilde{\nu}\right)\mathrm{d}p^2 + 2\left(\widetilde{\mu} - s\widetilde{\lambda}q\right)\mathrm{d}p\mathrm{d}q + \widetilde{\lambda}\mathrm{d}q^2$$

  where $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$, being the restrictions to S of λ, μ, ν, are constants, and

$$K(S) = \widetilde{\lambda}s^2\left(\widetilde{\mu}^2 - \widetilde{\lambda}\widetilde{\nu}\right)^{-1}.$$

2.2. Einstein Metrics When $g(Y, Y) \neq 0$

In the considered class of metrics, vacuum Einstein equations, $R_{\mu\nu} = 0$, can be completely solved [16]. If the Killing field Y is not of *light type*, *i.e.*, $g(Y, Y) \neq 0$, then in the adapted coordinates (x, y, p, q) the general solution is

$$g = f(\mathrm{d}x^2 \pm \mathrm{d}y^2) + \beta^2[(s^2k^2q^2 - 2slq + m)\mathrm{d}p^2 + 2(l - skq)\mathrm{d}p\mathrm{d}q + k\mathrm{d}q^2] \quad (1)$$

where $f = -\triangle_\pm \beta^2/2s^2k$, and $\beta(x, y)$ is a solution of the *tortoise equation*

$$\beta + A\ln|\beta - A| = u(x, y)$$

where A is a constant and the function u is a solution either of the Laplace or the d'Alembert equation, $\triangle_\pm u = 0$, $\triangle_\pm = \partial^2_{xx} \pm \partial^2_{yy}$, such that $(\partial_x u)^2 \pm (\partial_y u)^2 \neq 0$. The constants k, l, m are constrained by $km - l^2 = \mp 1, k \neq 0$ for Lorentzian metrics or by $km - l^2 = \pm 1, k \neq 0$ for Kleinian metrics.

Ricci flat manifolds of Kleinian signature possess a number of interesting geometrical properties and undoubtedly deserve attention in their own right. Some topological aspects of these manifolds were studied for the first time in [12], [13] and then in [11]. In recent years the geometry of these manifolds has seen a revival of interest. In part, this is due to the emergence of some new applications in physics.

2.2.1. Canonical Form of Metrics When $g(Y,Y) \neq 0$

The gauge freedom of the above solution, allowed by the function u, can be locally eliminated by introducing the coordinates (u,v,p,q), the function $v(x,y)$ being conjugate to $u(x,y)$, *i.e.* $\triangle_{\pm} v = 0$ and $u_x = v_y, u_y = \mp v_x$. In these coordinates the metric g takes the form

$$g = \frac{\exp\frac{u-\beta}{A}}{2s^2k\beta}(\mathrm{d}u^2 \pm \mathrm{d}v^2) + \beta^2[(s^2k^2q^2 - 2slq + m)\mathrm{d}p^2 + 2(l - skq)\mathrm{d}p\mathrm{d}q + k\mathrm{d}q^2]$$

with $\beta(u)$ a solution of $\beta + A\ln|\beta - A| = u$.

2.2.2. Normal Form of Metrics When $g(Y,Y) \neq 0$

In **geographic coordinates** (ϑ, φ) along Killing leaves one has

$$g|_S = \beta^2\left[\mathrm{d}\vartheta^2 + \mathcal{F}(\vartheta)\,\mathrm{d}\varphi^2\right]$$

where $\mathcal{F}(\vartheta)$ is equal either to $\sin h^2\vartheta$ or $-\cosh^2\vartheta$, depending on the signature of the metric. Thus, in the **normal coordinates**, $(r = 2s^2k\beta, \tau = v, \vartheta, \varphi)$, the metric takes the form (local "Birkhoff's theorem")

$$g = \varepsilon_1\left(\left[1 - \frac{A}{r}\right]\mathrm{d}\tau^2 \pm \left[1 - \frac{A}{r}\right]^{-1}\mathrm{d}r^2\right) + \varepsilon_2 r^2\left[\mathrm{d}\vartheta^2 + \mathcal{F}(\vartheta)\,\mathrm{d}\varphi^2\right] \qquad (2)$$

where $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$.

The geometric reason for this form is that, when $g(Y,Y) \neq 0$, a third Killing field exists which together with X and Y constitute a basis of $\mathfrak{s}\ (2,1)$. The larger symmetry implies that the geodesic equations describe a *non-commutatively integrable system* [15], and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves.

The above local form does not allow, however, to treat properly the singularities appearing inevitably in global solutions. The metrics (1), although they all are locally diffeomorphic to (2), play a relevant role in the construction of new global solutions as described in [17,18].

2.3. Einstein metrics when $g(Y,Y)=0$

If the Killing field Y is of **light type**, then the general solution of vacuum Einstein equations, in the adapted coordinates (x,y,p,q), is given by

$$g = 2f(\mathrm{d}x^2 \pm \mathrm{d}y^2) + \mu[(w\,(x,y) - 2sq)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}q] \tag{3}$$

where $\mu = A\Phi + B$ with $A, B \in \mathbb{R}$, Φ is a non constant harmonic function of x and y, $f = (\nabla\Phi)^2\sqrt{|\mu|}/\mu$, and $w\,(x,y)$ is solution of the **$\boldsymbol{\mu}$ - deformed Laplace equation**

$$\Delta_{\pm}w + (\partial_x \ln|\mu|)\,\partial_x w \pm (\partial_y \ln|\mu|)\,\partial_y w = 0$$

where Δ_+ (Δ_-) is the Laplace (respectively d'Alembert) operator in the $(x,y)-$ plane. Metrics (3) are Lorentzian if the orthogonal leaves are conformally Euclidean, *i.e.*, the positive sign is chosen, and Kleinian if not. Only the Lorentzian case will be analyzed and these metrics will be called of $(\mathcal{G}_2, 2)$ - **isotropic type**.

In the particular case $s=1$, $f=1/2$ and $\mu=1$, the above (Lorentzian) metrics are locally diffeomorphic to a subclass of the vacuum Peres solutions [14], that for later purpose we rewrite in the form

$$g = \mathrm{d}x^2 \pm \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + 2(\varphi_x\mathrm{d}x + \varphi_y\mathrm{d}y)\mathrm{d}u. \tag{4}$$

The correspondence between (3) and (4) depends on the special choice of the function $\varphi(x,y,u)$ (which, in general, is harmonic in x and y arbitrarily dependent on u); in our case

$$x \to x, \qquad y \to y, \qquad u \to u, \qquad v \to v + \varphi\,(x,y,u)$$

with $h = \varphi_u$.

In the case $\mu = \mathrm{const}$, the μ-deformed Laplace equation reduces to the Laplace equation. For $\mu = 1$, in the harmonic coordinates system (x,y,z,t) defined [4], for $|z-t| \neq 0$, by

$$\begin{aligned}
x &= x\\
y &= y\\
z &= \tfrac{1}{2}\left[(2q - w\,(x,y))\exp(-p) + \exp(p)\right]\\
t &= \tfrac{1}{2}\left[(2q - w\,(x,y))\exp(-p) - \exp(p)\right]
\end{aligned}$$

the Einstein metrics (3) take the particularly simple form

$$g = 2f(\mathrm{d}x^2 \pm \mathrm{d}y^2) + \mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}\,(w)\,\mathrm{d}\,(\ln|z-t|)\,. \tag{5}$$

This shows that, when w is constant, the Einstein metrics given by equation (5) are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when w is not constant, gravitational fields (5) look like a *disturbance* propagating at light velocity along the z direction on the Killing leaves (integral two dimensional submanifolds of $\mathcal{D}$). They represent, indeed, gravitational waves having the light as one of possible sources [4–6, 19].

3. Global Solutions

Here, we will give a coordinate-free description of previous local Ricci-flat metrics, so that it becomes clear what variety of different geometries, in fact, is obtained. We will see that with any of the found solutions a pair, consisting of a *ζ-complex curve* $\mathcal{W}$ and a *ζ-harmonic* function u on it, is associated. If two solutions are equivalent, then the corresponding pairs, say $(\mathcal{W}, u)$ and $(\mathcal{W}', u')$, are related by an invertible ζ-holomorphic map $\Phi : (\mathcal{W}, u) \longrightarrow (\mathcal{W}', u')$ such that $\Phi^* (u') = u$. Roughly speaking, the *moduli space* of the obtained geometries is surjectively mapped on the *moduli space* of the pairs $(\mathcal{W}, u)$.

Further parameters, distinguishing the metrics we are analyzing, are given below. Before that, however, it is worth to underline the following common peculiarities of these metrics

- they have, in the adapted coordinates, a block diagonal form whose upper block does not depend on the last two coordinates so that orthogonal leaves are totally geodesic.
- they possess a non trivial Killing field. Geodesic flows, corresponding to metrics, admitting three dimensional Killing algebras, are non-commutatively integrable. The existence of a non trivial Killing field is obvious from the description of model solution given in next section. For what concerns geodesic flows, they are integrated explicitly for model solution in next section, and the general result follows from the fact that any solution is a pull-back of a model one.

Solutions of the Einstein equations previously described manifest an interesting common feature. Namely, each of them is determined completely by a choice of

1) a solution of the wave, or the Laplace equation,

and either by

2′) a choice of the constant A and one of the branches, for β as function of u, of the tortoise equation

$$\beta + A \ln |\beta - A| = u \tag{6}$$

if $g(Y, Y) \neq 0$, or by

2″) a choice of a solution of one of the two equations

$$\left[\mu\left(\partial_y^2 - \partial_x^2\right) + \mu_y \partial_y - \mu_x \partial_x\right] w = 0, \qquad \Box\mu = 0 \tag{7}$$

$$\left[\mu\left(\partial_y^2 + \partial_x^2\right) + \mu_y \partial_y + \mu_x \partial_x\right] w = 0, \qquad \triangle\mu = 0 \tag{8}$$

in the case $g(Y, Y) = 0$.

They have a natural *fibered structure* with the Killing leaves as fibers. The wave and Laplace equations, mentioned above in 1), are in fact defined on the two dimensional manifold $\mathcal{W}$ which parameterizes the Killing leaves. These leaves themselves are two dimensional Riemannian manifolds and, as such, are geodesically complete.

For this reason the problem of the extension of decribed local solutions, is reduced to that of the extension of the base manifold $\mathcal{W}$. Such an extension should carry a geometrical structure that gives an intrinsic sense to the notion of the wave or the Laplace equation and to equations (7) and (8) on it. A brief description of how this can be done is the following.

3.1. $\zeta-$complex Structures

It is known there exist three different isomorphism classes of two dimensional commutative unitary algebras. They are

$$\mathbb{C} = \mathbb{R}[x] / \left(x^2+1\right), \qquad \mathbb{R}_{(2)} = \mathbb{R}[x] / \left(x^2\right), \qquad \mathbb{R}\oplus\mathbb{R} = \mathbb{R}[x] / \left(x^2-1\right)$$

Elements of this algebra can be represented in the form $a + \zeta b$, $a, b \in \mathbb{R}$, with $\zeta^2 = -1, 0$, or 1, respectively. For a terminological convenience we will call them **ζ- complex numbers**. Of course, ζ-complex numbers for $\zeta^2 = -1$ are just ordinary complex numbers. Furthermore, we will use the unifying notation $\mathbb{R}^2_\zeta$ for the algebra of ζ-complex numbers. For instance $\mathbb{C} = \mathbb{R}^2_\zeta$ for $\zeta^2 = -1$.

In full parallel with ordinary complex numbers, it is possible to develop a *ζ-complex analysis* by defining **ζ- holomorphic** functions as $\mathbb{R}^2_\zeta$-valued differentiable functions of the variable $z = x + \zeta y$. Just as in the case of ordinary complex numbers, the function $f(z) = u(x, y) + \zeta v(x, y)$ is ζ-holomorphic *iff* the ζ-Cauchy-Riemann conditions hold

$$u_x = v_y, \qquad u_y = \zeta^2 v_x. \tag{9}$$

The compatibility conditions of the above system requires that both u and v satisfy the **ζ- Laplace equation**, that is

$$-\zeta^2 u_{xx} + u_{yy} = 0, \qquad -\zeta^2 v_{xx} + v_{yy} = 0.$$

Of course, the ζ-Laplace equation reduces for $\zeta^2 = -1$ to the ordinary Laplace equation, while for $\zeta^2 = 1$ to the wave equation. The operator $-\zeta^2\partial_x^2 + \partial_y^2$ will be called the **ζ- Laplace operator**.

In the following a **ζ- complex structure** on $\mathcal{W}$ will denote an endomorphism $J : D(\mathcal{W}) \to D(\mathcal{W})$ of the $C^\infty(\mathcal{W})$ module $D(\mathcal{W})$ of all vector fields on $\mathcal{W}$, with $J^2 = \zeta^2 I$, $J \neq 0, I$, and vanishing Nijenhuis torsion, i.e., $[J, J]^{FN} = 0$, where

$[\cdot,\cdot]^{FN}$ denotes for the Frölicher-Nijenhuis bracket. A two dimensional manifold $\mathcal{W}$ supplied with a $\zeta-$complex structure is called a **ζ- complex curve**.

Obviously, for $\zeta^2 = -1$ a ζ-complex curve is just an ordinary one dimensional complex manifold (curve).

By using the endomorphism J the ζ-Laplace equation can be written intrinsically as

$$\mathrm{d}\left(J^*\mathrm{d}u\right) = 0$$

where $J^* : \Lambda^1(\mathcal{W}) \longrightarrow \Lambda^1(\mathcal{W})$ is the *adjoint to J endomorphism* of the $C^\infty(\mathcal{W})$ module of one forms on $\mathcal{W}$.

Given a two dimensional smooth manifold $\mathcal{W}$, an atlas $\{(U_i, \Phi_i)\}$ on $\mathcal{W}$ is called *ζ-complex iff*

i) $\Phi_i : U_i \longrightarrow \mathcal{W}, \quad U_i$ is open in $\mathbb{R}^2_\zeta$

ii) the transition functions $\Phi_j^{-1} \circ \Phi_i$ are ζ-holomorphic.

Two ζ-complex atlases on $\mathcal{W}$ are said to be *equivalent* if their union is again a ζ-complex atlas.

A class of ζ-complex atlases on $\mathcal{W}$ supplies, obviously, $\mathcal{W}$ with a ζ-complex structure. Conversely, given a ζ-complex structure on $\mathcal{W}$ there exists a ζ-complex atlas on $\mathcal{W}$ inducing this structure. Charts of such an atlas will be called **ζ- complex coordinates** on the corresponding ζ- complex curve. In ζ-complex coordinates the endomorphism J and its adjoint J^* are described by the relations

$$J\left(\partial_x\right) = \partial_y, \qquad J\left(\partial_y\right) = \zeta^2\partial_x$$
$$J^*\left(\mathrm{d}x\right) = \zeta^2\mathrm{d}y, \qquad J^*\left(\mathrm{d}y\right) = \mathrm{d}x.$$

If $\zeta^2 \neq 0$, the functions u and v in the equation (9) are said to be **conjugate**.

Alternatively, a ζ-complex curve can be regarded as a two dimensional smooth manifold supplied with a specific atlas whose transition functions

$$(x, y) \longmapsto (\xi(x, y), \eta(x, y))$$

satisfy to ζ-Cauchy-Riemann relations (9).

As it is easy to see, the ζ-Cauchy-Riemann relations (9) imply that

$$\partial_\eta^2 - \zeta^2\partial_\xi^2 = \frac{1}{\xi_x^2 - \zeta^2\xi_y^2}\left(\partial_y^2 - \zeta^2\partial_x^2\right)$$

and also

$$\mu\left(\partial_\eta^2 - \zeta^2\partial_\xi^2\right) + \mu_\eta\partial_\eta - \zeta^2\mu_\xi\partial_\xi = \frac{1}{\xi_x^2 - \zeta^2\xi_y^2}\left[\mu\left(\partial_y^2 - \zeta^2\partial_x^2\right) + \mu_y\partial_y - \zeta^2\mu_x\partial_x\right].$$

This shows that equation (7) (respectively, (8)) is well-defined on a ζ-complex curve with $\zeta^2 = 1$ (respectively, $\zeta^2 = -1$). The manifestly intrinsic expression for these equations is

$$\mathrm{d}\left(\mu J^{*}\mathrm{d}w\right) = 0.$$

We will refer to it as the *μ-deformed ζ-Laplace equation.*

A solution of the ζ-Laplace equation on $\mathcal{W}$ will be called **ζ- harmonic**. We can see that in the case $\zeta^2 \neq 0$ the notion of *conjugate ζ-harmonic function* is well defined on a ζ-complex curve. In addition, notice that the metric field $\mathrm{d}\xi^2 - \zeta^2\mathrm{d}\eta^2$, η being ζ-conjugate with ξ, is canonically associated with a ζ-harmonic function ξ on $\mathcal{W}$.

A map $\Phi : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ connecting two ζ- complex curves will be called *ζ- holomorphic* if $\varphi \circ \Phi$ is locally ζ -holomorphic for any local ζ-holomorphic function φ on $\mathcal{W}_2$. Obviously, if Φ is ζ- holomorphic and u is a ζ-harmonic function on $\mathcal{W}_2$, then $\Phi^*(u)$ is ζ-harmonic on $\mathcal{W}_1$.

It is worth noting that the *standard ζ- complex curve* is $\mathbb{R}^2_\zeta = \{(x + \zeta y)\}$, and the *standard ζ-harmonic function* on it is given by x, whose conjugated is y. The pair $\left(\mathbb{R}^2_\zeta, x\right)$ is **universal** in the sense that for a given ζ-harmonic function u on a ζ-complex curve $\mathcal{W}$ there exists a ζ -holomorphic map $\Phi : \mathcal{W} \longrightarrow \mathbb{R}^2_\zeta$ defined uniquely by the relations $\Phi^*(x) = u$ and $\Phi^*(y) = v$, v being conjugated with u.

3.2. Global Properties of Solutions

The above discussion shows that any global solution, that can be obtained by matching together local solutions desribed in Section 1, is a solution whose base manifold is a ζ-complex curve $\mathcal{W}$ and which corresponds to a ζ-harmonic function u on $\mathcal{W}$.

A solution of Einstein equations corresponding to $\mathcal{W} \subseteq \mathbb{R}^2_\zeta$, $u \equiv x$ will be called a *model*. Notice that there exist various model solutions due to various options in the choice of parameters appearing in $2'$) and $2''$) at the beginning of this section. An important role played by the model solutions is revealed by the property [18] that

Any solution of the Einstein equation which can be constructed by matching together local solutions described in Section 1 *is the pullback of a model solution via a $\zeta-$holomorphic map from a ζ-complex curve to* $\mathbb{R}_{\zeta^2}$.

We distinguish between the following two qualitatively different cases:

- I metrics admitting a normal three dimensional Killing algebra with two dimensional leaves
- II metrics admitting a normal two dimensional Killing algebra that does not *extend* to a larger algebra having the same leaves and whose distribution orthogonal to the leaves is integrable.

It is worth mentioning that the distribution orthogonal to the Killing leaves is automatically integrable in Case I [17]. In Case II the two dimensionality of the Killing leaves is guaranteed by Proposition 2 of [17].

Any Ricci-flat manifold (M, g), we are analyzing, is fibered over a ζ-complex curve $\mathcal{W}$

$$\pi : M \longrightarrow \mathcal{W}$$

whose fibers are the Killing leaves and as such are two dimensional Riemann manifolds of constant Gauss curvature.

Below, we shall call π the *Killing fibering* and assume that its fibers are *connected and geodesically complete*. Therefore, maximal (*i.e., non-extendible*) Ricci-flat manifolds, of the class we are analyzing in the paper, are those corresponding to maximal (*i.e., non-extendible*) pairs $(\mathcal{W}, u)$, where $\mathcal{W}$ is a ζ-complex curve and u is ζ-harmonic function on $\mathcal{W}$.

4. Examples

In this Section, we illustrate the previous general results with a few examples using the fact that any solution can be constructed as the pullback of a model solution *via* a ζ-holomorphic map Φ of a ζ-complex curve $\mathcal{W}$ to $\mathbb{R}^2_\zeta$. Recall that in the pair $(\mathcal{W}, u)$, describing the so obtained solution, $u = \mathrm{Re}(\Phi)$.

4.1. A Star "Outside" the Universe

The Schwarzschild solution shows a "star" generating a space "around" itself. It is an $\mathfrak{s}\ (3)$-invariant solution of the vacuum Einstein equations. On the contrary, its $\mathfrak{s}\ (2,1)$-analogue shows *a "star" generating the space only on "one side of itself". More exactly, the fact that the space in the Schwarzschild universe is formed by a one-parametric family of "concentric" spheres allows one to give a sense to the adverb "around". In the* $\mathfrak{s}\ (2,1)$-case the space is formed by a one-parameter family of "concentric" hyperboloids. The adjective "concentric" means that the curves orthogonal to hyperboloids are geodesics and metrically converge to a singular point. This explains in what sense this singular point generates the space only on "one side of itself".

4.2. Kruskal-Szekeres Type Solutions

We describe now a family of solutions which are of the *Kruskal-Szekeres type*, namely, that are characterized as being maximal extensions of the local solutions determined by an affine parametrization of null geodesics, and also by the use of more than one interval of monotonicity of $u\,(\beta)$.

Consider the ζ-complex curve

$$\mathcal{W}=\left\{(z=x+\zeta y)\in\mathbb{R}^2_\zeta;\, y^2-x^2<1\right\},\qquad \zeta^2=1$$

and the ζ-holomorphic function $\Phi:\mathcal{W}\to\mathbb{R}^2_\zeta$

$$\Phi(z)=A\ln\left(|A|\,z^2\right)=A\left(\ln\left|A\left(x^2-y^2\right)\right|+\zeta\ln\left|\frac{x+y}{x-y}\right|\right).$$

Thus, in the pair $(\mathcal{W},u)$ the ζ-harmonic function u is given by

$$u=A\ln\left|A\left(x^2-y^2\right)\right|.$$

Let us decompose $\mathcal{W}$ in the following way

$$\mathcal{W}=\mathcal{U}_1\cup\mathcal{U}_2$$

where

$$\begin{aligned}\mathcal{U}_1&=\left\{(z=x+\zeta y)\in\mathbb{R}^2_\zeta;\, 0\le y^2-x^2<1\right\}\\ \mathcal{U}_2&=\left\{(z=x+\zeta y)\in\mathbb{R}^2_\zeta;\, y^2-x^2\le 0\right\}.\end{aligned}$$

Consider now the solution defined as the pull back with respect to $\Phi|_{\mathcal{U}_1}$ and $\Phi|_{\mathcal{U}_2}$ of the model solutions determined by the following data: in the case of $\Phi|_{\mathcal{U}_1}$, $\mathcal{G}=\mathfrak{s}\ (3)$ or $\mathcal{G}=\mathfrak{s}\ (2,1)$, characterized by $F(\vartheta)=\sin^2\vartheta$ or $F(\vartheta)=\sin h^2\vartheta$ respectively, $\epsilon_1=\epsilon_2=1$, $A>0$, and for $\beta(u)$ the interval $]0,A]$. In the case of $\Phi|_{\mathcal{U}_2}$ the same data except for $\beta(u)$ which belongs to the interval $[A,\infty[$. The case $F(\vartheta)=\sin^2\vartheta$, corresponding to $\mathfrak{s}\ (3)$, will give the Kruskal-Szekeres solution. The case $F(\vartheta)=\sin h^2\vartheta$, corresponding to $\mathfrak{s}\ (2,1)$, will differ from the previous one in the geometry of the Killing leaves, which will now have a negative constant Gaussian curvature. The metric g has the following local form

$$g=4A^3\frac{\exp\frac{\beta}{A}}{\beta}\left(\mathrm{d}y^2-\mathrm{d}x^2\right)+\beta^2\left[\mathrm{d}\vartheta^2+F(\vartheta)\,\mathrm{d}\varphi^2\right]$$

with singularity $\beta=0$ occurring at $y^2-x^2=1$.

Acknowledgment

This work is partially supported by Agenzia Spaziale Italiana (ASI) (grant WP5000), and by the Italian Ministero Istruzione Università e Ricerca (MIUR) through the PRIN 2008 grant. The results exposed here have been obtained in collaboration with my colleagues F. Canfora, G. Sparano, A. Vinogradov and P. Vitale.

References

[1] Aliev B. and Leznov A., *Exact Solutions of the Vacuum Einstein's Equations Allowing for two Noncommutative Killing Vectors Type G_2II of Petrov classification*, J . Math. Phys. **33** (1992) 2567-2573.

[2] Bächtold M., *Ricci Flat Metrics with Bidimensional Null Orbits and Non-Integrable Orthogonal Distribution*, Diff. Geom. Appl. **25** (2007) 167-176

[3] Belinsky V. and Zakharov V., *Integration of the Einstein Equations by Means of the Inverse Scattering Problem Technique and Construction of the Exact Soliton Solutions*, Sov. Phys. JETP **48** (1978) 985-994.

[4] Canfora F., Vilasi G. and Vitale P., *Nonlinear Gravitational Waves and Their Polarization*, Phys. Lett. B **545** (2002) 373-378.

[5] Canfora F., Vilasi G. and Vitale P., *Spin-1 Gravitational Waves*, Int. J. Mod. Phys. B **18** (2004) 527-540.

[6] Canfora F. and Vilasi G., *Spin-1 Gravitational Waves and Their Natural Sources,* Phys. Lett. B **585** (2004) 193-199.

[7] Catalano-Ferraioli D. and Vinogradov A., *Ricci-flat 4-Metrics with Bidimensional Null Orbits, Part I. General Aspects and Nonabelian Case*, Acta. Appl. Math. **92** (2006) 209-225; *Ricci-flat 4-Metrics with Bidimensional Null Orbits, Part II. Abelian Case*, Acta. Appl. Math. **92** (2006) 226-239.

[8] Chinea F., *New First Integral for Twisting Type-N Vacuum Gravitational Fields with Two Non-Commuting Killing Vectors*, Class. Quantum Grav. **15** (1998) 367-371.

[9] Einstein A. and Rosen N., *On Gravitational Waves*, J. Franklin Inst. **223** (1937)43-54.

[10] Hallisoy M., *Studies in Space-Times Admitting Two Spacelike Killing Vectors*, J. Math. Phys. **29** (1988) 320-326.

[11] Law P., *Neutral Einstein Metrics in Four Dimensions*, J. Math. Phys. **32** (1991) 3039-3042.

[12] Matsushita Y., *On Euler Characteristics of Compact Einstein 4-Manifolds of Metric Signature* $(++--)$, J. Math. Phys. **22** (1981) 979-982.

[13] Matsushita Y., *Thorpe-Hitchin Inequality for Compact Einstein 4-Manifolds of Metric Signature* $(++--)$ *and the Generalized Hirzebruch Index Formula*, J. Math. Phys. **24** (1983) 36-40.

[14] Peres A., *Some Gravitational Waves*, Phys. Rev. Lett. **3** (1959) 571-572.

[15] Sparano G. and Vilasi G., *Noncommutative Integrability and Recursion Operators*, J. Geom. Phys. **36** (2000) 270-284.

[16] Sparano G., Vilasi G. and Vinogradov A., *Gravitational Fields with a Non-Abelian, Bidimensional Lie Algebra of Symmetries,* Phys. Lett. B **513** (2001) 142-146.

[17] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves. I. Local Aspects,* Diff. Geom. Appl. **16** (2002) 95-120.

[18] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves. II. Global Aspects*, Diff. Geom. Appl. **17** (2002) 15-35.

[19] Vilasi S., Canonico R., Sparano G. and Vilasi G., *Is the Light too Light?,* J. Geom. Symm. Phys. **19** (2010) 87-98.

Twelfth International Conference on
Geometry, Integrability and Quantization
June 4–9, 2010, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2011, pp 342–353

POISSON-NIJENHUIS STRUCTURE FOR GENERALIZED ZAKHAROV-SHABAT SYSTEM IN POLE GAUGE ON THE LIE ALGEBRA $\mathfrak{sl}(3,\mathbb{C})$

ALEXANDER B. YANOVSKI

Department of Mathematics & Applied Mathematics, University of Cape Town 7700 Rondebosch, Cape Town, South Africa

Abstract. We consider the recursion operator approach to the soliton equations related to a $\mathfrak{sl}(3,\mathbb{C})$ generalized Zakharov-Shabat auxiliary linear system in pole gauge and show that the recursion operator can be identified with the dual to a Nijenhuis tensor for a Poisson-Nijenhuis structure on the manifold of potentials.

1. Introduction

The soliton equations or completely integrable equations have been object of intense study even from their discovery. Their most essential property is that they admit a Lax representation $[L,A]=0$. In it L,A are linear operators on ∂_x,∂_t depending also on some functions $q_i(x,t)$, $1\leq i\leq s$ ('potentials') and a spectral parameter λ. The equation $[L,A]=0$ should be satisfied identically in λ and in this way the Lax equation $[L,A]=0$ is equivalent to a system of partial differential equations for $q_i(x,t)$. Usually one fixes the linear problem $L\psi=0$ (auxiliary linear problem) and considers all the evolution equations (of certain form of course) one can obtain changing the operator A. These equations are called nonlinear evolution equations (NLEEs) associated (related) with L (or with the linear system $L\psi=0$). There are several different schemes to resolve them but the essential point is that the Lax representation permits to pass from the original evolution defined by the equation to the evolution of some spectral data related to the problem $L\psi=0$ which is linear and consequently easily found. From this data the potentials can be recovered by a process called Inverse Scattering Method, see the monograph books [4, 6].

The **Generalized Zakharov-Shabat** (GZS) system presented below is a paradigm of auxiliary linear problem. It can be written as follows

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = 0. \tag{1}$$

Here $q(x)$ and J belong to some fixed simple Lie algebra $\mathfrak{g}$ in some finite dimensional irreducible representation. The element J is regular, that is the kernel of ad_J ($\mathrm{ad}_J(X) \equiv [J, X]$, $X \in \mathfrak{g}$) is the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The **potential** $q(x)$ belongs to the orthogonal completion $\mathfrak{h}^\perp$ of $\mathfrak{h}$ with respect to the Killing form

$$\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}_X\,\mathrm{ad}_Y), \qquad X, Y \in \mathfrak{g}. \tag{2}$$

Therefore $q(x) = \sum_{\alpha\in\Delta} q_\alpha E_\alpha$ where E_α are the root vectors, Δ is the root system of $\mathfrak{g}$. The scalar functions $q_\alpha(x)$ defined on $\mathbb{R}$, are complex valued, smooth and rapidly vanishing for $x \to \pm\infty$, we can assume that $q_\alpha(x)$ are of Schwartz type. The functions q_α are called also 'potentials' and we shall consider $q(x)$ as a point in an infinite dimensional manifold - the manifold of potentials. The classical Zakharov-Shabat system is obtained for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $J = \mathrm{diag}(1,-1)$.

Remark 1. We assume that the basic properties of the semisimple Lie algebras (real and complex) are known. All definitions and normalizations we use coincide with those made in [11] and are almost universally accepted.

Remark 2. When Generalized Zakharov-Shabat systems on different algebras are involved we say that we have Generalized Zakharov-Shabat $\mathfrak{g}$-system to underline it is on the algebra $\mathfrak{g}$, but when we work on a fixed algebra its symbol is usually omitted.

Referring for the details to [10] we simply remind that the adjoint solutions of GZS operator L are functions of the type $w = mXm^{-1}$ where X is a constant element from $\mathfrak{g}$ and m is fundamental solution of $Lm = 0$. Let us denote by w^{a} and w^{d} the orthogonal projection (with respect to the Killing form) of w over $\mathfrak{h}^\perp$ and $\mathfrak{h}$ respectively. If one denotes the orthogonal projector on $\mathfrak{h}^\perp$ by π_0 then of course $w^{\mathrm{a}} = \pi_0 w$ and $w^{\mathrm{d}} = (\mathbf{1} - \pi_0)w$. One of the most important facts from the theory of GZS system is that if a suitable set of adjoint solutions $(w_i(x,\lambda))$ is taken then roughly speaking for λ belonging to the spectrum of L the functions $w_i^{\mathrm{a}}(x,\lambda)$ form a complete sets in the space of potentials. If one expands a potential over the subset of the adjoint solutions as coefficients one gets the minimal scattering data for L. Thus passing from the potentials to the scattering data can be considered as a sort of Fourier transform, called generalized Fourier transform. For this transform the functions $w_i^{\mathrm{a}}(x,\lambda)$ play the role the exponents play in the usual Fourier transform. This interpretation was given for the first time in [1] and after that has been developed in a number of works, see for example the monograph books [6, 12] for comprehensive study of $\mathfrak{sl}(2,\mathbb{C})$-case and bibliography, [2, 10] for more general situations.

I. The recursion operators (generating operators, Λ-operators) are the operators for which the functions $w_i^{\mathrm{a}}(x,\lambda)$ are eigenfunctions and therefore for the generalized Fourier transform they play the same role as the differentiation operator plays in the usual Fourier transform method. Their explicit form can be found in a number of articles, and books, see for example [6]. For the above reasons the recursion operators $\Lambda_\pm$ (usually one says just recursion operator) play important role in the theory of soliton equations - it is a theoretical too which apart from explicit solutions can give most of the information about the NLEEs, [6, 23]. In particular, through them can be obtained:

i) The hierarchies of the nonlinear evolution equations solvable through L

ii) The conservation laws for these NLEEs

iii) The hierarchies of Hamiltonian structures for these NLEEs.

There is another important trend in the theory of the recursion operators, it is related with the study of the recursion operators related to gauge-equivalent systems. Taking as example the GZS system, assume that we make a gauge transformation of the type $\psi \mapsto \psi_0^{-1}\psi = \tilde{\psi}$ where ψ_0 is a fundamental solution to GZS system corresponding to $\lambda = 0$. Then if we denote $S = \psi_0^{-1}J\psi_o$ and the orbit of the coadjoint representation of the Lie group G corresponding to $\mathfrak{g}$ by $\mathcal{O}_J$ we shall obtain that $\tilde{\psi}$ is a solution of the following linear problem

$$\tilde{L}\tilde{\psi} = \mathrm{i}\partial_x\tilde{\psi} - \lambda S\tilde{\psi} = 0, \qquad S \in \mathcal{O}_J. \tag{3}$$

One can choose different fundamental solutions ψ_0 and one will obtain different behavior for S when $x \mapsto \pm\infty$ but usually for ψ_0 is taken the Jost solution that satisfies $\lim_{x\to-\infty}\psi_0 = \mathbf{1}$. The system (3) is called GZS system in pole gauge in contrast to the system (1) which is called GZS system in canonical gauge.

The theory of the NLEEs related with the GZS auxiliary problem in canonical gauge (L) is in direct connection with the theory of the NLEEs related with the GZS auxiliary problem in pole gauge ($\tilde{L}$). The NLEEs for both systems are in one-to-one correspondence and are called gauge-equivalent equations. This beautiful construction has been discovered for the first time in the famous work of Zakharov and Takhtadjan, [22] in which there has been proved the gauge-equivalence of two famous equations - the Heisengerg ferromagnet equation and the nonlinear Schrödinger equation.

In fact the constructions for the system L and its gauge equivalent $\tilde{L}$ are in complete analogy. Instead of the fixed Cartan subalgebra $\mathfrak{h} = \ker \mathrm{ad}_J$ we have 'moving' Cartan subalgebra $\mathfrak{h}_S(x) = \ker \mathrm{ad}_{S(x)}$, 'moving' space $\mathfrak{h}_S^{\perp}(x)$ orthogonal (with respect to the Killing form) to $\mathfrak{h}_S(x)$ (and consequently moving projector $\pi_S(x)$) etc. We have the corresponding adjoint solutions $\tilde{m} = \tilde{\psi}X\tilde{\psi}^{-1}$ where $\tilde{\psi}$ is a solution of $\tilde{L}\tilde{\psi} = 0$ and X is a constant element in $\mathfrak{g}$. If we denote by $\tilde{m}^{\mathrm{a}}$ and $\tilde{m}^{\mathrm{d}}$

the projections of $\tilde{m}(x)$ on $\mathfrak{h}_S^{\perp}(x)$ and $\mathfrak{h}_S(x)$ respectively then the corresponding recursion operators are constructed using the fact that the functions $\tilde{m}^{\mathrm{a}}$ must be eigenfunctions for them.

Let us make the following agreement. Though the Cartan subalgebra $\mathfrak{h}_S(x)$, its orthogonal space $\mathfrak{h}_S^{\perp}(x)$ and the projector $\pi_S(x)$, depend on x we shall not write it explicitly unless there is a posibility of confusion. So for example in the case of a function $X(x)$ that is defined on $\mathbb{R}$ and such that $X(x) \in \mathfrak{h}_S^{\perp}(x)$ we shall write simply $X \in \mathfrak{h}_S^{\perp}$, for two functions $X(x)$ and $Y(x)$ we shall write instead of $X(x) = Y(x)$ simply $X = Y$ and so on.

For GZS system in pole gauge everything is easily reformulated and the only real difficulty is to calculate all the quantities that are expressed through q and its derivative through S and its derivatives. There is a clear procedure how to achieve that goal but in each particular case it requires new calculations. The procedure has been developed in detail in our PhD thesis [20], outlined in [7, 8] (for the $\mathfrak{sl}(2,\mathbb{C})$ case) and in more general cases in [9]. In the case of $\mathfrak{sl}(3)$ the procedure has been carried out in detail in [21] - for all these references see also [6].

II. The recursion operators for GZS have also beautiful geometric meaning. It can be shown that their adjoint operators can be interpreted as Nijenhuis tensors on the manifolds of ‘potentials’ where the evolution defined by $[L, A] = 0$ occurs. The point is that one of characteristic properties of the soliton equations is that they are not simply Hamiltonian but they are Hamiltonian with respect to two different compatible Poisson structures. The property is called bi-Hamiltonian property of the NLEEs solvable through the corresponding linear problem. A Poisson structure on a manifold $\mathcal{M}$ is a field of linear maps $m \mapsto P_m : T_m^*(\mathcal{M}) \mapsto T_m(\mathcal{M})$ such that for any two smooth functions f, g the expression $\{f, g\}(m) = \langle \mathrm{d}g_m, P_m(\mathrm{d}f)_m\rangle$ is a Poisson bracket. (Here $\langle\, ,\rangle$ is the canonical pairing between $T_m(\mathcal{M})$ and $T_m^*(\mathcal{M})$ - the tangent and cotangent spaces at $m \in \mathcal{M}$). Compatible Poisson structures are called such Poisson structures P, Q for which their linear combination $aP + bQ$ (where a, b are constants) is also a Poisson tensor. It turns out that compatible Poisson structures give rise to Nijenhuis tensors in case one of it is invertible.

Indeed, if Q is invertible, then one can define $N = P \circ Q^{-1}$ and N is a field of linear maps $m \mapsto N_m : T_m(\mathcal{M}) \mapsto T_m(\mathcal{M})$ such that the so called Nijenhuis bracket $[N, N]$ of N is zero. Then the manifold of potentials is endowed with a very special geometric structure - Poisson-Nijenhuis (P-N) structure of coupled Poisson tensor and a Nijenhuis tensor. The properties of the P-N structure are responsible to the fact that the symmetries of the soliton equations have ‘hereditary’ properties and that there are infinitely many Hamiltonian structures for the corresponding NLEEs. This interpretation was found by F. Magri in his pioneer works [13, 14], one can see all the details of the theory in [3] or in [6], we shall assume that it is known and shall not describe it here.

As a matter of fact there is a nice picture of the relation of the P-N structures on the manifold of potentials for the GZS system in canonical gauge, the manifold of potentials for the same system in pole gauge and the manifold of the corresponding Jost solutions, see [6, Ch. 15].

Together with the possibility to calculate the recursion operators for GZS system in pole gauge through the gauge transformation, there exists another option - to calculate directly the P-N structure on the manifold of potentials using the compatible Poisson structures and then to find the conjugate to the Nijenhuis tensor. In this work we shall use it and then shall compare our result with the Recursion Operator already known in the case $\mathfrak{sl}(3,\mathbb{C})$, see [21]. Our motivation comes from the fact that there has been some renewed interest in the GZS system in pole gauge and its reductions recently, see [5].

2. P-N Structure for GZS Pole Gauge Hierarchy. The $\mathfrak{sl}(3,\mathbb{C})$ Case

Consider the GZS pole gauge $\mathfrak{sl}(3,\mathbb{C})$-system in general position - that is the smooth function $S(x)$ with domain $\mathbb{R}$, see (3), is subject only to the requirements that $S(x)\in\mathcal{O}_J$ and $S(x)$ tends fast enough to some constant values when $x\mapsto\pm\infty$. For J we shall assume that $J=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, $\sum_i\lambda_i=0$, where all $\lambda_i\neq 0$. Of course J must be regular, so that $\ker\mathrm{ad}_J$ coincides with the Cartan subalgebra of the diagonal matrices in $\mathfrak{sl}(3,\mathbb{C})$.

Let us consider a more general case then in the above when the algebra $\mathfrak{g}$ is arbitrary simple algebra. Let $S(x)$ is smooth, have values in $\mathfrak{g}$ and when $x\to\pm\infty$ the function $S(x)$ tend fast enough to constant values. These functions of this type form an infinite dimensional manifold which we shall denote by $\mathcal{M}$. Then it is reasonable to assume that the tangent space $T_S(\mathcal{M})$ at S consists of all the smooth functions $X:\mathbb{R}\mapsto\mathfrak{g}$ vanishing fast enough when $x\mapsto\pm\infty$. We denote that space by $\mathfrak{F}(\mathfrak{g})$. We shall also assume that the 'dual space' $T^*_S(\mathcal{M})$ is equal to $\mathfrak{F}(\mathfrak{g})$ and if $\alpha\in T^*_S(\mathcal{M})$, $X\in T_S(\mathcal{M})$ then

$$\alpha(X)=\langle\langle\alpha,X\rangle\rangle\equiv\int_{-\infty}^{+\infty}\langle\alpha(x),X(x)\rangle\mathrm{d}x \tag{4}$$

where $\langle\,,\rangle$ is the Killing form of $\mathfrak{sl}(3,\mathbb{C})$.

Remark 3. In other words, we identify $T^*_S(\mathcal{M})$ and $T_S(\mathcal{M})$ using the bi-linear form $\langle\langle\,,\rangle\rangle$. We do not want to make the definitions more precise, since we will speak rather about a geometric picture then about precise results. Such results can be obtained only after profound study of the spectral theory of L and $\tilde{L}$. In particular, we put dual space in quotation marks because it is clearly not equal to the dual of $\mathfrak{F}(\mathfrak{g})$. We mention however that the term 'allowed' functional H means that $\dfrac{\delta H}{\delta S} \in T^*_S(\mathcal{M}) \sim T_S(\mathcal{M})$.

First we note that the operators

$$\alpha \mapsto P(X) = \mathrm{i}\partial_x \alpha, \qquad \alpha \in T^*_S(\mathcal{M}) \tag{5}$$

$$\alpha \mapsto Q(\alpha) = \mathrm{ad}_S(\alpha), \qquad S \in \mathcal{M}. \tag{6}$$

It is a fact from the general theory that these Poisson tensors are compatible, [6, Ch. 15]. In other words $P+Q$ is also a Poisson tensor. Let us also mention that the tensor Q is the canonical Kirillov tensor which acquires the above form because the algebra is simple and coadjoint and adjoint representation are equivalent.

Now let $\mathcal{O}_J$ be the orbit of the coadjoint representation of G (the group that corresponds to $\mathfrak{g}$) passing through J. Let us consider the set of smooth functions $f : \mathbb{R} \mapsto \mathcal{O}_J$ such that when $x \to \pm\infty$ they tend fast enough to constant values. The set of this functions is denoted by $\mathcal{N}$ and clearly can be considered as submanifold of $\mathcal{M}$. If $S \in \mathcal{N}$ the tangent space $T_S(\mathcal{N})$ consists of all smooth functions X, tending to zero fast enough when $x \mapsto \pm\infty$ and such that $X(x) \in T_{S(x)}(\mathcal{O}_J)$ (Recall that $\mathcal{O}_J$ is a smooth manifold in a classical sense.) We again assume that $T^*_S(\mathcal{N}) \sim T_S(\mathcal{N})$ and that these spaces are identified via $\langle\langle\, ,\rangle\rangle$.

The Poisson tensors P and Q can be restricted from $\mathcal{M}$ to $\mathcal{N}$. The question how to restrict a Poisson tensor on submanifold has been considered in detail in the literature, see for example [17] and [18, 19]. We shall use a simplified version of the results obtained in these papers, proved in [15, 16]. We call it first restriction theorem.

Theorem 1. *Let $\mathcal{M}$ be Poisson manifold with Poisson tensor P and $\bar{\mathcal{M}} \subset \mathcal{M}$ be a submanifold. Let us denote by j the inclusion map of $\bar{\mathcal{M}}$ into $\mathcal{M}$, by $\mathcal{X}^*_P(\bar{\mathcal{M}})_m$ the subspace of covectors $\alpha \in T^*_m(\mathcal{M})$ such that*

$$P_m(\alpha) \in \mathrm{d}j_m(T_m(\bar{\mathcal{M}})) = \mathrm{Im}(\mathrm{d}j_m), \qquad m \in \bar{\mathcal{M}} \tag{7}$$

here Im *denotes the image and $T^\perp(\bar{\mathcal{M}})_m$ – the set of all covectors at $m \in \mathcal{M}$ vanishing on the subspace* $\mathrm{Im}(\mathrm{d}j_m)$*, $m \in \bar{\mathcal{M}}$ also called the annihilator of* $\mathrm{Im}(\mathrm{d}j_m)$ *in $T^*_m(\mathcal{M})$. Let the following relations hold*

$$\mathcal{X}^*_P(\bar{\mathcal{M}})_m + T^\perp(\bar{\mathcal{M}})_m = T^*_m(\mathcal{M}), \qquad m \in \bar{\mathcal{M}} \tag{8}$$

$$\mathcal{X}^*_P(\bar{\mathcal{M}})_m \cap T^\perp(\bar{\mathcal{M}})_m \subset \ker(P_m). \tag{9}$$

Then there exists unique Poisson tensor $\bar{P}$ on $\bar{\mathcal{M}}$, j-related with P, that is

$$P_m = \mathrm{d}j_m \circ \bar{P}_m \circ (\mathrm{d}j_m)^*. \tag{10}$$

The proof of the theorem is constructive. First, one takes $\beta \in T^*_m(\bar{\mathcal{M}})$, then represents $(j^*\beta)_m$ as $\alpha_1 + \alpha_2$ where $\alpha_1 \in \mathcal{X}^*_P(\bar{\mathcal{M}})_m$, $\alpha_2 \in T^\perp(\bar{\mathcal{M}})_m$ and finally puts $\bar{P}_m(\beta) = P_m(\alpha_1)$ (we identify m and $j(m)$ here).

Restricting the Poisson tensor Q is easy, one readily get that the restriction $\bar{Q}$ is given by the same formula as before

$$\alpha \mapsto \bar{Q}(\alpha) = \mathrm{ad}_S(\alpha), \qquad S \in \mathcal{N}, \qquad \alpha \in T^*_S(\mathcal{N}). \tag{11}$$

The tensor P is a little harder to restrict. The restriction we present below has been preformed in various works in the simplest case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, see for example [16]. We do it now in the case $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, in other words starting from here the algebra $\mathfrak{g}$ will be $\mathfrak{sl}(3,\mathbb{C})$.

First, let us introduce some facts and notation. Since J is a regular element from the Cartan subalgebra $\mathfrak{h}$ then each element S from the orbit $\mathcal{O}_J$ is also regular, $\mathfrak{h}_S(x) = \ker \mathrm{ad}_{S(x)}$ is a Cartan subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ and we have

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{h}_S(x) \oplus \mathfrak{h}^\perp_S(x) \tag{12}$$

($\mathfrak{sl}(3,\mathbb{C})$ is constant, so we do not write $\mathfrak{sl}(3,\mathbb{C})(x)$).

If $X \in T_S(\mathcal{N}) = \mathfrak{h}^\perp_S$ then $X(x) \in \mathfrak{h}^\perp_S(x)$ (we recall that these spaces depend on x) but in addition X is smooth and vanishes rapidly when $x \mapsto \pm\infty$. We shall denote the set of these functions by $\mathfrak{F}(\mathfrak{h}^\perp_S)$. So according to our notation $X \in \mathfrak{h}^\perp_S$ and $X \in \mathfrak{F}(\mathfrak{h}^\perp_S)$. Using the same logic, for $X \in \mathfrak{F}(\mathfrak{h}^\perp_S)$ we write $\mathrm{ad}_S(X)$ which means the function $\mathrm{ad}_{S(x)} X(x)$ belonging to $\mathfrak{F}(\mathfrak{h}^\perp_S)$.

We have some facts about J that we introduce in the below propositions. For the proofs see [21].

Proposition 1. The matrices J and $J_1 = J^2 - \frac{1}{3}\mathrm{tr}(J^2)\mathbf{1}$ span the Cartan subalgebra $\mathfrak{h} = \mathrm{ker}\mathrm{ad}_J$.

As a consequence, for $S \in \mathcal{O}_J$ the matrices S and $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ span the Cartan subalgebra $\mathfrak{h}_S$ of $\mathfrak{sl}(3,\mathbb{C})$. On $\mathfrak{h}^\perp_S$ the operator ad_S is invertible.

Proposition 2. The matrix J satisfies the equation

$$J^3 = \frac{1}{2}C_2 J + \frac{1}{3}C_3\mathbf{1}, \qquad C_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad C_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3. \tag{13}$$

Proposition 3. If $S \in \mathcal{O}_J = \{\tilde{X};\, \tilde{X} = gJg^{-1}, g \in \mathrm{SL}(3,\mathbb{C})\}$ then S satisfies (13), that is $S^3 = \frac{1}{2}C_2 S + \frac{1}{3}C_3\mathbf{1}$. If in addition for all λ_i, $\lambda_i \neq 0$ the inverse is also true, that is any S that satisfies the equation $S^3 = \frac{1}{2}C_2 S + \frac{1}{3}C_3\mathbf{1}$ belongs to the orbit.

The Killing form of $\mathfrak{sl}(3,\mathbb{C})$ is equal to $6\mathrm{tr}\, XY$ and one has the following useful identities

$$\langle J, J\rangle = 6C_2, \qquad \langle J_1, J_1\rangle = C_2^2, \qquad \langle J, J_1\rangle = 6C_3. \tag{14}$$

The Killing form is invariant with respect to the adjoint action, so we also have

$$\langle S, S\rangle = 6C_2, \qquad \langle S_1, S_1\rangle = C_2^2, \qquad \langle S, S_1\rangle = 6C_3. \tag{15}$$

The **Gram matrix**

$$T = \begin{pmatrix} \langle J, J\rangle & \langle J, J_1\rangle \\ \langle J_1, J\rangle & \langle J_1, J_1\rangle \end{pmatrix} = \begin{pmatrix} 6C_2 & 6C_3 \\ 6C_3 & C_2^2 \end{pmatrix} \tag{16}$$

has determinant $d_1 = 6(C_2^3 - 6C_3^2)$. Of course $d_1 \neq 0$. One can show that

$$d_1 = 12(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2 \equiv 12d. \tag{17}$$

Therefore

$$T^{-1} = \frac{1}{12d}\begin{pmatrix} \langle J_1, J_1\rangle & -\langle J, J_1\rangle \\ -\langle J_1, J\rangle & \langle J, J\rangle \end{pmatrix} = \frac{1}{12d}\begin{pmatrix} C_2^2 & -6C_3 \\ -6C_3 & 6C_2 \end{pmatrix}. \tag{18}$$

Now we are in position to perform the restriction of P on $\mathcal{N}$. For $S \in \mathcal{N}$ we have

$$\mathcal{X}_P^*(\mathcal{N})_S = \{\alpha;\, \mathrm{i}\partial_x\alpha \in \mathfrak{F}(\mathfrak{h}^\perp)\} \tag{19}$$

$$T^\perp(\mathcal{N})_S = \{\alpha;\, \langle\langle\alpha, X\rangle\rangle = 0,\ X \in \mathfrak{F}(\mathfrak{h}_S^\perp)\}. \tag{20}$$

We see that $T^\perp(\mathcal{N})_S$ is the set of smooth functions $\alpha(x)$ such that $\alpha \in \mathfrak{h}_S$ tends to zero fast enough when $x \mapsto \pm\infty$. We shall denote this space by $\mathfrak{F}(\mathfrak{h}_S)$. Naturally, $\mathfrak{F}(\mathfrak{h}_S) \subset \mathfrak{F}(\mathfrak{h}_S)_0$, where the space $\mathfrak{F}(\mathfrak{h}_S)_0$ consists of all smooth functions $X(x)$ such that $X \in \mathfrak{h}$ and such that X tends to some constant values when $x \mapsto \pm\infty$. Since S and S_1 span $\mathfrak{h}_S$, we have that $S, S_1 \in \mathfrak{F}(\mathfrak{h}_S)_0$ and

$$\begin{aligned} \mathfrak{F}(\mathfrak{h}_S)_0 = \{X;\, & X = a(x)S(x) + b(x)S_1(x), \quad a(x), b(x) - \text{smooth}, \\ & a(x), b(x) \text{ tend to some constant values when } x \mapsto \pm\infty\} \end{aligned} \tag{21}$$

$$\begin{aligned} \mathfrak{F}(\mathfrak{h}_S) = \{X;\, & X = a(x)S(x) + b(x)S_1(x), \quad a(x), b(x) - \text{smooth}, \\ & \lim_{x\to\pm\infty} a(x) = \lim_{x\to\pm\infty} b(x) = 0\} \end{aligned} \tag{22}$$

Let us consider now $\mathcal{X}_P^*(\mathcal{N})_S \cap T^\perp(\mathcal{N})_S$. It consists of elements

$$\alpha = a(x)S(x) + b(x)S_1(x)$$

such that $\mathrm{i}\partial_x\alpha \in \mathfrak{F}(\mathfrak{h}_S^\perp)$. But

$$\mathrm{i}\partial_x\alpha = \mathrm{i}a(x)S_x + \mathrm{i}b(x)(S_1)_x + \mathrm{i}a_xS(x) + \mathrm{i}b_xS_1(x)$$

so we must have $\langle \mathrm{i}\partial_x\alpha(x), S(x)\rangle = \langle \mathrm{i}\partial_x\alpha(x), S_1(x)\rangle = 0$. Now, let us note that from (15) follows that

$$\langle S(x), S_x(x)\rangle = \langle S_1(x), (S_1)_x(x)\rangle = 0,\ \langle S_1(x), S_x(x)\rangle = -\langle (S_1)_x(x), S_x(x)\rangle.$$

Next

$$\langle S_1, S_x\rangle = 6\,\mathrm{tr}(S_xS^2) = 2\,\mathrm{tr}(S^3)_x.$$

Using Proposition 3 we get that $\langle S_1, S_x\rangle$ is proportional to $\mathrm{tr}\, S_x = 0$. In this way we see that $S_x, (S_1)_x$ belong to $\mathfrak{F}(\mathfrak{h}_S^\perp)$ and therefore $a_x = b_x = 0$. Then a and b can be only identically zero and

$$\mathcal{X}_P^*(\mathcal{N})_S \cap T^\perp(\mathcal{N})_S = \{0\} \subset \ker P_S.$$

Consider now arbitrary $\alpha \in T^*(\mathcal{N})_S$. We want to represent it as $\alpha_1 + \alpha_2$, where $\alpha_1 \in \mathcal{X}^*(\mathcal{N})_S$, $\alpha_2 \in T^\perp(\mathcal{N})_S$. Therefore, $\alpha_2 = A(x)S(x) + B(x)S_1$ with $A(x), B(x)$ vanishing when $x \mapsto \pm\infty$. In addition, we must have

$$\mathrm{i}\partial_x\alpha = \mathrm{i}\partial_x\alpha_1 + \mathrm{i}A(x)S_x + \mathrm{i}B(x)(S_1)_x + \mathrm{i}A_xS(x) + \mathrm{i}B_xS_1 \qquad (23)$$

where $\mathrm{i}\partial_x\alpha_1 \in \mathfrak{F}(\mathfrak{h}_S^\perp)$. Taking the Killing form with S and S_1 we get the system

$$\langle\partial_x\alpha, S(x)\rangle = A_x\langle J, J\rangle + B_x\langle J, J_1\rangle \qquad (24)$$

$$\langle\partial_x\alpha, S_1(x)\rangle = A_x\langle J, J_1\rangle + B_x\langle J_1, J_1\rangle$$

$$(25)$$

and therefore

$$\begin{pmatrix} A_x \\ B_x \end{pmatrix} = T^{-1}\begin{pmatrix} \langle\partial_x\alpha, S(x)\rangle \\ \langle\partial_x\alpha, S_1(x)\rangle \end{pmatrix} \qquad (26)$$

where T is the Gram matrix introduced earlier. So we obtain

$$\begin{pmatrix} A \\ B \end{pmatrix} = T^{-1}\begin{pmatrix} \partial_x^{-1}\langle\partial_x\alpha, S(x)\rangle \\ \partial_x^{-1}\langle\partial_x\alpha, S_1(x)\rangle \end{pmatrix}. \qquad (27)$$

Remark 4. In all the theory of the recursion operators and their geometric interpretation usually the expressions on which the operator ∂_x^{-1} acts are total derivatives. Thus the same results will be obtained choosing for ∂_x^{-1} any of the following operators

$$\int_{-\infty}^{x} .\,\mathrm{d}y, \qquad \int_{+\infty}^{x} .\,\mathrm{d}y, \qquad \frac{1}{2}\left(\int_{-\infty}^{x} .\,\mathrm{d}y + \int_{+\infty}^{x} .\,\mathrm{d}y\right). \qquad (28)$$

However, one uses more frequently the third expression when one writes the corresponding Poisson tensors in order to make them explicitly skew-symmetric.

Returning to our task, for $\alpha \in T^*(\mathcal{N})_S$ let us put

$$\alpha = \alpha_1 + \alpha_2 \qquad (29)$$

$$\alpha_1 = \alpha - \alpha_2 \qquad (30)$$

$$\alpha_2 = (S, S_1)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\langle\partial_x\alpha, S(x)\rangle \\ \partial_x^{-1}\langle\partial_x\alpha, S_1(x)\rangle \end{pmatrix}. \qquad (31)$$

One checks that α_1, α_2 lie in the spaces $\mathcal{X}^*(\mathcal{N})_S, T^\perp(\mathcal{N})_S$ respectively. Thus the conditions of the first restriction theorem are fulfilled. Noting that for $\beta \in T_S^*(\mathcal{N})$

we have $\mathrm{d}j_S^*\beta = \pi_S(\beta)$ we find that the restriction $\bar{P}$ of P on $\mathcal{N}$ has the form

$$\bar{P}(\beta) = \mathrm{i}\pi_S\partial_x\beta - \mathrm{i}\,(S_x,(S_1)_x)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\langle\partial_x\beta, S(x)\rangle \\ \partial_x^{-1}\langle\partial_x\beta, S_1(x)\rangle \end{pmatrix}. \tag{32}$$

The Poisson tensor $\bar{Q}$ is invertible on $\mathcal{N}$, so one can construct a Nijenhuis $N = \bar{P}\circ \mathrm{ad}_S^{-1}$ tensor which evaluated at $X \in \mathfrak{F}(\mathfrak{h}_S^{\perp})$ gives

$$N(X) = \mathrm{i}\pi_S\partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\,(S_x,(S_1)_x)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\langle\partial_x(\mathrm{ad}_S^{-1}X), S(x)\rangle \\ \partial_x^{-1}\langle\partial_x(\mathrm{ad}_S^{-1}X), S_1(x)\rangle \end{pmatrix}. \tag{33}$$

Taking into account that $\langle \mathrm{ad}_S^{-1}(X), S\rangle = \langle \mathrm{ad}_S^{-1}(X), S_1\rangle = 0$ the above can be cast into equivalent form

$$N(X) = \mathrm{i}\pi_S\partial_x(\mathrm{ad}_S^{-1}X) + \mathrm{i}\,(S_x,(S_1)_x)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\langle \mathrm{ad}_S^{-1}X)S_x(x)\rangle \\ \partial_x^{-1}\langle \mathrm{ad}_S^{-1}X,(S_1)_x(x)\rangle \end{pmatrix}. \tag{34}$$

From the general theory of the compatible Poisson tensors now follows that

Theorem 2. *The Poisson tensor field $\bar{Q}$ and the Nijenhuis tensor field N endow the manifold $\mathcal{N}$ with a* P-N structure.

The final step is to calculate the dual of the tensor N with respect to the pairing $\langle\langle\,,\rangle\rangle$. A quick calculation, taking into account that ad_S is skew-symmetric with respect to the Killing form, gives for $\alpha \in \mathfrak{F}(\mathfrak{h}_S^{\perp})$

$$N^*(\alpha) = \mathrm{i}\,\mathrm{ad}_S^{-1}\left[\pi_S\partial_x\alpha + (S_x,(S_1)_x)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\langle\alpha S_x(x)\rangle \\ \partial_x^{-1}\langle\alpha,(S_1)_x(x)\rangle \end{pmatrix}\right] \tag{35}$$

or equivalently

$$N^*(\alpha) = \mathrm{i}\,\mathrm{ad}_S^{-1}\left[\pi_S\partial_x\alpha - (S_x,(S_1)_x)\,T^{-1}\begin{pmatrix} \partial_x^{-1}\partial_x\langle\alpha S(x)\rangle \\ \partial_x^{-1}\langle\partial_x\alpha, S_1(x)\rangle \end{pmatrix}\right]. \tag{36}$$

But if we write the above in components we shall see that these are the recursion operators $\tilde{\Lambda}_{\pm}$ for the GZS system in pole gauge, see [21]. Thus our results confirm the idea that the recursion operators and the Nijenhuis tensors are dual objects.

3. Conclusion

In this article we have found the P-N structure on the manifold of potentials $\mathcal{N}$ for the GZS system in pole gauge on the algebra $\mathfrak{sl}(3,\mathbb{C})$ obtaining geometric interpretation of the recursion operators.

Acknowledgment

This work has been essentially supported by the incentive grant of NRF South Africa IRF2010042600017.

References

[1] Ablowitz M., Kaup D., Newell A. and Segur H., *The Inverse Scattering Problem - Fourier Analysis for Nonlinear Problems*, Studies in Appl. Math. **53** (1974) 249–315.

[2] Beals R. and Coifman R., *Scattering and Inverse Scattering for First Order Systems*, Comm. Pure & Apppl. Math. **37** (1984) 39–89.

[3] Casati P., Falqui G., Magri F. and Pedroni M., *Eight Lectures on Integrable Systems. Integrability of Nonlinear Systems*, In: Lecture Notes in Physics **495**, Y. Kosmann-Schwarzbach, B. Grammaticos and K. Tamizhmani (Eds), Springer, Berlin, 2004, pp 209–250.

[4] Faddeev L. and Takhtadjan L., *Hamiltonian Method in the Theory of Solitons*, Springer, Berlin, 1987.

[5] Gerdjikov V., Mikhailov A. and Valchev T., *Reductions of Integrable Equations on A.III - Symmetric Spaces*, J. Phys. A: Math. Theor. **43** (2010) 434015.

[6] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies - Spectral and Geometric Methods*, Springer, Heidelberg, 2008.

[7] Gerdjikov V. and Yanovski A., *Gauge-Covariant Theory of the Generating Operator. I.*, Comm. Math. Phys. **103** (1986) 549–56.

[8] Gerdjikov V. and Yanovski A., *Gauge-Covariant Formulation of the Generating Operator. 1. The Zakharov-Shabat System*, Phys. Lett. A **103** (1984) 232–236.

[9] Gerdjikov V. and Yanovski A., *Gauge-Covariant Formulation of the Generating Operator. 2. Systems on Homogeneous Spaces*, Phys. Lett. A **110** (1985) 53–57.

[10] Gerdjikov V. and Yanovski A., *Completeness of the Eigenfunctions for the Caudrey-Beals-Coifman System*, J. Math. Phys. **35** (1994) 3687–3721.

[11] Goto M. and Grosshans F., *Semisimple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics vol.**38**, M. Dekker, New York, 1978.

[12] Iliev I., Khristov E. and Kirchev K., *Spectral Methods in Soliton Equations*, Pitman Monographs and Surveys in Pure and Applied Mathematics vol.**73**, John Wiley & Sons, New-York, 1994.

[13] Magri F., *A Simple Model of the Integrable Hamiltonian Equations*, J. Math. Phys. **19** (1978) 1156–1162.

[14] Magri F., *A Geometrical Approach to the Nonlinear Solvable Equations*, In: Lecture Notes in Physics vol.**120**, Springer, Berlin, 1980.

[15] Magri F. and Morosi C., *A Geometrical Characterization of Integrable Hamiltonian Systems Through the Theory of Poisson-Nijenhuis Manifolds*, Quaderni del Dipartimento di Matematica, Università di Milano 1984.

[16] Magri F., Morosi C. and Ragnisco O., *Reduction Techniques for Infinite-Dimensional Hamiltonian Systems: Some Ideas and Applications*, Comm. Math. Phys. **99** (1985) 115–140.

[17] Marsden J. and Ratiu T., *Reduction of Poisson Manifolds*, Lett. Math. Phys. **11** (1986) 161–169.

[18] Ortega J. and Ratiu T., *Momentum Maps and Hamiltonian Reduction*, Progress in Mathematics vol. **222**, Birkhäuser, Boston, 2004.

[19] Ortega J. and Ratiu T., *Singular Reduction of Poisson Manifolds*, Lett. Math. Phys. **46** (1998) 59–372.

[20] Yanovski A., *Gauge-Covariant Approach to the Theory of the Generating Operators for Soliton Equations*, PhD thesis: 5-87-222, Joint Institute for Nuclear Research (JINR), 1987.

[21] Yanovski A., *Generating Operators for the Generalized Zakharov-Shabat System and its Gauge Equivalent System in* $\mathfrak{sl}(3,\mathbb{C})$ *Case*, Universität Leipzig, Naturwissenchaftlich Theoretisches Zentrum Report # 20, 1993, http://cdsweb.cern.ch/record/256804/files/P00019754.pdf.

[22] Zakharov V. and Takhtadjan L., *Equivalence Between Nonlinear Schrödinger Equation and Heisenberg Ferromagnet Equation*, TMF (Theoretical and Mathematical Physics) **38** (1979) 26–35.

[23] Zakharov V. and Konopelchenko B., *On the Theory of Recursion Operator*, Comm. Math. Phys. **94** (1985) 483-510.

SUBJECT INDEX

E-MAIL ADDRESSES

Canonico, Rosangela – *Salerno*	canonico@sa.infn.it
Christov, Christo – *Lafayette*	christov@louisiana.edu
Courbage, Maurice – *Paris*	courbage@ccr.jussieu.fr
Djondjorov, Peter – *Sofia*	padjon@imbm.bas.bg
Dobrogowska, Alina – *Białystok*	alaryzko@alpha.uwb.edu.pl
Dobrowolski, Tomasz – *Cracow*	dobrow@up.krakow.pl
Donev, Stoil – *Sofia*	sdonev@inrne.bas.bg
Dragović, Vladimir – *Belgrade*	vladad@mi.sanu.ac.rs
Georgieva, Bogdana – *Sofia*	georgieva@fmi.uni-sofia.bg
Gerdjikov, Vladimir – *Sofia*	gerjikov@inrne.bas.bg
Gołubowska, Barbara – *Warsaw*	bgolub@ippt.gov.pl
Gueorguiev, Vesselin – *Sofia*	vgueorguiev.ucm@gmail.com
Hadzhilazova, Mariana – *Sofia*	murryh@obzor.bio21.bas.bg
Ikawa, Osamu – *Iwaki*	ikawa@fukushima-nct.ac.jp
Kasparian, Azniv – *Sofia*	kasparia@fmi.uni-sofia.bg
Katz, Alexander – *Staten Island*	katza@stjohns.edu
Kovalchuk, Vasyl – *Warsaw*	vkoval@ippt.gov.pl
Kyuldjiev, Assen – *Sofia*	kyuljiev@inrne.bas.bg
Martens, Agnieszka – *Warsaw*	amartens@ippt.gov.pl
Mladenov, Ivaïlo – *Sofia*	mladenov@bas.bg
Ludu, Andrei – *Natchitoches*	ludua@nsula.edu
Odzijewicz, Anatol – *Białystok*	aodzijew@uwb.edu.pl
Parisi, Luca – *Salerno*	parisi@sa.infn.it
Quintino, Aurea – *Lisbon*	aurea@ptmat.fc.ul.pt
Rożko, Ewa – *Warsaw*	erozko@ippt.gov.pl
Sansonetto, Nicola – *Verona*	nicola.sansonetto@gmail.com
Sławianowski, Jan Jerzy – *Warsaw*	jslawian@ippt.gov.pl
Valchev, Tihomir – *Sofia*	valtchev@inrne.bas.bg
Vassilev, Vassil – *Sofia*	vasilvas@imbm.bas.bg
Vilasi, Gaetano – *Salerno*	vilasi@sa.infn.it
Yanovski, Alexander –*Cape Touwn*	alexandar.ianovsky@uct.uc.za
Yoshioka, Akira – *Tokyo*	yoshioka@rs.kagu.tus.ac.jp

ANNOUNCEMENT

THE THIRTEENTH INTERNATIONAL CONFERENCE ON GEOMETRY, INTEGRABILITY AND QUANTIZATION

June 3–8, 2011
Sts. Constantine and Elena resort (near Varna), Bulgaria

organized by Ivaïlo M. Mladenov (Sofia), Gaetano Vilasi (Salerno)
and Akira Yoshioka (Tokyo)
with the help of

Advisory Committee

V. Dragovic – *Belgrade*
O. Garay – *Bilbao*
A. Ludu – *Natchitoches*
I. Vaisman – *Haifa*

Standing Committee

V. Gerdjikov – *Sofia*
M. Gürses – *Ankara*
G. Naber – *Philadelphia*
A. Odzijewicz – *Białystok*

with the Financial Support of

BULGARIAN ACADEMY OF SCIENCES
Further information about this meetings can be found at the Conference website:
http://www.bio21.bas.bg/conference

Geometry, Integrability and Quantization XII

Edited by
Ivaïlo M. Mladenov, Gaetano Vilasi and Akira Yoshioka

ISSN 1314–3247

Avangard Prima, Sofia, Bulgaria