DIFFERENTIAL GEOMETRY OF MOVING SURFACES AND ITS RELATION TO SOLITONS

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Abstract. In this article we present an introduction in the geometrical theory of motion of curves and surfaces in $\mathbb{R}^3$, and its relations with the nonlinear integrable systems. The working frame is the Cartan's theory of moving frames together with Cartan connection. The formalism for the motion of curves is constructed in the Serret-Frenet frames as elements of the bundle of adapted frames. The motion of surfaces is investigated in the Gauss-Weingarten frame. We present the relations between types of motions and nonlinear equations and their soliton solutions.

CONTENTS

1. Introduction

Realistic models for many-body or collective interactions involve nonlinear dynamics therefore a large part of interesting and intriguing phenomena cannot be

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explained or predicted by the corresponding linear approximations. Nonlinearity of the dynamics involves, among other things, a weaker type of uniqueness of solutions especially when the solitary waves have compact support (e.g. compactons) or when the configuration space is a compact manifold (e.g. circle, sphere). The most useful nonlinear systems are of course the integrable ones, i.e., those solvable by inverse scattering theory. These particular systems have soliton solutions and infinite number of conservation laws. The traditional nonlinear systems: Korteweg-de Vries, modified Korteweg-de Vries, sine-Gordon, Schrödinger nonlinear equation and Kadomtsev-Petviashvili were investigated in numerous works and books (see for example the following books and the references listed herein [1,6,10–12,17,22,41]).

Many of the integrable nonlinear systems have equivalent representations in terms of differential geometry of curves and surfaces in Riemannian spaces. Such geometric realizations provide a deeper insight into the structure of integrable equations, as well as new physical interpretations [19]. That is why the theory of motions of curves and surfaces, which includes vortices, filaments, and interface dynamics, represents an important emerging field for mathematics, engineering and physics. Other examples of applications include liquid droplets, quantum Hall electron droplets in high magnetic field, nonlinear nuclear surfaces, growth of dendritic crystals, dynamics of polymers, vortex structures in geophysical fluid dynamics and plasma, and motile cells [24].

The occurrence of nonlinearities in the contour dynamics problems involves the connection between this dynamics and the integrable evolution equations. This leads to the existence of soliton-like solutions in the motion of curves, as well as the existence of infinite number of conservation laws that can be put into relation with global geometric quantities.

The problem of the dynamics of moving curves and surfaces is not completely solved. There are systems, especially in the world of microorganisms with complicated moving shapes, where the interaction between the two-dimensional contours (like the cell membrane) and one-dimensional attachments (like flagella, cilia, etc.) cannot be neglected in order to understand the physics of their exquisite motility. A general model for such type of interaction should lie somewhere between the geometry of curves and surfaces, like for example the geometry of a $(1 + \epsilon)$-dimensional manifold.

The structure of the paper is the following. After few Lie groups and geometry prerequisites presented in Section 2 we introduce in Section 3 the Cartan theory of frames in relation to the theory of connection. In Section 4 we derive the theory of motion of curves based on differentiable forms and Cartan connection theory with applications to three-dimensional curves, and relations to soliton theory. Based on these results, we discuss in Section 5 the theory of motion of surfaces, and we

relate it to integrable systems. In Section 6 we present applications of the theory of motion of surfaces.

2. Prerequisites

We assume the reader familiar with elements of topology and differential geometry, for example in the spirit of the monographs [7, 34, 35, 43] or even [31] for direct physics applications. We denote the set of all homeomorphisms between two topological spaces $\mathbb{X}$, $\mathbb{Y}$ by $\mathrm{Hom}(\mathbb{X}, \mathbb{Y})$. In some cases we may want to loosen up the property of homeomorphism by

Definition 1. *A* local homeomorphism *is a function defined on a topological space such that any point from its domain of definition has an open neighborhood on which the function is a homeomorphism.*

Obviously, homeomorphism implies local homeomorphism. As a direct application we mention that any local homeomorphism from a compact space to a connected space is a covering. The proof is based on the fact that the local homeomorphism still preserves the property of being open, and the compactness property insures that we can always choose a finite sub-cover from any open cover of it. Being finite, we can always choose its neighborhoods small enough to be pairwise disjoint, so all the conditions of being a covering map can be accomplished.

An open map is a function between two topological spaces which maps open sets to open sets. Likewise, a closed map is a function which maps closed sets to closed sets. The open or closed maps are not necessarily continuous. A continuous function between topological spaces is called proper if inverse images of compact subsets are compact. An embedding between two topological spaces is a homeomorphism onto its image.

If a topological group G acts on a topological space $\mathbb{X}$ (from the left) with the continuous map $m : G \times \mathbb{X} \to \mathbb{X}$ we denote the triple $(\mathbb{X}, G, m)$ and call it G-space. For a quick introduction in the theory of group actions from the differential geometry point of view we recommend the text [13], while for more technical details and applications we recommend [32]. We have the following definitions

Definition 2. *The set* $G_x = \{g \in G \, ; \, m(g, x) = x\}$ *is called isotropy group of* x *(or stabilizer subgroup of* x*). The set* $O_x = \{m(g, x) \, ; \, g \in G\}$ *is called the orbit of* x*. The set of all orbits is denoted* $\mathbb{X}/G$ *and it is called orbit space and it is a topological space through the quotient induced topology with respect to the canonic projection* $x \to O_x$*.*

The group actions on topological spaces can be classified as follows

Definition 3. *The action of G on $\mathbb{X}$ is free if the isotropy group is trivial for all $x \in \mathbb{X}$. The action of G on $\mathbb{X}$ is proper if the map $\theta : G \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ given by $(g, x) \to (x, m(g, x))$ is a proper function. The action of G on $\mathbb{X}$ is transitive if it possesses only a single group orbit, i.e., if all elements are equivalent. The G-space $(\mathbb{X}, G, m)$ is a homogeneous space if G acts in a transitive way.*

The principal homogeneous space (or torsor) of G is a homogeneous space $\mathbb{X}$ such that the isotropy group of any point is trivial. Equivalently, a principal homogeneous space for a group G is a topological space $\mathbb{X}$ on which G acts freely and transitively, so that for any $x, y \in \mathbb{X}$ there exists a unique $g \in G$ such that $m(g, x) = y$. If $\mathbb{X}$ is a G-space with proper action the quotient space $\mathbb{X}/G$ is Hausdorff.

All these properties and definitions can be extended if the space $\mathbb{X}$ is a differentiable manifold, and G is a Lie group acting on $\mathbb{X}$, case in which the structure $(\mathbb{X}, G, m)$ is called a G-manifold. Moreover, if the action of G is proper and free $\mathbb{X}/G$ has a differentiable manifold structure and the canonical projection $\mathbb{X} \to \mathbb{X}/G$ is a submersion.

A **submersion** is a differentiable map $f : M \to N$ between differentiable manifolds whose differential is everywhere surjective. An **immersion** is a differentiable map between differentiable manifolds whose derivative is everywhere injective (an immersion does not need to be injective itself). The concepts of submersion and immersion are dual to each other. That is they are maximal rank maps such that if $\dim(M) < \dim(N)$ we have an immersion, while if $\dim(M) > \dim(N)$ we have a submersion. A smooth embedding is an injective immersion and a topological embedding (i.e., homeomorphism onto its image) at the same time. An immersion (submersion) maps the coordinates in a faithful way, while an embedding is in addition topological or geometrical structure preserving [5, 18, 20, 40].

For a given differential manifold $\mathbb{X}$ we denote by $T_x\mathbb{X}$, and $T_x^*\mathbb{X}$ the tangent and co-tangent spaces of $\mathbb{X}$ at $x \in \mathbb{X}$, respectively, and in general we denote k-forms by $\theta, \omega \in \Omega^k T_x^*\mathbb{X}$, where Ω^k is the space of skew-symmetric linear forms [2, 32]. If $\{\mathbf{v}_i\}_{i=1,2,\ldots,n} \in T\mathbb{X}$ are vector fields, then the action of an $n-$form ω on them is denoted $\omega(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. By $\mathrm{d}, \mathrm{D}, \imath, \wedge$ we denote the exterior differentiation, covariant exterior derivative, contraction and exterior product with vector fields, respectively, of differential forms. We denote the Lie derivative of a differential form ω at $x \in \mathbb{X}$, with respect to $\mathbf{v} \in T_x\mathbb{X}$ by $\mathbf{v}(\omega)$, and we will use the relation

$$\mathrm{d}\omega(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i-1} \mathbf{v}_i(\omega(\mathbf{v}_1, \ldots, \hat{\mathbf{v}_i}, \ldots, \mathbf{v}_{k+1}))$$

$$+ \frac{1}{k+1} \sum_{i=1, i<j}^{j=k+1} (-1)^{i+j} \omega([\mathbf{v}_i, \mathbf{v}_j], \mathbf{v}_1, \ldots, \hat{\mathbf{v}_i}, \ldots, \hat{\mathbf{v}_j}, \ldots, \mathbf{v}_{k+1}).$$

(1)

The hat placed on a vector means that vector should be omitted from the counting. This expression is important in two cases. First, when a one-form is valued in a Lie algebra of a Lie group, and the two vector fields are invariant to this group. In this case the first term in the RHS is zero, and we have the **Maurer-Cartan equation**

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0. \tag{2}$$

In the second case the form $\mathrm{d}\omega + \omega \wedge \omega$ represents the curvature two-form of a linear connection, also called the first Cartan structure equation.

We denote a fiber bundle by the quadruple $\mathbb{E}(\mathbb{X}, \mathbb{F}, \pi, G)$, where $\mathbb{X}$ is the base space, $\mathbb{F}$ is the standard fiber, π is the canonic projection, and G is the structure Lie group [18, 40]. The inverse image $\pi^{-1}(x) = \mathbb{E}_x$ is called the fiber at x. A cross-section in a bundle is a differentiable injective map $\phi : \mathbb{X} \to \mathbb{E}$ so that $\pi\phi = \mathrm{Id}_{\mathbb{X}}$. Any local fiber $\mathbb{E}_x$ is isomorphic to the standard fiber vector space $\mathbb{F} = \mathbb{V}_n(\mathbb{R})$, and the corresponding isomorphisms depend smoothly on x in the base space. A typical example of vector bundle is the tangent bundle $T\Sigma$ of a parameterized differentiable surface Σ in $\mathbb{R}^3$. The base space is the surface itself, and the tangent bundle is the set of all tangent vectors at all points of the surface. The projection is the assignment for each vector of its initial point. The fiber at x is the tangent plane at x and is a topological vector space. Choosing a unique representative $\mathbb{F} = \mathbb{R}^2$, linear correspondences $\mathbb{E}_x \to \mathbb{F}$ can be constructed, but not uniquely. In this case the structure group G is the full linear group operating on $\mathbb{F}$. A cross-section here is just a differentiable vector field over the surface.

3. Cartan Theory of Frames and Connection

Many differential geometry objects originate directly from the theory of Lie groups and algebras. In the following   will represent an n-dimensional Lie algebra associated to the Lie group G, and $\mathbf{A}, \mathbf{B}, \cdots \in$. A function is called left invariant if it commutes with the left group translations, or with their adjoint representation. In a Lie algebra we can define two important objects which later on will become handy in the definitions of vector bundles and connections [18]. A canonical one-form θ defined on G is a left invariant -valued one-form uniquely determined by the invariance relation $\theta(\mathbf{A}) = \mathbf{A}$. A left invariant one-form ω defined on   fulfils the equation of Maurer-Cartan

$$\mathrm{d}\omega(\mathbf{A}, \mathbf{B}) = -\frac{1}{2}\omega([\mathbf{A}, \mathbf{B}])$$

for any $\mathbf{A}, \mathbf{B} \in$, see also equation (2). As a consequence, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for   we can write

$$\theta = \theta^i \mathbf{e}_i, \qquad \mathrm{d}\theta^i = -\frac{1}{2}C^i_{jk}\theta^j \wedge \theta^k \tag{3}$$

where $[\mathbf{e}_i, \mathbf{e}_j] = C_{ij}^k \mathbf{e}_k$, $k = 1, \ldots, n$ define the structure relations (constants).

When a Lie group G acts on differential manifolds it induces orbits, see Section 1. However, its Lie algebra   is local, and in that it cannot act at different points on a manifold, like G does, except on G itself. In order to generalize this action we enrich the manifold with a fiber bundle structure. In a fiber bundle we have vertical and horizontal displacements by use of the covariant derivative and the connection form, respectively [8, 9].

Definition 4. *A principal bundle over the base space* $\mathbb{X}$ *with structure group* G *is a fiber bundle* $\mathbb{P}(\mathbb{X}, G)$ *on which* G *acts freely* (*on the right*) *and* $\mathbb{X} = \mathbb{P}/G$.

Every fiber $\pi^{-1}(x)$ of a principal bundle is diffeomorphic to G, and actually the base space is just the space of all orbits of the action of G on $\mathbb{P}$. For any element $\mathbf{A} \in$ we can construct a **fundamental vector field** $\mathbf{A}^* : \mathbb{X} \to T\mathbb{X}$ defined by some $x_0 \in \mathbb{X}$, $\mathbf{A}^* = \mathrm{d}[\exp(t\mathbf{A})x_0]/\mathrm{d}t \in T_{x_0}\mathbb{X}$, that is the vector field tangent to the one-parameter Lie subgroups generated by $\mathbf{A}$. The fundamental vector field is tangent to each fiber at each point of $\mathbb{P}$. The best example of principal bundle is the bundle of linear frames (or simply frames) over an n-dimensional manifold $\mathbb{X}$. It is the principal bundle $\mathbb{F}\mathbb{X} = \mathbb{P}(\mathbb{X}, \mathrm{GL}(n, \mathbb{R}))$ which consists of ordered bases in $T_x\mathbb{X}$ defined at each x, namely linear frames.

Theorem 1. *If* $\dim \mathbb{X} = n$ *a linear frame* $v \in \mathcal{F}(\mathbb{X})$ *can be also understood as a linear mapping of some canonical basis of a vector space* $\mathbb{R}^3$ *in* $T\mathbb{X}$, *i.e.,* $u(\mathbf{e}_i) = \mathbf{X}_i$, $i = 1, \ldots, n$.

Moreover, by using the natural inner product of vectors in $\mathbb{R}^n$, we define the bundle $\mathcal{O}\mathcal{F}(\mathbb{X}) = \mathbb{P}(\mathbb{X}, \mathrm{O}(n, \mathbb{R}))$ called **bundle of orthonormal frames** over $\mathbb{X}$.

The bundle of frames explains how the frames at a given point of $\mathbb{X}$ change under the action of a group, but does not relate this to the possible change of the point x itself under the action of the group. In order to combine these two actions, if the manifold $\mathbb{X}$ is n-dimensional we need the concept of associated vector bundle to the principal bundle $\mathbb{P}$. To construct it we begin with $\mathbb{P}(\mathbb{X}, G)$ and use a finite dimensional vector space called standard fiber $\mathbb{F}$ ($\mathbb{F}$ in isomorphisms with some $\mathbb{R}^n$). The new vector bundle is denoted $\mathbb{E}(\mathbb{X}, \mathbb{R}^n, \pi_{\mathbb{E}}, G; \mathbb{P})$, its canonical projection is $\pi_{\mathbb{E}}$, and its space is nothing but the quotient space $\mathbb{E} = (\mathbb{P} \times \mathbb{F})/G$. The tangent bundle $T\mathbb{X} = \bigcup_{x \in \mathbb{X}} T_x\mathbb{X}$ is the associated vector bundle to the principal bundle $\mathcal{F}(\mathbb{X})$ of frames. Basically, the space of frames over $\mathbb{X}$, $\mathcal{F}(\mathbb{X}) = \mathbb{P}(\mathbb{X}, \mathrm{GL}(n, \mathbb{R}))$ with canonical projection π is mapped into the space of directions in $\mathbb{X}$, $T\mathbb{X} = \mathbb{E}(\mathbb{X}, \mathbb{R}^n, \mathrm{GL}(n, \mathbb{R}), \pi_{\mathbb{E}}; \mathbb{F}(\mathbb{X}))$ with canonical projection $\pi_{\mathbb{E}}$.

Now Theorem 1 can be better understood, i.e., the bundle $\mathbb{P} = \mathcal{F}(\mathbb{X})$ consists in frames, and the bundle $\mathbb{E} = T\mathbb{X}$ consists in vectors placed in frames modulo action of G. The local character of each such element is given by the canonical

projections. However, the manifold generated by a fixed frame (at a point) and al possible vectors (at the same point) is a fiber in $T\mathbb{X}$ and it is isomorphic to the generic fiber $\mathbb{F}$. So, any frame $u \in \mathcal{F}(\mathbb{X})$ generates an isomorphisms $\pi^{-1}(x) \ni u :$ $\mathbb{F} \to \pi_{\mathbb{E}}^{-1}(x)$, that is, u gives to any abstract vector from $\mathbb{FX}$ a set of components and places it in a frame. The frame u maps this abstract vector into the tangent space $T\mathbb{X}$ and gives it geometrical meaning. This construction can be seen in parts of Fig. 1. If instead of tangent spaces we use affine spaces constructed upon the tangent spaces, the vector bundle of linear frames becomes the bundle of affine frames.

The quintessence of the vector/frame duality can be presented in a nut-shell by introducing the one-form called the **canonical form** $\theta \in \Omega^1(\mathcal{F}(\mathbb{X}))$ on the principal bundle of frames $\mathcal{F}(\mathbb{X})$ with values in the standard fibre $\mathbb{F}$, see equation (3). The action of the canonical form on a vector $\mathbf{X} \in T\mathcal{F}(\mathbb{X})$ is $\theta(\mathbf{X}) = u^{-1} \circ d\pi(\mathbf{X}) \in \mathbb{F}$. If $\mathbb{X}$ is a n-dimensional affine space, then a point $x \in \mathbb{X}$ is represented by a position vector $\mathbf{r} = x^i \mathbf{e}_i$ whose components are given in a certain frame $\{\mathbf{e}_i\}_{i=1,\ldots,n} = u \in$ $\pi^{-1}(x) \in \mathcal{F}(\mathbb{X})$. The question is: how does this position vector changes with $d\mathbf{r}$ by infinitesimally moving the frame. The answer is given by the canonical form, that is by

$$d\mathbf{r} = \theta(\mathbf{X}) = \theta^i(\mathbf{X})\mathbf{e}_i$$

where $\mathbf{X} \in T_u\mathcal{F}(\mathbb{X})$ describes this infinitesimal motion of the frame in the tangent space to the bundle of frames.

The bundle of frames does not provide a recipe of how frames transform when the base point moves through the base space. In order to provide such a law we need an extra construction which is the Cartan connection on $\mathbb{X}$. It will provide the infinitesimal transformation of a point in the vector bundle when we perform an infinitesimal move in the base. Since the infinitesimal transformations are described by vectors in the tangent space, the Cartan connection will map a point (to be moved) in the vector bundle to a vector in the tangent bundle to the vector bundle (how this point transforms), map depending on a vector in the tangent space of the base (the direction of moving).

Let $\mathbb{M}$ be a differential manifold and $\mathbb{P}(\mathbb{M}, G)$ its principal bundle.

Definition 5. *A connection* Γ *in* $\mathbb{P}(M, G)$ *is the assignment of an* G-*invariant subspace* $\mathbb{H}_p \subset T_p\mathbb{P}$, *for any* $p \in \mathbb{P}$ *and depending differentiable on* p, *called horizontal subspace.*

The orthogonal complement of $\mathbb{H}_p$ is called vertical subspace, it is denoted by $\mathbb{V}_p$, and we have $T_p\mathbb{P} = \mathbb{V}_p \oplus \mathbb{H}_p$. Any vector $\mathbf{X} \in T_p\mathbb{P}$ can be uniquely decomposed in two orthogonal components $\mathbf{X} = v\mathbf{X} + h\mathbf{X}$ each in the corresponding sub space $v\mathbf{X} \in \mathbb{V}_p, h\mathbf{X} \in \mathbb{H}_p$. A horizontal lift of a vector field on is the unique horizontal vector field on $\mathbb{P}$ such that the differential of the canonical projection

on $d\pi : T\mathbb{P} \to T\mathbb{M}$ maps it to the initial vector field. Any parameterized curve in $\mathbb{M}$, and any point $p \in \mathbb{P}$ provide a *lift* of this curve to a unique horizontal (with horizontal tangent vectors) curve in $\mathbb{P}$, to which it canonically projects. As an example, imagine $\mathbb{P}$ as the orthonormal frame bundle over $\mathbb{R}^3$, and a curve in this space. At any point in the base space we can choose a variety of frames, any frame from the local fiber. But there only one which may be a Serret-Frenet frame to that curve, let it be p_0. When we move along the curve that Serret-Frenet frame from the initial point moves from fiber to fiber in a "parallel" way following the curve through its lifted image.

The existence of the Cartan connection on the principal bundle $\mathbb{P}$ allows us to "flag" elements of $\mathbb{P}$ and watch their evolution according to a certain law imposed by this connection, when we move in the base space along some curve. This is the parallel displacement along a certain curve in the base space. We consider x_0 the starting point of a parameterized curve $\gamma \subset \mathbb{X}$, and its local fiber $\pi^{-1}(x_0) \subset \mathbb{P}$. Through any point p_0 in this fiber we can built a unique horizontal lift of γ which canonically maps back on γ. When we move to a different point on γ the intersection between the fiber over this new point and the horizontal lift of γ through p_0 is a unique point of this new fiber. Doing this transport now for various $p_0 \in \pi^{-1}(x_0)$ it is like we map all points p_0 of a fiber into all points of another fiber following the curve. This mapping is actually a fiber isomorphisms, and it is call the parallel displacement of the fibers along the curve.

One of the most important results of differential geometry is that to each connection we can associate a -valued one-form on $\mathbb{P}$

Definition 6. *A connection form ω of a given connection Γ is a differentiable one-form on $\mathbb{P}$ with values in   such that for each $\mathbf{X} \in T_p\mathbb{P}$ we have $\omega(\mathbf{X}) = \{\mathbf{A} \in   ; \mathbf{A}^* = v\mathbf{X}\}$.*

In other words, a connection form maps a vector field $\mathbf{V}$ on $\mathbb{P}$ to a Lie algebra vector whose fundamental vector field is exactly the vertical component of $\mathbf{V}$. In a physicist language a connection form is a vector field defined on a bundle of frames such that its directional derivatives in any directions provide one-dimensional Lie algebras of symmetry (flows) in the vertical component of those directions.

The exterior covariant derivative of the connection form is the **curvature form** $D\omega = \Omega$, and we have the structure equation

$$d\omega = -\frac{1}{2}[\omega,\omega] + \Omega \tag{4}$$

acting on any pair of vector fields on $\mathbb{P}$. The proof is immediate and it is based on equation (1), and on the vertical/horizontal direct sum properties. A connection is flat if and only if its curvature form is null. In a similar manner we define the
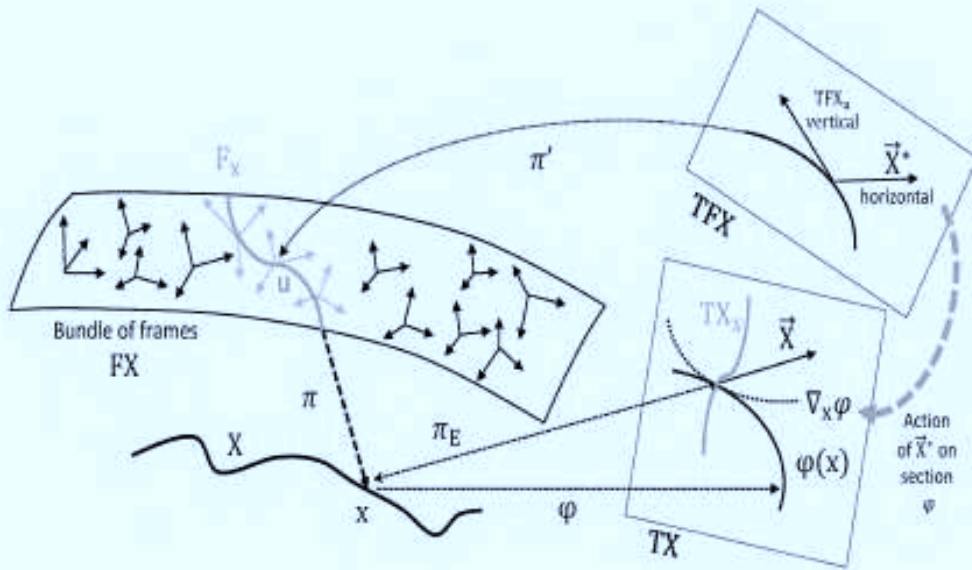
Figure 1. Pictorial interpretation of the covariant derivative. We have the principal bundle of frames $\mathcal{F}(\mathbb{X})$ and its projection π on top of the manifold $\mathbb{X}$, and the tangent bundles to each of these: $T\mathbb{X}$, and $T\mathcal{F}(\mathbb{X})$, respectively, with their projections π_E, π'. We also represented the local fibres. At $T\mathbb{X}_x = \pi_E^{-1}(x) \in T\mathbb{X}$ we have two vectors: the arbitrary direction $\mathbf{X}$, and the vector cross section φ. The first one is horizontally lifted in $T\mathcal{F}(\mathbb{X})$ as $\mathbf{X}^*$ and then acts upon φ generating its covariant derivative $\nabla_{\mathbf{X}}\phi$, as a new cross section (dashed line) in $T\mathbb{X}$.

torsion form $\Theta = D\,\theta$ and we have another structure equation [18, 40]

$$\mathrm{d}\theta = -\frac{1}{2}[\omega, \theta] + \Theta. \tag{5}$$

A connection defined in the bundle of linear frames is a linear connection, and if it is defined in a bundle of affine frames it is an affine connection. On any manifold of positive dimension there are infinitely many affine connections. The choice of an affine connection is equivalent to prescribing a way of differentiating vector fields which satisfies several reasonable properties (linearity and the Leibniz rule). This yields a possible definition of an affine connection as a covariant derivative or (linear) connection on the tangent bundle. A choice of affine connection is also equivalent to a notion of parallel transport, which is a method for transporting tangent vectors along curves. This also defines a parallel transport on the frame bundle. In the bundle of orthonormal frames we have a metric induced by the action of the orthogonal group. So, we define a Riemannian connection (or Levi-Civita connection) a linear connection with zero torsion.

In order to build the covariant derivative of a cross section $\varphi : \mathbb{X} \rightarrow T\mathbb{X}$ in the $\mathbf{X} \in T\mathbb{X}$ direction we have to lift this last vector to its horizontal component $\mathbf{X}^* \in$

$\mathbb{H} \subset T\mathcal{F}(\mathbb{X})$. Following the projections we have $\mathcal{F}(\mathbb{X}) \ni u \to x = \pi(u) \to \varphi(x)$ which actually defines a cross section in $\mathcal{F}(\mathbb{X})$. So, we can apply the directional derivative $\mathbf{X}^*(\varphi(x(u))) = \nabla_{\mathbf{X}}\varphi$, and this is the requested covariant derivative, see Fig. 1. Basically, it is the horizontal component of the directional derivative.

In order to express the connection form ω and consequently its covariant derivative in components us first need to define a canonical basis $\{\mathbf{e}_i\}_{i=1,\ldots,n}$ in the standard fiber $\mathbb{F} \sim \mathbb{R}^n$, and a canonical basis $\{\mathbf{E}_{ij}\}_{i,j=1,\ldots,n}$ for the Lie algebra $(n,\mathbb{R})$. Since the canonical form θ is $\mathbb{R}^n$-valued, and the connection form ω is $(n,\mathbb{R})$-valued we have

$$\theta = \theta^i \mathbf{e}_i, \qquad \omega = \omega^{ij}\mathbf{E}_{ij} \tag{6}$$

while the two structure equations (4-5), can be written now

$$\mathrm{d}\theta^i = -\omega^{ij} \wedge \theta^j + \Theta^i$$
$$\mathrm{d}\omega^{ij} = -\omega^{ik} \wedge \omega^{kj} + \Omega^{ij}. \tag{7}$$

Obviously, for Riemannian connections on manifolds imbedded in flat spaces the structure equations reduce to

$$\mathrm{d}\theta = -\omega \wedge \theta, \qquad \mathrm{d}\omega = -\omega \wedge \omega \tag{8}$$

with the simple interpretation, [39], that the canonical form, equation (6) accounts for the position changes at a change of frame, and the connection form accounts for the twisting of the frames when we move the point

$$\mathrm{d}\mathbf{r} = \theta^i \mathbf{e}_i \quad \text{change of position}$$
$$\mathrm{d}\mathbf{e}_i = \omega^{ij}\mathbf{e}_j \quad \text{change of frame}. \tag{9}$$

Let us assign local coordinates in the n-dimensional space $\mathbb{X}$ in the form $x \leftrightarrow (x^i)$. The coordinates in the tangent bundle are covariant vectors $\partial/\partial x^i$, a frame in $\mathcal{F}(\mathbb{X})$ is described by the vector fields $\mathbf{X} = X_j^i(x)\partial/\partial x^i$, and the local coordinates in the bundle of frames are (x^i, X_j^i), namely a point and a basis of n-vector fields. Consequently, a frame $u \in \mathcal{F}(\mathbb{X})$ is represented by the components of the basis fields $u \leftrightarrow X_j^i$ which is exactly the $n \times n$ linear isomorphism u from $\mathbb{F}$ onto $T_x\mathbb{X}$. The canonical one-form and the connection one-form can be written

$$\theta = (X^{-1})_j^i \mathrm{d}x^j \mathbf{e}_i$$
$$\omega = ((X^{-1})_k^i \mathrm{d}X_j^k + (X^{-1})_k^i \Gamma_{ml}^k X_j^l \mathrm{d}x^m)\mathbf{E}_i^j$$

where the connection coefficients Γ are the Christoffel's symbols. The basis vectors $\partial/\partial x^j$ in $T\mathbb{X}$ can be horizontally lifted to

$$\left(\frac{\partial}{\partial x^j}\right)^* = \frac{\partial}{\partial x^j} - \Gamma_{jk}^i X_l^k \frac{\partial}{\partial X_l^i}$$

that is we subtract from the tangent vector its vertical component, which is represented by its connection part (ω or Γ). The covariant derivative acts on the basis (covariant) vectors as follows

$$\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i} = \Gamma^k_{ji} \frac{\partial}{\partial x^k}. \tag{10}$$

Equation (10) and the linearity of the covariant derivative direct us to the coordinate expression of the covariant derivative of a vector field $\mathbf{V} = V_i \partial/\partial x^i$ defined on $\mathbb{X}$ with respect to the directions of the local frame

$$\nabla_j V_i = \frac{\partial V_i}{\partial x^j} - \Gamma^k_{ij} V_k.$$

We illustrate these constructions with an example. Let us have a unit radius spherical surface $\mathbb{X} = \mathbb{S}^2$ embedded in $\mathbb{R}^3$ with coordinates $x^1 = \theta \in [0, \pi], x^2 = \phi \in [0, 2\pi)$. The tangent space is $T\mathbb{S}^2$ generated by the basis vectors $\{\mathbf{e}_\theta, \mathbf{e}_\phi\}$. The bundle of the orthonormal frames $\mathcal{O}(\mathbb{S}^2)$ has coordinates $(\theta, \phi, \hat{R}(\alpha))$ where the last one represents an element of the Lie structure group $O(2, \mathbb{R})$, i.e., a rotation of angle α of the tangent frame around the normal to the sphere. The covariant derivatives have the form

$$\nabla_{\mathbf{e}_\theta} \mathbf{e}_\theta = 0, \qquad \nabla_{\mathbf{e}_\phi} \mathbf{e}_\theta = \mathbf{e}_\phi \cot \theta, \qquad \nabla_{\mathbf{e}_\phi} \mathbf{e}_\phi = \mathbf{e}_\theta \sin \theta \cos \theta$$

and the horizontal lift of the basis vectors is

$$\mathbf{e}^*_\theta = \mathbf{e}_\theta - \mathbf{n} \cos \theta, \qquad \mathbf{e}^*_\phi = \mathbf{e}_\phi - \mathbf{n} \sin \theta \cos \theta.$$

We can check this by noticing that at $\theta = \pi/2$ the covariant derivatives cancel, as well as the vertical projections, which is correct since this equatorial circle is actually a geodesic and performs a parallel transport for the tangent vectors. If we want to find, for example (see [39] pp. 66), how is parallel-transported a tangent vector field we can choose a vector which is $\mathbf{e}_\phi$ at an initial point, and we transport it along a parallel to the sphere at $\theta = \theta_0$, parameterized by $t \in [0, 2\pi)$. The resulting parallel-translating vector is

$$\mathbf{V}(t) = \sin(\theta_0) \sin(t \cos \theta_0) \mathbf{e}_\theta + \cos(t \cos \theta_0) \mathbf{e}_\phi, \qquad \nabla_{\mathbf{e}_\phi} \mathbf{V} = 0.$$

Corollary 1. *In a Riemannian manifold, that is on a manifold* $(\mathbb{X}, g_{jk})$ *endowed with a* $(0, 2)$ *type of symmetric nonsingular tensor field* $g_{ij}(x)$ *of class at least* $C_1(\mathbb{X})$, *is to obtain the Christoffel's symbols of the first kind from the metric*

$$\Gamma^{(g)}_{ijk} = \frac{1}{2} \left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right)$$

and

$$\Gamma^{(g)i}_{jk} = g^{li} \Gamma^{(g)}_{jlk}.$$

4. The Theory of Motion of Curves

In the following we use the traditional definition of a parameterized curve from [5, 20, 39].

Definition 7. *A parametrized curve is a differentiable (class C_k) map $\mathbf{r}(u)$ from the open real interval $u \in I = (a, b) \subset \mathbb{R}$ into $\mathbb{R}^3$. If $k = \infty$ the parameterized curve is smooth.*

The metric of a parametrized curve is

$$g(u) = \frac{\partial x^i}{\partial u} \frac{\partial x^i}{\partial u} = \mathbf{r}_u \cdot \mathbf{r}_u.$$

The corresponding Serret-Frenet formulas are

$$\begin{pmatrix} \mathbf{t}_s \\ \mathbf{n}_s \\ \mathbf{b}_s \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{11}$$

Here κ, τ are the curvature and torsion and the arc-length and unit tangent are given by

$$s(u) = \int_0^u \sqrt{g(\tilde{u})} d\tilde{u}, \qquad \mathbf{t} = \frac{\partial \mathbf{r}}{\partial s}.$$

A moving parameterized curve $\gamma(t) \subset \mathbb{R}^3$, which can be described at any moment of time by the Serret-Frenet frames, generates a set of points Σ_γ. Any parameterized surface $\Sigma \subset \mathbb{R}^3$ can be described by its tangent bundle $T\Sigma$, but we need a more sophisticated vector bundle to describe the hypothetical surface obtained through the curve motion than the available tangent bundle $T\Sigma_\gamma$. Moreover, in order to approach a moving curve as a regular surface some restrictions should apply to this motion. The curve should not self-intersect during the motion in order to have fulfilled the immersion condition for a regular surface. The time dependence of the position of any point on the curve should be a differentiable function, which requests some extra structure relations (or compatibility equations) between the mixed time and arc-length second order derivatives. In conclusion, the surface obtained by the motion of the curve has to fulfil some extra constraints.

In order to define the differentiable motion of a curve in arbitrary direction, like for example along $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$, we have to define vector fields along the curve that do not belong only to the tangent space of the curve $T\gamma$. However, it would be simpler if we could describe such vector fields in the moving Serret-Frenet frames. For that we have to immerse the local Serret-Frenet frames in the frame bundle for the affine space $\mathbb{R}^3$.

The immersion can be obtained by mapping different vector bundles over orthogonal groups $O(n, \mathbb{R})$ into vector sub-bundles over orthogonal subgroups, correspondingly. Then, the homomorphisms between different orthogonal groups provide the requested mappings between the frame bundles. If such mappings are constructed, by using their pull-backs, the covariant derivative in $\mathbb{R}^3$ induces a covariant derivative in the curve. This allows us to define vertical and horizontal vector spaces for the vector bundle of the frames along the curve. Consequently we can identify "orthogonal" spaces to the curve, and the vectors in these spaces will provide the local directions of motion of the curve.

The imbedded parameterized curve γ is a Riemannian sub-manifold of $\mathbb{R}^3$, and it has a natural Riemannian connection defined on it. Let $x \in \gamma$ and we have the vector subspace relation $T_x\gamma \lhd T_x\mathbb{R}^3$. We denote by $(T_x\mathbb{R}^3)^\iota$ the orthogonal complement of $T_x\gamma$ in $T_x\mathbb{R}^3$ which is called the normal space to the immersion γ at x. We can build the following two orthogonal frame bundles, and when we denote them we skip from the notation the structure groups, which obviously are the corresponding orthogonal groups. We have $\mathcal{OF}(\gamma)$ over γ with canonical projection π', and $\mathcal{OF}(\mathbb{R}^3)$ over $\mathbb{R}^3$ with canonical projection π. Also, we can factorize $\mathcal{OF}(\mathbb{R}^3)/\gamma = \{v \in \mathcal{OF}(\mathbb{R}^3); \pi(v) \in \gamma\}$ which is a principal bundle of orthonormal frames over γ with symmetry group the orthogonal real Lie group $O(3, \mathbb{R})$.

Definition 8. *We define the bundle of adapted frames $\mathcal{OF}(\mathbb{R}^3, \gamma)$ over γ with symmetry group* $O(2, \mathbb{R}) \times O(1, \mathbb{R})$.

This is actually a sub-bundle of $\mathcal{OF}(\mathbb{R}^3)/\gamma$ obtained through the map i (see the diagram in equation (2) in a natural way: it contains the frames over $\mathbb{R}^3$ which are also frames over the curve, and have one axis along the tangent to the curve. The $O(2, \mathbb{R})$ part in the symmetry group takes care of the possible rotations of this frames around the curve tangent, while the $O(1, \mathbb{R}) = \{1, -1\}$ part describes the two possible chiralities along the curve. Mapping of the three-dimensional vectors along the curve, and in the normal plane induces two orthogonal Lie groups natural homomorphisms $h' : O(2, \mathbb{R}) \times O(1, \mathbb{R}) \rightarrow O(1, \mathbb{R})$ and $h'' : O(1, \mathbb{R}) \times O(2, \mathbb{R}) \rightarrow O(2, \mathbb{R})$, which induce on their own two corresponding fiber bundles homomorphisms which we denoted with same letters, see Theorem 2.

Theorem 2. *The bundle of adapted frames in* Definition 8 *can be constructed with the homomorphisms* h', h'', i, π *according to the following diagram*

$$
\begin{array}{ccc}
\mathcal{OF}(\mathbb{R}^3) & \xrightarrow[\;O(3,\mathbb{R})\;]{\pi} & \mathbb{R}^3 \\
\;\uparrow{\scriptstyle j} & & \\
\mathcal{OF}(\mathbb{R}^3)/\gamma & \xrightarrow[\;O(3,\mathbb{R})\;]{\pi} & \gamma \\
\;\uparrow{\scriptstyle i} & & \\
\end{array}
$$

$$
\mathcal{OF}(\gamma) = \mathcal{OF}(\mathbb{R}^3,\gamma)/O(2,\mathbb{R}) \xleftarrow{\;h'\;} \mathcal{OF}(\mathbb{R}^3,\gamma) \xrightarrow{\;h''\;} \mathcal{OF}(\mathbb{R}^3,\gamma)/O(1,\mathbb{R})
$$

$$
\begin{array}{ccccc}
O(1,\mathbb{R})\Big\downarrow{\scriptstyle \pi'} & & O(1,\mathbb{R})\times O(2,\mathbb{R})\Big\downarrow{\scriptstyle \pi} & & O(2,\mathbb{R})\Big\downarrow{\scriptstyle \pi''} \\
\gamma & & \gamma & & \gamma
\end{array}
$$

Now we can construct the vector normal bundle of γ as $T(\gamma)^i = \bigcup_{x\in\gamma}(T_x\gamma)^i$ associated to the bundle of normal frames, with standard fibre $\mathbb{R}^2$ and group $O(2,\mathbb{R})$. If we denote by Γ_3 the Riemannian connection form on $\mathcal{OF}(\mathbb{R}^3)$ then the composite pull-back $i^*j^*\Gamma_3$ is the connection form in $\mathcal{OF}(\mathbb{R}^3,\gamma)$. Geometrically this connection form defines parallel displacement of the normal space $T_x\gamma^i$ onto the normal space $T_y\gamma^i$ along the curve γ.

In the following we express the covariant derivative for the curve. We denote the directional and covariant derivatives in $\mathbb{R}^3$ along $\mathbf{v} \in T\mathbb{R}^3$ by $D_{\mathbf{v}} = \nabla_{\mathbf{v}}$, and we assign a basis $\{\mathbf{e}_i\}$ in $T\mathbb{R}^3$. We need the expression of the covariant derivative $\nabla_i = \nabla_{\mathbf{e}_i}$ from equation (10). For imbedded manifolds the connection Γ simply becomes the second fundamental form defined on the submanifold (see [18], Chapter VII, [39] pp. 64, or [5] Section 4-4) and the result is called Gauss' formula, or Weingarten's formula, function if $\mathbf{V}$ belongs to the tangent or normal space, respectively

$$
\nabla_{\mathbf{e}_i}\mathbf{v} = D_{\mathbf{e}_i}\mathbf{v} - \mathbf{\Pi}(\mathbf{e}_i,\mathbf{v}). \tag{12}
$$

The vector $\mathbf{\Pi}$ is the vertical component of the directional derivative, usually called the **second fundamental form** defined on $\mathbb{X}$ with values in the vertical space (we remember that if X is a surface with unit normal $\mathbf{n}$ we have $\mathbf{\Pi} = \Pi\mathbf{n}$. For any two vector fields $\mathbf{v}, \mathbf{w} \in T\gamma$ we define the covariant derivative associated to the (natural) Riemannian connection of γ at a point $x \in \gamma$, equation (10)

$$
(\nabla_{\mathbf{v}}\mathbf{w})_x = (D_{\mathbf{v}}\mathbf{w})_x - \mathbf{\Pi}_x(\mathbf{v},\mathbf{w}) \in T_x\gamma. \tag{13}
$$

Here $\mathbf{\Pi}_x(\mathbf{v},\mathbf{w}) \in T_x\gamma^i$ is the second fundamental form of γ at x, i.e., a symmetric bilinear differential form with values in the normal space to γ. The vector second fundamental form $\mathbf{\Pi}$ allows us to define directional derivatives along the normal space to γ at points on γ.

In the following we give an example in coordinates. We know we can always choose two differential orthonormal fields of vectors ξ_1, ξ_2 (i.e., two sections)
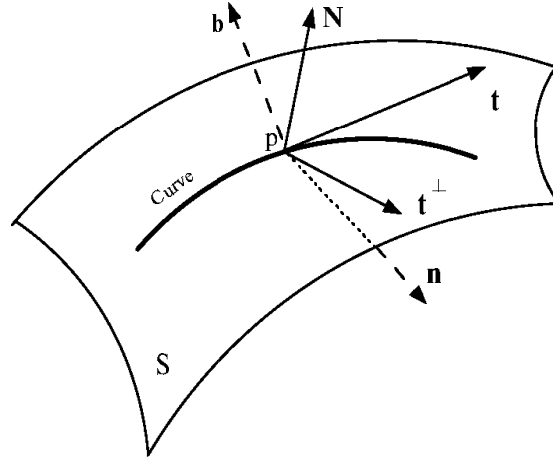
Figure 2. A curve on a surface generates a Darboux frame formed by the vector fields $\{\mathbf{t}, \mathbf{N}, \mathbf{t}^i, \}$, the unit tangent, the principal normal, and their cross product, respectively.

of the normal bundle $T\gamma^i$. Let us also choose $x_0 \in \gamma$ and note that it is always possible to choose an adapted orthogonal frame with a system of normal coordinates $\{y^1, y^2, y^3\}$ with origin in x_0 such that $(\partial/\partial y^1)_{x_0}$ spans $T_{x_0}\gamma$ and $\{\xi_1 = (\partial/\partial y^2)_{x_0}, \xi_2 = (\partial/\partial y^3)_{x_0}\}$ spans $T_{x_0}\gamma^i$. Let s be the arc-length in a neighborhood $U(x_0) \subset \gamma$ and let $y^i = y^i(s)$ be the equations describing the imbedding of U into $\mathbb{R}^3$. We have the action of the second fundamental form Π on tangent vectors of γ given by

$$\Pi\left(\left.\frac{\partial}{\partial s}\right|_{x_0}, \left.\frac{\partial}{\partial s}\right|_{x_0}\right) = \left(\frac{\partial^2 y^1}{\partial s^2}\right)_{x_0}\frac{\partial}{\partial y^1} + \left(\frac{\partial^2 y^2}{\partial s^2}\right)_{x_0}\frac{\partial}{\partial y^2}. \tag{14}$$

The proof is simple and it is based on direct calculation of the Hessian of transformation from x to y coordinates, and on the fact that the Christoffel symbols for the Riemannian connection in $\mathbb{R}^3$ are zero (see e.g. the second volume of [18], Chapter VII). It is easy to check that equation (14) includes the Serret-Frenet relations (11), namely equation (14) represents $\Pi(\mathbf{t}, \mathbf{t}) = \kappa \mathbf{n}$. Let us choose $y^1 = s, y^2 = -\mathbf{r}(s_0) \cdot \mathbf{n}(s_0)$, and $y^3 = \mathbf{r}(s_0) \cdot \mathbf{b}(s_0)$. We have

$$\left.\frac{\partial^2 y^2}{\partial s^2}\right|_{s_0} = -\frac{\partial}{\partial s}(\mathbf{r}_s \cdot \mathbf{n} + \mathbf{r} \cdot \mathbf{n}_s)_{s_0} = (\tau_s y^3 + \tau^2 y^2 - \kappa_s y^1 + \kappa + \kappa^2 y^2)_{s_0} = \kappa.$$

In the same way we obtain $\partial^2 y^3/\partial s^2 = 0$ at s_0, which proves the affirmation. In the following we relate the general frame bundle formalism developed in Section 3 to three-dimensional curve motions in space. On each point of arc-length coordinate s along the parameterized curve γ we define the adapted (orthonormal) Serret-Frenet frame $\{\mathbf{e}_i\}_{i=1,2,3} = \{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of vectors in the principal bundle $\mathcal{OF}(\mathbb{R}^3, \gamma)$ over γ, equation (2). Let be (s, n, b) the local coordinates in this frames, and $(s, n, b, \alpha_1, \alpha_2, \alpha_3)$ local coordinates in the principal bundle, where α_i

represent the three angles of frame rotations in $O(3, \mathbb{R})$. The canonical one-form has the generic expression

$$\theta = \theta_1 \mathrm{d}s + \theta_2 \mathrm{d}n + \theta_3 \mathrm{d}b + \sum_{i=1}^{3} \theta_i \mathrm{d}\alpha_i.$$

Its action on tangent vectors from the principal bundle is given by equations (10) in the form

$$\mathrm{d}\mathbf{r} = \theta^i(\mathbf{X})\mathbf{e}_i = W\mathbf{t} + U\mathbf{n} + B\mathbf{b} \tag{15}$$

with W, U, B arbitrary one-form coefficients. When we consider the time motion of the curve these coefficients become the pull-back one-forms of a cross-section in the principal bundle determined by γ. Namely, they are the coefficients of the velocity of the curve in the local Serret-Frenet frames

$$\mathrm{d}\mathbf{r} = \mathbf{V}(s,t)\mathrm{d}t = \frac{\partial \mathbf{r}}{\partial t}\mathrm{d}t = (W\mathrm{d}t)\mathbf{t} + (U\mathrm{d}t)\mathbf{n} + (B\mathrm{d}t)\mathbf{b}$$

according to the definition of curve velocity introduced, for example, in [14,21,23, 29,36]. We mention that there should be no notation confusion between t as time parameter and t as tangent unit vector. Let us denote by Γ_{ij}^k the Christoffel symbols associated with the connection defined on this principal bundle. We determine them by using equation (13)

$$\mathrm{D}_{\mathbf{t}}\,\mathbf{t} = \kappa\mathbf{n} \to \nabla_1\mathbf{e}_1 = \mathrm{D}_1\,\mathbf{e}_1 - \mathbf{\Pi}(\mathbf{e}_1, \mathbf{e}_1) = 0, \quad \text{so} \quad \Gamma_{11}^1 = 0$$

$$\mathrm{D}_{\mathbf{t}}\,\mathbf{n} = -\kappa\mathbf{t} + \tau\mathbf{b} \to \nabla_1\mathbf{e}_2 = \mathrm{D}_1\,\mathbf{e}_2 - \mathbf{\Pi}(\mathbf{e}_1, \mathbf{e}_2) = -\kappa\mathbf{e}_1, \quad \text{so} \quad \Gamma_{12}^1 = -\kappa$$

$$\dots$$

$$\mathrm{D}_{\mathbf{b}}\,\mathbf{b} = -\mathbf{b}\cdot\frac{\partial\mathbf{t}}{\partial b}\mathbf{t} - \mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial b}\mathbf{n} \to \nabla_3\mathbf{e}_3 = \mathrm{D}_3\,\mathbf{e}_3 - \mathbf{\Pi}(\mathbf{e}_3, \mathbf{e}_3)$$

$$= -\mathbf{b}\cdot\frac{\partial\mathbf{t}}{\partial b}\mathbf{e}_1, \quad \text{so} \quad \Gamma_{33}^1 = -\mathbf{b}\cdot\frac{\partial\mathbf{t}}{\partial b}.$$

In order to obtain the connection form, in addition to the Christoffel symbols, we need the transformations of the orthonormal adapted frames in the bundle of frames in the form of three 2×2 rotation matrices $\hat{R}$ as one-parameter Lie subgroups of $O(2, \mathbb{R})$

$$\frac{\partial\mathbf{e}_i}{\partial x^q} = \hat{R}_q^{ij}\mathbf{e}_j$$

with $i = 2, 3$, $q = 1, 2, 3$ and $x^1 = s, x^2 = n, x^3 = b$. For $q = 1$ we have obviously

$$\hat{R}_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}.$$

By applying the structure conditions equations (7) in the form of equations (8) we obtain the relations describing the change of frames along the local frame directions, that is the Gauss-Weingarten equations (10), in the form

$$d\mathbf{e}_i = \omega_q^{ij}\,dx^q\mathbf{e}_j.$$

There is a simple curvilinear coordinates-like language in which the connection form coefficients have an intuitive form [36]

$$\frac{\partial}{\partial n}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & -\Gamma_{22}^1 & -\Gamma_{23}^1\\ \Gamma_{22}^1 & 0 & \mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial n}\\ \Gamma_{23}^1 & -\mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial n} & 0 \end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} \tag{16}$$

$$\frac{\partial}{\partial b}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & -\Gamma_{32}^2 & -\Gamma_{33}^1\\ \Gamma_{32}^2 & 0 & \mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial b}\\ \Gamma_{33}^1 & -\mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial b} & 0 \end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}. \tag{17}$$

Of course the derivatives with respect of s are the Serret-Frenet relations equation. Moreover, by defining the vector field

$$\mathbf{X} = \mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n} + \mathbf{b}\frac{\partial}{\partial b} \in T\mathcal{OF}(\mathbb{R}^3,\gamma)$$

we can construct the other curvilinear differential operators like the curvilinear divergence of the tangent

$$\operatorname{div}\mathbf{t} = \mathbf{n}\cdot\frac{\partial\mathbf{t}}{\partial n} + \mathbf{b}\cdot\frac{\partial\mathbf{t}}{\partial b}$$

where we used $\mathbf{t}\cdot\partial\mathbf{t}/\partial s = 0$

$$\operatorname{div}\mathbf{n} = -\kappa + \mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial b}, \qquad \operatorname{div}\mathbf{b} = -\mathbf{b}\cdot\frac{\partial\mathbf{n}}{\partial n}.$$

The curvilinear curl has the form

$$\operatorname{rot}\mathbf{t} = \mathbf{t}\times\frac{\partial\mathbf{t}}{\partial s} + \mathbf{n}\times\frac{\partial\mathbf{t}}{\partial n} + \mathbf{b}\times\frac{\partial\mathbf{t}}{\partial b}$$

$$= \kappa\mathbf{b} + \mathbf{n}\times\left(\frac{\partial\mathbf{t}}{\partial n}\cdot\mathbf{b}\right)\mathbf{b} + \mathbf{b}\times\left(\frac{\partial\mathbf{t}}{\partial b}\cdot\mathbf{n}\right)\mathbf{n} = \kappa\mathbf{b} + \Omega_s\mathbf{t}$$

where $\Omega_s = \mathbf{t}\cdot(\operatorname{rot}\mathbf{t})$ is called the total moment of the $\mathbf{t}$ field or abnormality. Similarly we have

$$\operatorname{rot}\mathbf{n} = -(\operatorname{div}\mathbf{b})\mathbf{t} + \Omega_n\mathbf{n} - \Gamma_{22}^1\mathbf{b}, \qquad \operatorname{rot}\mathbf{b} = (\kappa + \operatorname{div}\mathbf{n})\mathbf{t} + \Gamma_{33}^1\mathbf{n} + \Omega_b\mathbf{b}$$

with $\Omega_n = \Gamma_{32}^2 - \tau$, $\Omega_b = -\Gamma_{23}^1 - \tau$ being the other two abnormalities.

It is interesting to mention a relation between the three rotational abnormalities

$$\Omega_s - \tau = \frac{1}{2}(\Omega_s + \Omega_n + \Omega_b).$$

According to [36] this relation is a consequence of the Dupin's theorem (i.e., the intersections of surfaces of orthogonal curvilinear coordinates are lines of curvature). Expressing the motion of three-dimensional curves through the abnormalities forms has the advantage of classification of motions in three categories, function of which abnormality we choose to keep zero. For example, the well known binormal motion happens when the normal abnormality vanishes, $\Omega_n = 0$ which is typical vortex filament motion. In the binormal motion the s—lines and b—lines are contained in a one-parameter surface U = constant, perpendicular on $\mathbf{n}$ = $\operatorname{grad} U/|\operatorname{grad} U|$. Consequently, the normal field is quasi-potential (is derived as the product between a scalar function and a gradient). All equations and forms of the surface generated by a binormal motion can be easy calculated. For example, following the Weatherburn theorem ([42] XII, 121) $K = \mathbf{N} \cdot \operatorname{rot}_U \mathbf{t} \times \operatorname{rot}_U \mathbf{b}$, we have the Gaussian and mean curvature in the form

$$K = -\kappa(\kappa + \operatorname{div} \mathbf{n}) - \tau^2, \qquad H = \operatorname{div} \mathbf{n}$$

respectively, while the Gauss-Codazzi equations and Gauss' Theorema Egregium are encapsulated in a very simple expression

$$K = \frac{\partial \Gamma^1_{33}}{\partial s} + (\Gamma^1_{33})^2.$$

In the case when the b parameter can be considered time (the so-called pure binormal motions) it results that $\mathbf{r}_b = \mathbf{r}_t = g^{1/2}\mathbf{b}$ and, most importantly, $s_t = 0$ which draws the conclusion

Corollary 2. *Pure binormal motions are possible only for inextensible curves.*

This could be the geometrical insight of the strong stability of vortex filaments having this type of motion.

From the structure equations for the connection form $d\omega = -\omega \wedge \omega + \Omega$ we obtain the expression of the curve motion in time, as function of the velocity. It is easy to note that $\partial b/\partial t = B, \partial n/\partial t = U$ and we have

$$\frac{\partial}{\partial s}\begin{pmatrix} W \\ U \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & -\Gamma^1_{22} & -\Gamma^2_{32} + \tau \\ 0 & -\Gamma^1_{23} - \tau & -\Gamma^1_{33} \end{pmatrix}\begin{pmatrix} W \\ U \\ B \end{pmatrix} + \begin{pmatrix} \frac{\dot{g}}{2g} \\ 0 \\ 0 \end{pmatrix}.$$

Here we note that the change in time of the arc-length accounts for a non-zero curvature of the connection. We can re-write Serret-Frenet equations by using equations (15-17), and we obtain the evolution of the frame in terms of the components

of the velocity of the curve

$$\frac{d\mathbf{t}}{dt} = \left(\frac{\partial U}{\partial s} - \tau B + \kappa W\right)\mathbf{n} + \left(\frac{\partial B}{\partial s} + \tau U\right)\mathbf{b}$$

$$\frac{d\mathbf{n}}{dt} = -\left(\frac{\partial U}{\partial s} - \tau B + \kappa W\right)\mathbf{t} + \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial B}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau B + \kappa W\right)\right]\mathbf{b}$$

$$\frac{d\mathbf{b}}{dt} = -\left(\frac{\partial B}{\partial s} + \tau U\right)\mathbf{t} - \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial B}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau B + \kappa W\right)\right]\mathbf{n}.$$

$$(18)$$

The kinematics of the metric is described by

$$\frac{dg}{dt} = 2g\left(\frac{\partial W}{\partial s} - \kappa U\right). \tag{19}$$

The total (material) time derivative can be broken into the partial derivative and an extra term

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(W - \int^s \kappa U d\tilde{s}\right)\frac{\partial}{\partial s}.$$

From the above relations we can derive the dynamical connections between the velocity components and curvature and torsion of γ

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 U}{\partial s^2} + (\kappa^2 - \tau^2)U + \frac{\partial \kappa}{\partial s}\int^s \kappa U d\tilde{s} - 2\tau\frac{\partial B}{\partial s} - B\frac{\partial \tau}{\partial s}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial B}{\partial s} + \tau U\right)\right. \tag{20}$$

$$\left. + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau B\right) + \tau\int^s \kappa U ds'\right] + \kappa\tau U + \kappa\frac{\partial B}{\partial s}.$$

On behalf of the fundamental theorem of curves once we integrate equations (18) and (19) and find κ, τ the curve is uniquely determined in the arc-length parametrization, up to rigid motions in space. Obviously, as a check, if we cancel the torsion we obtain the equations of motion for the two-dimensional curves. In conclusion we can formulate the following affirmation

Corollary 3. *The motion of a parametrized real curve γ can be described in two similar formalisms. One possibility is to integrate the nonlinear partial differential system of equations* (16-17) *and to obtain as solutions the parametric evolution of the Serret-Frenet frame in the principal bundle $\mathcal{OF}(\mathbb{R}^3, \gamma)$ as*

$$(\mathbf{t}(s, n, b),\ \mathbf{n}(s, n, b),\ \mathbf{b}(s, n, b)).$$

Another possibility is to consider the motion given by the time dependent functions (W, U, B) in equation (15), *and to integrate the time-dependent nonlinear differential equations* (20) *for curvature and torsion, equation* (19) *for the metric, and*

finally equations (18) *for the Serret-Frenet unit vectors*

$$(\kappa(s,t), \tau(s,t), g(s,t), \mathbf{t}(s,t), \mathbf{n}(s,t), \mathbf{b}(s,t).$$

In order to map the three-dimensional curve motion into a nonlinear integrable system we follow [15,21], as well as an older suggestion of Darboux, and we introduce the complex curvature–torsion function by the **Hasimoto transformation**

$$\Phi(s,t) = \kappa(s,t) \exp\left(i \int^s \tau(s',t)d\tilde{s}\right). \tag{21}$$

By coupling equations (18-19) with equation (21) we obtain a complex equation in the form

$$\frac{\partial \Phi}{\partial t} = \left[\frac{\partial^2}{\partial s^2} + |\Phi|^2 + i\Phi \int^s \tau \Phi^* d\tilde{s} + \frac{\partial \Phi}{\partial s} \int^s \Phi^* d\tilde{s}\right] U \exp\left(i \int^s \tau(\tilde{s},t)d\tilde{s}\right)$$

$$+ \left[i\frac{\partial^2}{\partial s^2} + i|\Phi|^2 + \Phi \int^s \tau \Phi^* d\tilde{s} - i\Phi \int^s \frac{\partial \Phi^*}{\partial \tilde{s}} d\tilde{s}\right] B \exp\left(i \int^s \tau(\tilde{s},t)d\tilde{s}\right) \tag{22}$$

where $*$ is complex conjugation, and the square parentheses are operators acting to the right. A simple example is immediate: if we choose a binormal type of motion with $B = \kappa$, and zero normal velocity $U = 0$, equation (22) reduces to the (focusing) version of the **nonlinear Schrödinger equation**

$$i\frac{\partial \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial s^2} + \frac{3}{2}|\Phi|^2\frac{\partial \Phi}{\partial s} = 0. \tag{23}$$

If we consider a more complex type of motion with $U = -\kappa_s$, and $B = -\kappa\tau$ we obtain instead the equation

$$\frac{\partial \Phi}{\partial t} + \frac{\partial^3 \Phi}{\partial s^3} + \frac{3}{2}|\Phi|^2\frac{\partial \Phi}{\partial s} = 0 \tag{24}$$

which is an **mKdV equation** for a complex function. Of course equations (23)–(24) reduce to the previously studied two-dimensional case if $\tau = 0$, i.e., the imaginary part of all equations vanishes.

Another example of mapping is provided by the binormal motion of curves with constant curvature, i.e., $\Omega_n = 0$ (or $\partial \mathbf{r}/\partial b = g^{1/2}\mathbf{b}$) and $\kappa =$const. The resulting equation for torsion can be mapped, after a scaling, into either the **Dym nonlinear equation**, or the **Camassa-Holm equation** from hydrodynamics. If the initial curve is a helix, a binormal motion with constant curvature generates the so-called soliton surfaces, [36], which are periodic surfaces of revolution representing the motion of a soliton along a circular helix.

Several examples of curve motions associated to integrable nonlinear systems are described and illustrated in [24]. There are many physical systems that can be described using the theory of curve motion. The most important, and directly related

to the integrable nonlinear equations are the application related to filaments, vortex filaments, and vortices either in fluid dynamics or in mesoscopic superconductivity [25]. More modern applications are related to curve diffusion, image and pattern processing and recognition.

5. Theory of Motion of Surfaces

In the following we consider a time parameterized family of regular surfaces defined by the immersions $\mathbf{r}(t, u^\alpha) : [0, \infty] \times U \subset \mathbb{R} \times \mathbb{R}^2 \to \Sigma(t) \subset \mathbb{R}^3$. We assume it is possible to define at any moment of time t an orthonormal basis $\{\mathbf{e}_\alpha, \mathbf{N}\}_{\alpha=1,2}$ in $\mathbb{R}^3$ where

$$\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \left| \frac{\partial \mathbf{r}}{\partial u^\alpha} \right|^{-1}.$$

In the following we denote by $g_{\mu\nu} = \mathbf{r}_u \cdot \mathbf{r}_v$ the first fundamental form. We apply the Cartan frame formalism described in Section 3 for the principal bundle of adapted frames $\mathcal{OF}(\mathbb{R}^3, \Sigma(t))$ over $\Sigma(t)$ which are actually the Darboux frames, Fig. 2., and from equations (6,10) we can write the canonical form

$$\theta(\mathbf{X}) = \mathrm{d}\mathbf{r} = \mathbf{r}_\mu \mathrm{d}u^\mu + W^\mu \mathbf{e}_\mu \mathrm{d}t + U\mathbf{N}\mathrm{d}t$$

$$= \sum_{\alpha=1}^{2} (\underbrace{\sqrt{g_{\alpha\alpha}}\mathrm{d}u^\alpha + W^\alpha \mathrm{d}t}_{\theta_\alpha})\mathbf{e}_\alpha + \underbrace{U\mathbf{N}\mathrm{d}t}_{\theta_3}.$$

(25)

We denote by W^α, U the tangent and normal components of surface velocity, respectively, see equation (15). By using the Gauss and Weingarten equations (12) and (13) we have

$$\mathbf{r}_{\mu\nu} = \Gamma^\lambda_{\mu\nu}\mathbf{r}_\lambda + \mathbf{\Pi}_{\mu\nu}, \qquad \mathbf{\Pi} = \Pi\mathbf{N}.$$

We use the definition of the principal normal to the surface in the form

$$\mathbf{N}_\mu = -g^{\lambda\nu}\mathbf{r}_\lambda\Pi_{\nu\mu}.$$

We also use the fact that the Christoffel symbols are derived from the Riemannian metric on $\Sigma(t)$, and we write the connection form, equation (6)

$$\omega(\mathbf{X})|_{T\Sigma} = \mathrm{d}\mathbf{r}_\mu = \Gamma^\lambda_{\mu\nu}\mathbf{r}_\lambda \mathrm{d}u^\nu + \mathbf{N}\Pi_{\mu\nu}\mathrm{d}u^\nu + \Upsilon_{\mu\nu}\mathbf{r}_\nu\mathrm{d}t + \Xi_\mu\mathbf{N}\mathrm{d}t$$

$$\mathrm{d}\mathbf{N}|_{T\Sigma^i} = -g^{\lambda\mu}\Pi_{\mu\nu}\mathbf{r}_\lambda\mathrm{d}u^\nu + \Upsilon_\mu\mathbf{r}_\mu\mathrm{d}t + \Xi\mathbf{N}\mathrm{d}t$$

(26)

where the one-forms $U\mathrm{d}t, W^\mu\mathrm{d}t, \Upsilon_{\mu\nu}\mathrm{d}t, \Upsilon_\mu\mathrm{d}t, \Xi_\mu\mathrm{d}t, \Xi\mathrm{d}t$ are responsible for the motion (tangent and normal) of the surface. By applying the structure conditions in equations (7-8) we obtain a partial differential system with eight equations for these nine unknown functions [26, 30]. The indeterminacy is related to the fact that there is no natural parametrization on the surface. Also, from the structure

equation (i.e., $d^2\mathbf{r} = 0$) we obtain six equations for the time dependence of the surface metric and of the second fundamental form

$$g_{\mu\nu,t} = g_{\mu\alpha}W_\nu^\alpha + g_{\nu\beta}W_\mu^\beta - 2\Gamma_{\mu\nu}^\lambda W^\alpha g_{\alpha\lambda} - 2\Pi_{\mu\nu}U$$

$$\Pi_{\mu\nu,t} = U_{,\mu\nu} + \Pi_{\mu\lambda}W_{,\nu}^\lambda + \Pi_{\nu\lambda}W_{,\mu}^\lambda + (\Pi_{\mu\lambda}\Gamma_{\rho\nu}^\lambda + \Pi_{\nu\lambda}\Gamma_{\rho\mu}^\lambda)W^\rho \qquad (27)$$

$$+ \Gamma_{\mu\nu}^\lambda U_{,\lambda} - g^{\rho\lambda}\Pi_{\rho\nu}\Pi_{\mu\lambda}U.$$

The coma subscript represents differentiation with respect to the variables written after this coma. Equations (27) represent the intrinsic formulation of surface motion, which (as opposed to the local formulation $\mathbf{r}(u^1, u^2, t)$) is not redundant and does not have the "z-axis" type of singularities. If we are given the surface velocity components, by integration of equations above we obtain the evolution of the surface at any moment of time, through the knowledge of its fundamental forms. Similar to the curve motion case, the W^α tangent velocity components are not essential: they just re-parameterize the surface, or "pushing" particles along the surface. We can note this by asking $U = 0$ for example and noticing that the resulting equations are linear in W components.

In order to verify if equations (27) describe the motion of the surface for real, we perform a limiting procedure reducing the surface to one of its curves of coordinates, and expecting to re-obtain the equations of motions for curves. However, like in any limiting process, we first have to write these equations in covariant form

$$g_{\mu\nu,t} = \nabla_\mu W_\nu + \nabla_\nu W_\mu - 2\Pi_{\mu\nu}U$$

$$\Pi_{\mu\nu,t} = \nabla_\mu(\nabla_\nu U) + (\Pi_{\mu\lambda}\nabla_\nu + \Pi_{\nu\lambda}\nabla_\mu)W^\lambda - g^{\rho\lambda}\Pi_{\rho\mu}\Pi_{\lambda\nu}U. \qquad (28)$$

Taken together equations (28) and the ten Gauss-Codazzi conditions ($d^2\mathbf{e}_\mu = d^2\mathbf{N} = 0$) provide sixteen equations for nine functions describing the surface and its motion: $E, F, G, e, f, g, W^1, W^2, U$.

We apply the following limiting verification procedure: if we make $\partial\mathbf{r}/\partial u^2 = 0$, and consequently the surface shrinks to some moving plane curve $\Sigma(t) \to \gamma(t)$, $\mathbf{N} \to \mathbf{n}$ we expect $g_{\mu\nu}(u^1, u^2, t) \to g(s, t)$, $\mathbf{W} \to W$, while U keeps having the same interpretation. Also, since the principal curvatures will approach $\kappa_1 \to \kappa$, $\kappa_2 \to 0$ we have

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{eG - 2fF + gE}{2(EG - F^2)} \to \frac{e}{2E}$$

that is $\Pi_{\mu\nu} \to g\kappa$. In this limit the first of the equations in (28) reduces to the regular time variation of the curve metric $g_t = 2g(W_s - \kappa U)$, namely equation (20). In the same plane curve limit we have the second relation in equations (28) approaching equation $\kappa_t = U_{ss} + \kappa^2 U + \int^s \kappa U d\tilde{s}$, which is the first relation in equations (19). These two approaches, obtain by writing the covariant form for
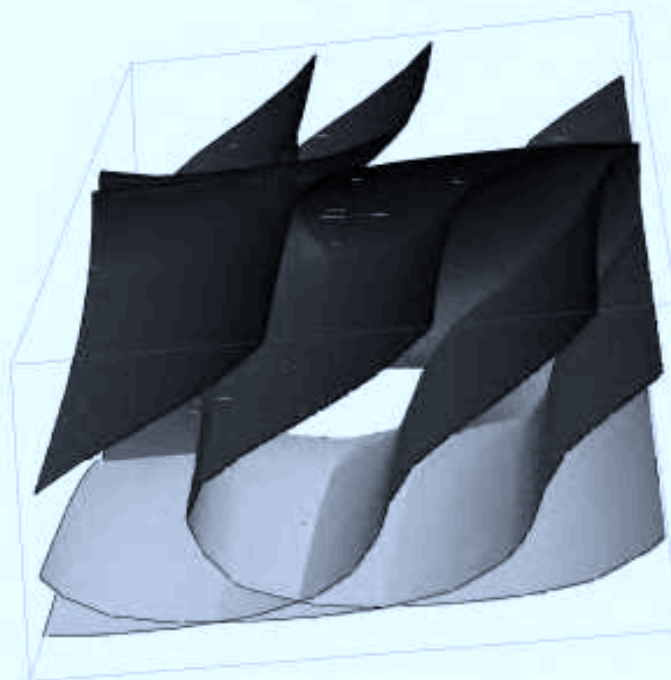
Figure 3. Moving developable surface as mKdV soliton solution of the Gauss-Weingarten equations (32).

the motion of surfaces represent a good check of correctness. We can make the statement

Corollary 4. *For a given velocity (W_1, W_2, U) as a function of (u, v) the solutions of the nonlinear differential system in equations (28), i.e., $(g_{\mu\nu}, \Pi_{\mu,\nu})$ as functions of (u, v), $\mu, \nu = 1, 2$ represent the first (metric) and the second fundamental forms of a surface with this velocity, modulo rigid motions, and according to Gauss' Theorema Egregium the surface is uniquely determined.*

6. Application to Motion of Surfaces

In literature there are basically three simplification approaches of the surface motion equation [26, 30]. The first one uses a sort of "diagonal philosophy" by using orthogonal particle-frozen coordinates in the surface that push back the particles in their original position when the surfaces changes. The other two approaches investigate particular cases of surfaces like developable surfaces ($K = 0$) or K-surfaces ($K < 0$ and constant). The physical applications range from diffusion processes, interface dynamics, motion of fluid sheets and vortices to swimming of motile cells and membrane theters [3, 16, 27, 28, 33, 37, 38]. In the first approach we use surface coordinates along the principal directions (the surface should have

no umbilical points, though!) in $\Sigma(t)$ such that

$$
g_{\mu\nu} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}, \qquad \Pi_{\mu\nu} = \begin{pmatrix} \kappa_1 e^{a_1} & 0 \\ 0 & \kappa_2 e^{a_2} \end{pmatrix} \tag{29}
$$

with $a_\mu, \kappa_\mu \in C^2(\mathbb{R}^2)$. The "frozen particles" rigidity constraints $g_{12,t} = \Pi_{12,t} = 0$ reduce the equation of motion equation (28) to a system of total differentials with respect to time for the unknown functions a_μ, κ_μ

$$
\left(\frac{\partial}{\partial t} - W^\mu \frac{\partial}{\partial u^\mu} \right) a_\nu = 2W^\nu_{,\nu} - 2\kappa_\nu U
$$

$$
\left(\frac{\partial}{\partial t} - W^\mu \frac{\partial}{\partial u^\mu} \right) \kappa_\nu = \kappa^2_\nu U + U_{,\sigma_\nu \sigma_\nu} + \frac{1}{2} e^{-a_{\nu'}} a_{\nu,\sigma_{\nu'}} U_{,\sigma_{\nu'}}. \tag{30}
$$

We take in equations (30) $\nu = 1, \nu' = 2$ or viceversa, without summations and we need to introduce the following coordinate transformation [26]

$$
\sigma_1 = \int^{u^1} \exp\left(\frac{1}{2} \ a_1(\tilde{u}^1, u^2) \right) d\tilde{u}^1.
$$

There is a similar expression for σ_2. The moving surface is then described by the following Gauss-Weingarten relations

$$
\frac{\partial}{\partial \sigma^1} \begin{pmatrix} \mathbf{r}_{\sigma^1} \\ \mathbf{r}_{\sigma^2} \\ \mathbf{N} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_{1,1} & -a_{1,2}\,e^{a_1-a_2} & 2\kappa_1 e^{a_1} \\ a_{1,2} & a_{2,1} & 0 \\ -2\kappa_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_{\sigma^1} \\ \mathbf{r}_{\sigma^2} \\ \mathbf{N} \end{pmatrix}
$$

$$
\frac{\partial}{\partial \sigma^2} \begin{pmatrix} \mathbf{r}_{\sigma^1} \\ \mathbf{r}_{\sigma^2} \\ \mathbf{N} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_{1,2} & a_{2,1} & 0 \\ -a_{2,1}\,e^{a_2-a_1} & a_{2,2} & 2\kappa_2 e^{a_2} \\ 0 & -2\kappa_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_{\sigma^1} \\ \mathbf{r}_{\sigma^2} \\ \mathbf{N} \end{pmatrix}. \tag{31}
$$

When we confine to developable surfaces, the kinematic equations for the surface simplify considerable because the Gauss-Weingarten equations reduce to a vector form from a two-tensor form. It is interesting that the motion of surfaces with constant non-positive Gauss curvature can be mapped into either the mKdV or sine-Gordon integrable systems [4].

In the following we present an example for a moving developable surface, parametrized by (u^1, u^2), that is a surface whose Gauss curvature is identical zero [30]. Among other solutions, the Gauss-Weingarten equations (28) provide a simple analytic solution in the form

$$
\mathbf{r}(u, s, t) = u\mathbf{f}(s, t) + \int \Phi(\tilde{s}, t) \frac{\partial \mathbf{f}}{\partial \tilde{s}} d\tilde{s} \tag{32}
$$

where we define a mKdV soliton type of solution by choosing

$$\Phi(u^2, t) = \frac{1}{2(1 - c^4)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1 + c^2}{1 - c^2}} \sinh[c(u^2 - (1 + c^2)t)]\right)$$

and the Euclidean components of the vector function **f** are

$$f_1(u^2, t) = \frac{1 - c^2}{1 + c^2} \cos[c(u^2 - (1 + c^2)t)]$$

$$- \frac{2c}{1 + c^2} \tanh[c(u^2 - (1 + c^2)t)] \sin[u^2 - (1 + c^2)t]$$

$$f_2(u^2, t) = \frac{1 - c^2}{1 + c^2} \sin[u^2 - (1 + c^2)t]$$

$$- \frac{2c}{1 + c^2} \tanh[c(u^2 - (1 + c^2)t)] \cos[u^2 - (1 + c^2)t]$$

$$f_3(u^2, t) = \frac{2c}{1 + c^2} \operatorname{sech}[c(u^2 - (1 + c^2)t)].$$

Here $|c| < 1$ is an arbitrary real parameter. Examples of time evolution of surface for $c = 0.12$ and $t = 1, 2, 3, 4$ and 5 are given in Fig. 3.

Acknowledgements

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