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CONSTANT MEAN CURVATURE SURFACES AT THE INTERSECTION OF INTEGRABLE GEOMETRIES

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Abstract. The constant mean curvature surfaces in three-dimensional space-forms are examples of isothermic constrained Willmore surfaces, characterized as the constrained Willmore surfaces in three-space admitting a conserved quantity. Both constrained Willmore spectral deformation and constrained Willmore Bäcklund transformation preserve the existence of a conserved quantity. The class of constant mean curvature surfaces in three-dimensional space-forms lies, in this way, at the intersection of several integrable geometries, with classical transformations of its own, as well as constrained Willmore transformations and transformations as a class of isothermic surfaces. Constrained Willmore transformation is expected to be unifying to this rich transformation theory.

1. Introduction

Minimal surfaces appear as the area-minimizing surfaces amongst all those spanning a given boundary. The Euler-Lagrange equation of the underlying variational problem turns out to be the zero mean curvature equation. A physical model of a minimal surface can be obtained by dipping a wire frame into a soap solution. The resulting soap film is minimal, in the sense that it always tries to organize itself so that its surface area is as small as possible whilst spanning the wire contour. This minimal surface area is reached for the flat position, which is also the position in which the membrane is the most relaxed, i.e., where the elastic energy is minimal - these surfaces are elastic energy extremals and, in this way, examples of Willmore surfaces. In fact, a classical result by Thomsen [23] characterizes isothermic Willmore surfaces in three-space as minimal surfaces in some three-dimensional space-form.

Unlike flat soap films, soap bubbles do not extremize the elastic energy - they exist under a certain surface tension, in an equilibrium where slightly greater pressure inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are examples of area-minimizing surfaces under the constraint of volume enclosed - these are surfaces of (non-zero) constant mean curvature and, therefore, examples of constrained Willmore surfaces (which are not Willmore surfaces), elastic energy extremals with respect to infinitesimally conformal variations (rather than with respect to all variations). Indeed, as established by Richter [18], **constant mean curvature** (CMC) surfaces in three-dimensional space-forms are, in particular, isothermic constrained Willmore surfaces.

In [16], a spectral deformation and a Bäcklund transformation of constrained Willmore surfaces are defined and a permutability between the two is established. It is shown that all these transformations corresponding to the zero multiplier preserve the class of Willmore surfaces. The class of CMC surfaces in three-dimensional space-forms is characterized as the class of constrained Willmore surfaces in three-space admitting a conserved quantity. It is shown that, for special choices of parameters, both spectral deformation and Bäcklund transformation preserve the class of constrained Willmore surfaces admitting a conserved quantity, and, in particular, the class of CMC surfaces in three-dimensional space-forms.

The class of constant mean curvature surfaces in three-space lies, in this way, at the intersection of several integrable geometries, with constrained Willmore spectral deformation and Bäcklund transformations, an isothermic spectral deformation (classically defined by Bianchi [2] and, independently, Calapso [10]), as well as a classical spectral deformation of its own (the Bonnet transformation [4]), and, in the Euclidean case, isothermic Darboux transformations (classically discovered by Darboux [12]) or, equivalently [15], Bianchi-Bäcklund transformation [1]. The isothermic spectral deformation is known to preserve the constancy of the mean curvature of a surface in some space-form, cf. [9]. In [16], it is shown that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation. These spectral deformations of CMC surfaces in three-dimensional space-forms are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation. In [14] it is shown that, for special choices of parameters, the Darboux transformation of isothermic surfaces in Euclidean three-space preserves the constancy of the mean curvature in $\mathbb{R}^3$, as well as the mean curvature itself. Isothermic Darboux transformation of a CMC surface in Euclidean three-space is expected to be obtained as a particular case of constrained Willmore Bäcklund transformation. Constrained Willmore transformation is in this way expected to be unifying to this rich transformation theory.

Our theory is local and, throughout this text, with no need for further reference, restriction to a suitable non-empty open set shall be underlying.

2. Constrained Willmore Surfaces

In modern literature on the elasticity of membranes, a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature is considered the elastic energy of a membrane. By neglecting the total mean curvature (by physical considerations) and having in consideration that the total Gaussian curvature of compact orientable Riemannian surfaces without boundary is a topological invariant, Willmore [25] defined the **Willmore (elastic) energy** of a compact oriented Riemannian surface, without boundary, isometrically immersed in $\mathbb{R}^3$, to be

$$\mathcal{W} = \int H^2 \mathrm{d}A$$

i.e., the total squared mean curvature. The Willmore functional "extends" (for more details, see [16]) to isometric immersions of compact oriented Riemannian surfaces in Riemannian manifolds by means of half of the total squared norm of Π^0, the trace-free part of the second fundamental form, which, in fact, amongst surfaces in $\mathbb{R}^3$, differs from $\mathcal{W}$ by the total Gaussian curvature, but still shares then the critical points with $\mathcal{W}$. And so does

$$\mathcal{W} = \int_M |\Pi^0|^2 \mathrm{d}A$$

which is what we consider as the **Willmore energy functional**.

By definition the **Willmore surfaces** are the extremals of the Willmore energy. The class of **constrained Willmore** (CW) **surfaces** appears as the generalization of the class of Willmore surfaces that arises when we consider extremals of the Willmore functional with respect to infinitesimally conformal variations - those satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(X^{1,0}, X^{1,0})_t = 0$$

fixing $X^{1,0}$ a $(1,0)$-vector field - rather than with respect to all variations (Note that conformal variations are characterized by $(X^{1,0}, X^{1,0})_t = 0$, fixing $X^{1,0}$ a $(1,0)$-vector field). Under a conformal change of the metric, the squared norm of the trace-free part of the second fundamental form and the area element change in an inverse way, leaving the Willmore energy unchanged. In particular, this establishes the class of (constrained) Willmore surfaces as a Möbius invariant class.

Our study is one of (constrained) Willmore surfaces in n-dimensional space-forms[1] with $n \geq 3$, which, in view of the conformal invariance mentioned above, we

[1]Throughout this text, we will, alternatively, use n-space to refer to n-dimensional space-form (Euclidean, spherical or hyperbolic).

approach as immersions

$$\Lambda : (M, \mathcal{C}_\Lambda) \to \mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$$

of an oriented compact[2] surface M into the conformal n-sphere, which we model on the projective space of the light-cone $\mathcal{L} \subset \mathbb{R}^{n+1,1}$, following Darboux [11], (for a modern account, see [5]) providing M with the conformal structure $\mathcal{C}_\Lambda$ induced from the one on $\mathbb{P}(\mathcal{L})$ (and with the canonical complex structure).

A fundamental construction in conformal geometry of surfaces is the mean curvature sphere congruence, the bundle of two-spheres tangent to the surface and sharing with it the mean curvature vector at each point (although the mean curvature vector is not conformally invariant, under a conformal change of the metric it changes in the same way for the surface and the osculating two-sphere). Let

$$S : M \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$$

be the mean curvature sphere congruence of Λ (the k-spheres of $\mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$ are exactly the manifolds $\mathbb{P}(\mathcal{L} \cap V)$ with V a $(k+1, 1)$-plane of $\mathbb{R}^{n+1,1}$ (see [5]).). We have a decomposition $\underline{\mathbb{R}}^{n+1,1} = S \oplus S^\perp$ and then a decomposition of the trivial flat connection d on $\underline{\mathbb{R}}^{n+1,1}$ as

$$\mathrm{d} = \mathcal{D} \oplus \mathcal{N}$$

for $\mathcal{D} = \nabla^S + \nabla^{S^\perp}$, with ∇^S and $\nabla^{S^\perp}$ the connections induced on S and $S^\perp$, respectively, by d. Set

$$\Lambda^{1,0} := \Lambda \oplus \mathrm{d}\sigma(T^{1,0}M), \qquad \Lambda^{0,1} := \Lambda \oplus \mathrm{d}\sigma(T^{0,1}M)$$

two subbundles of $S^{\mathbb{C}}$, defined independently of the choice of $\sigma \in \Gamma(\Lambda)$ never-zero, and then $\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}$.

In generalization of what is presented in [7] for the particular case of $n = 4$, we have (see [16])

$$\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (\mathrm{d}S \wedge *\mathrm{d}S)$$

a manifestly conformally invariant formulation of the Willmore energy. This formulation makes it clear that

$$\mathcal{W}(\Lambda) = E(S)$$

the Willmore energy of Λ coincides with the Dirichlet energy of S (with respect to any of the metrics in the conformal class on M). N. Ejiri [13] and independently, M. Rigoli [19] proved, furthermore, that

$$\Lambda \text{ Willmore} \Leftrightarrow S \text{ harmonic}$$

so that Λ is a Willmore surface if and only if $S : (M, \mathcal{C}_\Lambda) \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ is a harmonic map. According to Uhlenbeck [24], it follows that Λ is a Willmore

[2] A natural extension to surfaces that are not necessarily compact will take place at some point below.

surface if and only if $\mathrm{d}^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1}$ is a flat connection, for all λ in $\mathbb{S}^1$. More generally (for more details, see [16]) we have

Theorem 1 ([6]). *Λ is a constrained Willmore surface if and only if there exists a real one-form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that the connection*

$$\mathrm{d}^\lambda_q := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1} \tag{1}$$

*is flat, for all $\lambda \in \mathbb{S}^1$. In that case, q is said to be a multiplier to Λ and Λ is said to be a q-*CW *surface.*

In Theorem 1 and throughout this text, we consider the identification

$$\wedge^2\mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$$

of the exterior power $\wedge^2\mathbb{R}^{n+1,1}$ with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$ via

$$\omega \mapsto v_1 \wedge v_2(\omega) := (v_1, \omega)v_2 - (v_2, \omega)v_1$$

for $v_1, v_2, \omega \in \mathbb{R}^{n+1,1}$.

The characterization of constrained Willmore surfaces in space-forms presented in Theorem 1 provides a natural extension of the concept to surfaces that are not necessarily compact.

Willmore surfaces are the 0-CW surfaces. The zero multiplier is not necessarily the only multiplier to a CW surface with no constraint on the conformal structure, though. In fact, the uniqueness of multiplier characterizes non-isothermic constrained Willmore surfaces

Proposition 1 ([16]). A constrained Willmore surface has a unique multiplier if and only if it is not an isothermic surface.

A classical result by Thomsen [23] characterizes isothermic Willmore surfaces in three-space as minimal surfaces in some three-dimensional space-form. Constant mean curvature surfaces in three-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by Richter [18]. However, isothermic constrained Willmore surfaces in three-space are not necessarily CMC surfaces in some space-form, as established by an example due to Burstall [3], of a constrained Willmore cylinder that does not have constant mean curvature in any space-form.

For later reference, it is convenient to denote, alternatively, d^λ_q by $\mathrm{d}^{\lambda,q}_S$ and to use $\hat{\mathrm{d}}^{\lambda,q}_V$ for the analogue defined for a general non-degenerate subbundle V of $(\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}} = \underline{\mathbb{C}}^{n+2}$, provided with the complex bilinear extension of the metric on $\underline{\mathbb{R}}^{n+1,1}$, a general one-form q with values in $\wedge^2\underline{\mathbb{C}}^{n+2}$ and $\hat{\mathrm{d}}$ a general flat metric connection on $\underline{\mathbb{C}}^{n+2}$. The characterization of *q-constrained harmonicity* of the bundle S consisting of the flatness of $\mathrm{d}^{\lambda,q}_S$, for all λ in $\mathbb{S}^1$, extends naturally to a notion respecting a general non-degenerate subbundle V of $\underline{\mathbb{C}}^{n+2}$ and a general $q \in \Omega^1(\wedge^2\underline{\mathbb{C}}^{n+2})$, by means of the flatness of the connection $\mathrm{d}_V^{\lambda,q}$, for all $\lambda \in \mathbb{S}^1$. For more details, see [16].

3. Transformations of Constrained Willmore Surfaces

Constrained Willmore surfaces in space-forms form a Möbius invariant class of surfaces with strong links to the theory of integrable systems, admitting, amongst others, a spectral deformation, defined by the action of a loop of flat metric connections, and Bäcklund transformations, defined by applying a dressing action [16]. All these transformations are closely related and all those corresponding to the zero multiplier preserve the class of Willmore surfaces.

3.1. Spectral Deformation

For each λ in $\mathbb{S}^1$, the flatness of the metric connection d_q^λ establishes the existence of an isometry

$$\phi_q^\lambda : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$$

of bundles, defined on a simply connected component of M, preserving connections, unique up to a Möbius transformation. We use an interpretation of loop group theory by Burstall and Calderbank and define a spectral deformation of Λ which is supposed to be a $q-$CW surface into new constrained Willmore surfaces by setting, for each λ in $\mathbb{S}^1$

$$\Lambda_q^\lambda := \phi_q^\lambda \Lambda.$$

This comes as an immediate consequence of the fact that $(\mathrm{d}_q^\lambda)_{q_\lambda}^\mu = \mathrm{d}_q^{\lambda\mu}$, for $q_\lambda = \lambda^{-2}q^{1,0} + \lambda^2 q^{0,1}$ and for all $\lambda, \mu \in \mathbb{S}^1$, which readily establishes Λ_q^λ as a $\mathrm{Ad}_{\phi_q^\lambda}(q_\lambda)$-CW surface. In particular, spectral deformation corresponding to the zero multiplier preserves the class of Willmore surfaces. For each λ, we refer to Λ_q^λ as the transformation of Λ defined (in the ambit of Möbius geometry) by the flat metric connection d_q^λ.

3.2. Bäcklund Transformation

We use a version of the dressing action theory of Terng-Uhlenbeck [22] to define a transformation of Λ into new constrained Willmore surfaces. We start by defining a transformation on the level of constrained harmonic bundles. For that, we give conditions on a **dressing** $r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ such that the gauging

$$\hat{\mathrm{d}}_S^{\lambda,\tilde{q}} := r(\lambda) \circ \mathrm{d}_S^{\lambda,q} \circ r(\lambda)^{-1}$$

of d_q^λ by $r(\lambda)$, for each λ, establishes the constrained harmonicity of some bundle $\hat{S}$ from the constrained harmonicity of S, as follows. Define $\tilde{q} \in \Omega^1(\wedge^2\underline{\mathbb{C}}^{n+2})$ by setting

$$\tilde{q}^{1,0} := \mathrm{Ad}_{r(0)}q^{1,0}, \qquad \tilde{q}^{0,1} := \mathrm{Ad}_{r(\infty)}q^{0,1}.$$

Set, furthermore

$$\hat{q} = \mathrm{Ad}_{r(1)^{-1}}\tilde{q}$$

and

$$\hat{S} = r(1)^{-1}S.$$

Lemma 1 ([16]). *Let $\rho \in \Gamma(\underline{\mathbb{C}}^{n+2})$ be reflection across S, $\rho = \pi_S - \pi_{S^\perp}$, for π_S and $\pi_{S^\perp}$ the orthogonal projections of $\underline{\mathbb{C}}^{n+2}$ onto $S^{\mathbb{C}}$ and $(S^\perp)^{\mathbb{C}}$, respectively. Suppose $r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ is such that*

i) *$\lambda \mapsto r(\lambda)$ is holomorphic and invertible at $\lambda = 0$ and $\lambda = \infty$*
ii) *$\rho r(\lambda)\rho^{-1} = r(-\lambda)$, for all λ*
iii) *$\lambda \mapsto \hat{d}_S^{\lambda,\tilde{q}}$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$.*

Then, for $\hat{d} := \hat{d}_S^{1,\tilde{q}}$, the notation $\hat{d}_S^{\lambda,\tilde{q}}$ is not merely formal, that is, the connection denoted by $\hat{d}_S^{\lambda,\tilde{q}}$ is of the form (1).

Suppose that 1 is in the domain of r. In that case, and under the hypotheses of Lemma 1, it follows immediately, in view of the specific form of $\hat{d}_S^{\lambda,\tilde{q}}$, that

$$r(1)^{-1} \circ \hat{d}_S^{\lambda,\tilde{q}} \circ r(1) = d_{\hat{S}}^{\lambda,\hat{q}}$$

which establishes the $\hat{q}$-constrained harmonicity of $\hat{S}$ from the q-constrained harmonicity of S.

Now set

$$\hat{\Lambda}^{1,0} := r(1)^{-1}r(\infty)\Lambda^{1,0}, \qquad \hat{\Lambda}^{0,1} := r(1)^{-1}r(0)\Lambda^{0,1}$$

and

$$\hat{\Lambda} := \hat{\Lambda}^{1,0} \cap \hat{\Lambda}^{0,1}.$$

The isotropy of both $\Lambda^{1,0}$ and $\Lambda^{0,1}$ establishes that for both $\hat{\Lambda}^{1,0}$ and $\hat{\Lambda}^{0,1}$ and therefore, the nullity of the bundle $\hat{\Lambda}$. On the other hand, an extra condition on r, namely, $\det r(0)|_S = \det r(\infty)|_S$, establishes $\hat{\Lambda}$ as a line bundle. Actually, condition ii) in Lemma 1 establishes, in particular, that $r(0)|_S, r(\infty)|_S \in \Gamma(O(S))$. One verifies, furthermore, that, if $\hat{\Lambda} \subset (\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$ is a real bundle, then $\hat{S}$ is the mean curvature sphere congruence of the surface $\hat{\Lambda}$ and, ultimately, that if $\hat{q}$ is real, then $\hat{\Lambda}$ is a $\hat{q}$-CW surface.

Following the philosophy of Terng-Uhlenbeck [22], we then construct $r = r(\lambda)$ satisfying the conditions above, as well as establishing the reality of $\hat{\Lambda}$ and $\hat{q}$ from

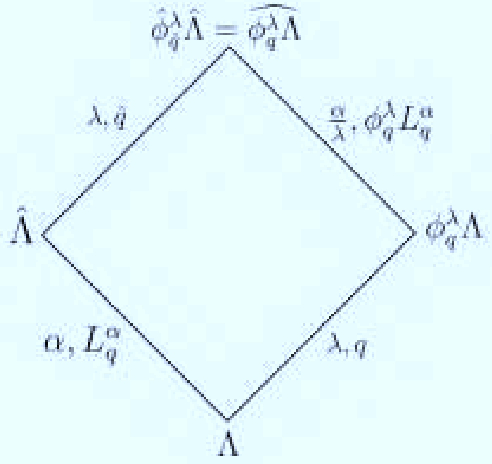

Figure 1. A permutability of spectral deformation and Bäcklund transformation of constrained Willmore surfaces.

the reality of Λ and q, respectively. We consider a two-step process of transformations of the type

$$r_{\alpha,L}^{(-)}(\lambda) := (-)\frac{\alpha-\lambda}{\alpha+\lambda}\,\pi_L + \pi_{(L\oplus\rho L)^{\perp}} + (-)\,\frac{\alpha+\lambda}{\alpha-\lambda}\,\pi_{\rho L}$$

(respectively), parametrized by $\alpha \in \mathbb{C}\backslash\mathbb{S}^1$ non-zero and $L = L_q^{\alpha} \subset \underline{\mathbb{C}}^{n+2}$ a $\mathrm{d}_S^{\alpha,q}$-parallel null line bundle such that $\rho L \cap L^{\perp} = \backslash\{0\}$. Namely, we consider

$$r = r_{\alpha,L^{\alpha}}r_{\beta,L^{\beta}}^{-}$$

for $\beta = \overline{\alpha}^{-1}$, $L^{\beta} = \overline{L}$ and $L^{\alpha} = r_{\beta,L^{\beta}}^{-}(\alpha)L$, with α and L as above. We refer to $\hat{\Lambda}$ as the Bäcklund transform of Λ of parameters α, L_q^{α}. Note that Bäcklund transformation corresponding to the zero multiplier preserves the class of Willmore surfaces.

3.3. Spectral Deformation vs Bäcklund Transformation

Spectral deformation and Bäcklund transformation of constrained Willmore surfaces permute, as follows

Theorem 2 ([16]). *Let α, L_q^{α} be Bäcklund transformation parameters to Λ corresponding to the multiplier q, let λ be in $\mathbb{S}^1$ and $\phi_q^{\lambda} : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_S^{\lambda,q}) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ and $\hat{\phi}_{\hat{q}}^{\lambda} : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_{\hat{S}}^{\lambda,\hat{q}}) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ be isometries preserving connections. Then the Bäcklund transform of parameters $\frac{\alpha}{\lambda}$, $\phi_q^{\lambda}L_q^{\alpha}$ of the spectral deformation $\phi_q^{\lambda}\Lambda$ of Λ, of parameter λ, corresponding to the multiplier q, coincides with the spectral deformation of parameter λ corresponding to the multiplier $\hat{q}$ of the Bäcklund transform of parameters α, L_q^{α} of Λ, i.e., the diagram in* Fig.1 *commutes.*

4. Conserved Quantities under CW Transformation

Suppose Λ is a q-CW surface. Let $p(\lambda) = \lambda^{-1}v + v_0 + \lambda\bar{v}$ be a Laurent polynomial with $v_0 \in \Gamma(S^{\mathbb{C}})$ real, $v \in \Gamma((S^{\mathbb{C}})^{\perp})$ and $v_\infty := p(1) \neq 0$. We say that $p(\lambda)$ is a **q-conserved quantity** of Λ if $\mathrm{d}_q^{\lambda} p(\lambda) = 0$, for all $\lambda \in \mathbb{C}\backslash\{0\}$ and then following the idea by Burstall and Calderbank we have

Lemma 2 ([16]). *$p(\lambda)$ is a q-conserved quantity of Λ if and only if*

$$\mathrm{d}v_\infty = 0, \quad \mathcal{D}^{0,1}v = 0, \qquad \mathcal{N}^{1,0}v + q^{1,0}v_0 = 0.$$

The characterization above, of a q-conserved quantity $p(\lambda)$ of Λ, shows, in particular, that $p(\lambda)$ determines q (for details, see [16]). There is then no ambiguity on referring to $p(\lambda)$ simply as a conserved quantity of Λ.

For special choices of parameters, both spectral deformation and Bäcklund transformation of constrained Willmore surfaces preserve the existence of a conserved quantity, as follows

Theorem 3 ([16]). *Let μ be in $\mathbb{S}^1$ and $\phi_q^{\mu} : (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d}_S^{\mu,q}) \to (\underline{\mathbb{R}}^{n+1,1}, \mathrm{d})$ be an isometry preserving connections. Suppose that either v_0 is non-zero or $\overline{\mu}v + \mu\overline{v}$ is non-zero. In that case, if $p(\lambda)$ is a q-conserved quantity of Λ, then $\phi_q^{\mu}p(\mu\lambda)$ is a $\mathrm{Ad}_{\phi_q^{\mu}}(q_\mu)$-conserved quantity of the spectral deformation $\phi_q^{\mu}\Lambda$ generated by the parameter μ of Λ.*

We have also

Theorem 4 ([16]). *Suppose $p(\lambda)$ is a q-conserved quantity of Λ. Let α, L_q^{α} be Bäcklund transformation parameters to Λ corresponding to the multiplier q and let r be the corresponding dressing. If $p(\alpha) \perp L_q^{\alpha}$, then*

$$\hat{p}(\lambda) := r(1)^{-1}\, r(\lambda)\, p(\lambda)$$

is a $\hat{q}$-conserved quantity of the Bäcklund transform $\hat{\Lambda}$ of Λ of parameters α, L_q^{α}.

5. Example: Constant Mean Curvature Surfaces in Three-dimensional Space-forms

The class of constant mean curvature surfaces in three-dimensional space-forms is characterized as the class of constrained Willmore surfaces in three-space admitting a conserved quantity. It follows that, for special choices of parameters, both spectral deformation and Bäcklund transformation of constrained Willmore surfaces preserve the class of CMC surfaces in three-dimensional space-forms. The class of CMC surfaces in three-dimensional space-forms lies, in this way, at the intersection of several integrable geometries, with classical transformations of its own, as well as constrained Willmore transformations and transformations as a class of isothermic surfaces. Constrained Willmore transformation is expected to be unifying to this rich transformation theory.

In contrast to constrained Willmore surfaces, constant mean curvature surfaces are not conformally-invariant objects, which requires carrying a distinguished space-form. Following [5] we start by realizing all space-forms as submanifolds of the light-cone, given $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero

$$S_{v_\infty} := \{v \in \mathcal{L}\,;\, (v, v_\infty) = -1\}$$

inherits from $\mathbb{R}^{n+1,1}$ a positive definite metric of (constant) sectional curvature $-(v_\infty, v_\infty)$. For each v_∞, the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defines a diffeomorphism

$$\pi_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L} \cap \langle v_\infty\rangle^\perp).$$

Let us consider the particular case $n = 3$. Let T and $\perp$ denote the orthogonal projections of $\underline{\mathbb{R}}^{4,1}$ onto S and $S^\perp$, respectively. Suppose the surface Λ is not contained in any two-sphere. This condition ensures (see [16]) that, given $v_\infty \in \mathbb{R}^{4,1}$ non-zero, Λ is (locally) a surface in $\mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L} \cap \langle v_\infty\rangle^\perp) \cong S_{v_\infty}$

$$\Lambda \cong (\pi_{S_{v_\infty}})^{-1} \circ \Lambda : M \to S_{v_\infty}$$

with mean curvature given, up to sign, by

$$H_\infty = (v_\infty^\perp, v_\infty^\perp)^{\frac{1}{2}}.$$

In particular, Λ is a minimal surface in the space-form S_{v_∞} (i.e., $H_\infty = 0$) if and only if $v_\infty \in \Gamma(S)$.

5.1. CMC Surfaces and Conserved Quantities

According to Lemma 2, the existence of a conserved quantity $p(\lambda)$ of Λ establishes, in particular, the constancy of $v_\infty := p(1)$. Furthermore we have

Theorem 5 ([16]). *If Λ is a* CW *surface and $p(\lambda)$ is a conserved quantity of Λ, then Λ has constant mean curvature in the space-form S_{v_∞}, for $v_\infty = p(1)$. Reciprocally, if H_∞ is constant, for some non-zero $v_\infty \in \mathbb{R}^{4,1}$, then*

$$p_\infty(\lambda) := \lambda^{-1}\frac{1}{2}v_\infty^\perp + v_\infty^T + \lambda\frac{1}{2}v_\infty^\perp$$

is a conserved quantity of the CW *surface Λ. Constant mean curvature surfaces in three-dimensional space-forms are the constrained Willmore surfaces in three-space admitting a conserved quantity.*

Next we establish a conserved quantity with respect to a general multiplier to a surface with constant mean curvature in some three-space. The conclusion that these surfaces allow CW spectral deformation and CW Bäcklund transformation into new ones will then follow from Theorems 3 and 4.

As suggested by Proposition 1, the characterization of the set of multipliers to a constrained Willmore surface is closely related to the isothermic condition. Isothermic surfaces are classically defined by the existence of conformal curvature line cooordinates, i.e., conformal coordinates with respect to which the second fundamental form is diagonal. This is a conformally-invariant condition, although the second fundamental form is not conformally-invariant, and it can be reformulated in a manifestly invariant way, as follows (This formulation is also discussed in [6] and [21].)

Theorem 6 ([8]). *Λ is isothermic if and only if there is a non-zero real closed form $\eta \in \Omega^1(\Lambda \wedge \Lambda^1)$. In that case, we say that (Λ, η) is isothermic.*

In the conditions of Theorem 6, the form η is unique up to non-zero constant real scale, cf. [21].

Following Proposition 1, we have, furthermore

Proposition 2 ([16]). Suppose (Λ, η) is an isothermic q-CW surface. Then the set of multipliers to Λ is the one-dimensional affine space $q + \langle *\eta \rangle_{\mathbb{R}}$.

Fix $v_\infty \in \mathbb{R}^{4,1}$ non-zero. Suppose Λ has constant mean curvature in S_{v_∞}. Define $N \in \Gamma(S^\perp)$ unit by setting $v_\infty^\perp = H_\infty N$ (in the particular case Λ is minimal in S_{v_∞}, N is defined only up to sign). Write σ_∞ for $(\pi_{S_{v_\infty}})^{-1} \circ \Lambda$. Set $\eta_\infty := \frac{1}{2}\sigma_\infty \wedge \mathrm{d}N$, a form derived by F. Burstall and D. Calderbank which establishes Λ as an isothermic surface and for which scaling by the mean curvature in S_{v_∞} setting

$$q_\infty := H_\infty \eta_\infty$$

provides a multiplier to Λ (see [16])

Proposition 3. (Λ, η_∞) is an isothermic q_∞-CW surface.

Proposition 3 makes it clear, in particular, that minimal surfaces in three-dimensional space-forms are examples of Willmore surfaces.

For each $t \in \mathbb{R}$, set

$$q_\infty^t := q_\infty + t * \eta_\infty .$$

Proposition 4 ([17]). For each $t \in \mathbb{R}$

$$p_\infty^t(\lambda) := \lambda^{-1}\frac{1}{2}\,(H_\infty - \mathrm{i}t)N + v_\infty^T + \lambda\frac{1}{2}\,(H_\infty + \mathrm{i}t)N$$

is a q_∞^t-conserved quantity of Λ.

5.2. CMC Surfaces at the Intersection of Integrable Geometries

The results in Section 5.1 combine to establish the following

Theorem 7 ([17]). *The class of* CMC *surfaces in three-dimensional space-forms is preserved by both* CW *spectral deformation and* CW *Bäcklund transformation, for special choices of parameters, with preservation of both the space-form and the mean curvature in the latter case.*

Fix $v_\infty \in \mathbb{R}^{4,1}$ non-zero and suppose Λ has constant mean curvature in S_{v_∞}. The fact that a Bäcklund transform of Λ still is a surface of constant mean curvature H_∞ in S_{v_∞}, as stated above, is not immediate from Theorem 4. In contrast, it is immediate from Theorem 3 that, given λ in $\mathbb{S}^1$ and $\phi^\lambda_{t,\infty} : (\underline{\mathbb{R}}^{4,1}, \mathrm{d}^\lambda_{q^t_\infty}) \to (\underline{\mathbb{R}}^{4,1}, \mathrm{d})$ an isometry preserving connections, the spectral deformation $\phi^\lambda_{t,\infty}\Lambda$ of Λ, of parameter λ, corresponding to the multiplier q^t_∞ has constant mean curvature

$$H^\lambda_{t,\infty} = \mid \mathrm{Re}\,(\lambda H_\infty + \frac{it}{2}(\lambda - \lambda^{-1})) \mid$$

in the space-form $S_{v^\lambda_{t,\infty}}$ for

$$v^\lambda_{t,\infty} := \phi^\lambda_{t,\infty}(v^T_\infty + ((\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))N).$$

Zero curvature representation provides a context in which Bonnet transformation [4] of CMC surfaces in $\mathbb{R}^3$ can be generalized to CMC surfaces in general three-space, as follows (*the classical* CMC *spectral deformation*). For each $\lambda \in \mathbb{S}^1$, set

$$\mathrm{d}^\lambda_\infty := \mathcal{D} + \lambda \mathcal{N}^{1,0} + \lambda^{-1}\mathcal{N}^{0,1} + 2(\lambda - 1)q^{1,0}_\infty + 2(\lambda^{-1} - 1)q^{0,1}_\infty.$$

Theorem 8 ([16]). *The connection* $\mathrm{d}^\lambda_\infty$ *is flat, for all* $\lambda \in \mathbb{S}^1$. *Besides, if for each* $\lambda \in \mathbb{S}^1$, $\phi^\lambda_\infty : (\underline{\mathbb{R}}^{4,1}, \mathrm{d}^\lambda_\infty) \to (\underline{\mathbb{R}}^{4,1}, \mathrm{d})$ *is an isometry preserving connections, then*

i) $v^\lambda_\infty := \phi^\lambda_\infty v_\infty$ *is a non-zero constant section of* $\underline{\mathbb{R}}^{4,1}$

ii) *the transformation* $\Lambda^\lambda_\infty := \phi^\lambda_\infty \Lambda$ *of* Λ, *defined by the flat metric connection* d^λ_∞, *has constant mean curvature* H_∞ *in the space-form* $S_{v^\lambda_\infty}$

iii) *the family* $\phi^\lambda_\infty \sigma_\infty$, *with* $\lambda \in \mathbb{S}^1$, *is a family of isometrical deformations of* σ_∞ *in a fixed space-form, preserving the mean curvature.*

In [8], the spectral deformation of isothermic surfaces in $\mathbb{R}^3$ (or, equivalently, in general three-space) classically discovered by Bianchi [2] and, independently, Calapso [10] is generalized to n-space, for general n, by means of zero curvature representation, as follows (*the isothermic spectral deformation*). Let η be a non-zero real one-form with values in $\Lambda \wedge \Lambda^{(1)}$. For each $t \in \mathbb{R}$, set

$$\mathrm{d}^\eta_t := \mathrm{d} + t\eta.$$

Theorem 9 ([8]). *(Λ, η) is isothermic if and only if d_t^η is a flat connection, for each $t \in \mathbb{R}$. In that case, the transformation Λ_t^η of Λ defined by the flat metric connection d_t^η is still isothermic, for each $t \in \mathbb{R}$.*

The isothermic spectral deformation is known [9] to preserve the constancy of the mean curvature in some three-dimensional space-form, defining then a transformation of CMC surfaces into new ones. In fact [16], given $t \in \mathbb{R}$ and $\phi_t^\infty : (\mathbb{R}^{4,1}, \mathrm{d}_t^{\eta_\infty}) \to (\mathbb{R}^{4,1}, \mathrm{d})$ an isometry preserving connections, the deformation $\phi_t^\infty \Lambda$ of Λ has constant mean curvature H_t^∞ in the space-form $S_{v_t^\infty}$, for

$$v_t^\infty := \phi_t^\infty (v_\infty + \frac{t}{2} N)$$

with

$$(H_t^\infty)^2 = (H_\infty + \frac{t}{2})^2.$$

Proposition 5 ([17]). The classical CMC spectral deformation of parameter other than -1 can be obtained as constrained Willmore spectral deformation

$$\mathrm{d}_\infty^\lambda = \mathrm{d}_{q_\infty^{t_\lambda}}^{\lambda^{-1}}$$

for $\lambda \neq -1$ in $\mathbb{S}^1$ and

$$t_\lambda := \mathrm{i} H_\infty \frac{1-\lambda}{1+\lambda} \in \mathbb{R}.$$

Furthermore: for all $\lambda \in \mathbb{S}^1$

$$\mathrm{d}_\infty^{\lambda^{-1}} = \mathrm{d}_{q_\infty}^\lambda + 2H_\infty (1 - \mathrm{Re}\,\lambda)\, \eta_\infty^\lambda$$

for $\eta_\infty^\lambda = \lambda^{-1} \eta_\infty^{1,0} + \lambda \eta_\infty^{0,1}$. Hence the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation and, in the particular case of a minimal surface, the classical CMC spectral deformation coincides, up to reparametrization, with constrained Willmore spectral deformation corresponding to the zero multiplier.

These spectral deformations of CMC surfaces in three-dimensional space-forms are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation, cf. Theorem 2.

CMC surfaces in Euclidean three-space enjoy, furthermore, Darboux transformation as isothermic surfaces or, equivalently [15], Bianchi-Bäcklund transformation, as discussed in [20] (cf. [1]). In fact, in [14], it is shown that, for special choices of parameters, the transformation of isothermic surfaces in $\mathbb{R}^3$ classically discovered by Darboux [12] preserves the constancy of the mean curvature in $\mathbb{R}^3$, as well as the mean curvature itself. This is also the case for constrained Willmore Bäcklund transformation, cf. Theorem 7. In [14], a description of Darboux transformation of constant mean curvature surfaces in Euclidean three-space is presented in the quaternionic setting. It is based on the solution of a Riccati equation and it displays a striking similarity with the Darboux transformation of constrained Willmore surfaces in four-space defined in [16]. Non-trivial Darboux transformation of constrained Willmore surfaces can be obtained as a particular case of constrained Willmore Bäcklund transformation, as established in [16]. We believe that isothermic Darboux transformation of a CMC surface in Euclidean three-space can be obtained as a particular case of constrained Willmore Bäcklund transformation.

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