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f-BIHARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract. We show that if ψ is an f-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then ψ is an f-harmonic map. We prove that if the f-tension field $\tau_f(\psi)$ of a map ψ of Riemannian manifolds is a Jacobi field and ϕ is a totally geodesic map of Riemannian manifolds, then $\tau_f(\phi \circ \psi)$ is a Jacobi field. We finally investigate the stress f-bienergy tensor, and relate the divergence of the stress f-bienergy of a map ψ of Riemannian manifolds with the Jacobi field of the $\tau_f(\psi)$ of the map.

1. Introduction

Harmonic maps between Riemannian manifolds were first established by Eells and Sampson in 1964. Afterwards, there are two reports and one survey paper by Eells and Lemaire [15–17] about the developments of harmonic maps up to 1988. Chiang, Ratto, Sun and Wolak also studied harmonic and biharmonic maps in [4–9]. f-harmonic maps which generalize harmonic maps, were first introduced by Lichnerowicz [25] in 1970, and were studied by Course [12,13] recently. The f-harmonic maps relate to the equation of the motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction in mathematical physics. Moreover, F-harmonic maps between Riemannian manifolds were first introduced by Ara [1,2] in 1999, which could be considered as the special cases of f-harmonic maps.

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. By definition the ***f*-biharmonic** maps between Riemannian manifolds are the critical points of ***f*-bienergy**

$$E_2^f(\psi) = \frac{1}{2}\int_{M_1} f|\tau_f(\psi|^2 \mathrm{d}v$$

where dv the volume form determined by the metric g. The f-biharmonic maps between Riemannian manifolds which generalized biharmonic maps by Jiang [20, 21] in 1986, were first studied by Ouakkas, Nasri and Djaa [27] in 2010.

In section two, we describe the motivation, and review f-harmonic maps and their relationship with F-harmonic maps. In Theorem 3.1, we show that if ψ is an f-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then ψ is an f-harmonic map. It is well-known from [18] that if ψ is a harmonic map of Riemannian manifolds and ϕ is a totally geodesic map of Riemannian manifolds, then $\phi \circ \psi$ is harmonic. However, if ψ is f-biharmonic and ϕ is totally geodesic, then $\phi \circ \psi$ is not necessarily f-biharmonic. Instead, we prove in Theorem 3.3 that if the f-tension field $\tau_f(\psi)$ of a smooth map ψ of Riemannian manifolds is a Jacobi field and ϕ is totally geodesic, then $\tau_f(\phi \circ \psi)$ is a Jacobi field. It implies Corollary 3.4 [8] that if ψ is a biharmonic map between Riemannian manifolds and ϕ is totally geodesic, then $\phi \circ \psi$ is a biharmonic map. We finally investigate the stress f-bienergy tensors. If ψ is an f-biharmonic of Riemannian manifolds, then it usually does not satisfy the conservation law for the stress f-bienergy tensor $S_2^f(\psi)$. However, we obtain in Theorem 4.2 that if $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds, then

$$\operatorname{div} S_2^f(Y) = \pm\langle J_{\tau_f(\psi)}(Y),\ d\psi(Y)\rangle \qquad \text{for all} \quad Y \in \Gamma(TM_1) \tag{1}$$

where div S_2^f is the divergence of S_2^f and $J_{\tau_f(\psi)}$ is the Jacobi field of $\tau_f(\psi)$ (there is a - or + sign convention in the formula). Hence, if $\tau_f(\psi)$ is a Jacobi field, then ψ satisfies the conservation law for S_2^f. It implies Corollary 4.4 [22] that if ψ is a biharmonic map between Riemannian manifolds, then ψ satisfies the conservation law for the stress bi-energy tensor $S_2(\psi)$. We also discuss a few applications concerning the vanishing of the stress f-bienergy tensor.

2. Preliminaries

2.1. Motivation

In mathematical physics, the equation of the motion of a continuous system of spins with inhomogeneous neighborhood Heisenberg interaction is

$$\frac{\partial \psi}{\partial t} = f(x)(\psi \times \triangle\psi) + \nabla f \cdot (\psi \times \nabla\psi) \tag{2}$$

where $\Omega \subset R^m$ is a smooth domain in the Euclidean space, f is a real-valued function defined on Ω, $\psi(x,t) \in S^2$, and $\times$ is the cross product in $\mathbb{R}^3$ and $\triangle$ is the **Laplace operator** in $\mathbb{R}^m$. Such a model is called the inhomogeneous Heisenberg ferromagnet [10, 11, 14]. Physically, the function f is called the coupling function, and is the continuum of the coupling constant between the neighboring spins. It is known from [18] that the tension field of a map ψ into S^2 is $\tau(\psi) = \triangle\psi + |\nabla\psi|^2\psi$. Observe that the right hand side of (2) can be expressed as

$$\psi \times (f\tau(\psi) + \nabla f \cdot \nabla\psi). \tag{3}$$

Hence, ψ is a smooth stationary solution (i.e., $\frac{\partial\psi}{\partial t} = 0$) of (2) if and only if

$$f\tau(\psi) + \nabla f \cdot \nabla\psi = 0 \tag{4}$$

i.e., ψ is an f-harmonic map. Consequently, there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (2) on the domain Ω and the set of f-harmonic maps from Ω into S^2. The inhomogeneous Heisenberg spin system (2) is also called inhomogeneous Landau-Lifshitz system (cf. [19, 23, 24]).

2.2. f-harmonic Maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. Many aspects of the f-harmonic maps which generalize harmonic maps, were studied in [12, 13, 19, 24] recently. Let $\psi : (M_1,\ g) \to (M_2,\ h)$ be a smooth map from an m-dimensional Riemannian manifold $(M_1,\ g)$ into an n-dimensional Riemannian manifold $(M_2,\ h)$. A map $\psi : (M_1, g) \to (M_2, h)$ is ***f*-harmonic** if and only if ψ is a critical point of the f-energy

$$E_f(\psi) = \frac{1}{2}\int_{M_1} f|\mathrm{d}\psi|^2 \mathrm{d}v.$$

In terms of the Euler-Lagrange equation, ψ is f-harmonic if and only if the ***f*-tension field**

$$\tau_f(\psi) = f\tau(\psi) + \mathrm{d}\psi(\mathrm{grad}\, f) = 0 \tag{5}$$

where $\tau(\psi) = \mathrm{Tr}_g\, D\mathrm{d}\psi$ is the tension field of ψ. In particular, when $f = 1$, $\tau_f(\psi) = \tau(\psi)$.

Let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F' > 0$ on $(0, \infty)$. F-harmonic maps between Riemannian manifolds were introduced in [1, 2]. For a smooth map $\psi : (M_1,\ g) \to (M_2, h)$ of Riemannian manifolds, the F-energy of ψ is defined by

$$E_F(\psi) = \int_{M_1} F(\frac{|\mathrm{d}\psi|^2}{2})\mathrm{d}v. \tag{6}$$

When $F(t) = t$, $\frac{(2t)^{p/2}}{p}$ $(p \geq 4)$, $(1 + 2t)^\alpha$ $(\alpha > 1, \dim M_1 = 2)$, and e^t, they are the energy, the p-energy, the α-energy of Sacks-Uhlenbeck [28], and the exponential energy, respectively. A map ψ is F-harmonic if and only if ψ is a critical point of the F-energy functional. In terms of the Euler-Lagrange equation,

$\psi : M_1 \to M_2$ is an $F - harmonic$ map if and only if the **F-tension field**

$$\tau_F(\psi) = F'(\frac{|d\psi|^2}{2})\tau(\psi) + \psi_* \left(\text{grad}(F'(\frac{|d\psi|^2}{2}))\right) = 0. \tag{7}$$

Proposition 1. 1) *If $\psi : (M_1, g) \to (M_2, h)$ an F-harmonic map without critical points (i.e., $|d\psi_x| \neq 0$ for all $x \in M_1$), then it is an f-harmonic map with $f = F'(\frac{|d\psi|^2}{2})$. In particular, a p-harmonic map without critical points is an f-harmonic map with $f = |d\psi|^{p-2}$.*
2) [15, 25]. *A map $\psi : (M_1^m, g) \to (M_2^n, h)$ is f-harmonic if and only if $\psi : (M_1^m, f^{\frac{2}{m-2}} g) \to (M_2^n, h)$ is a harmonic map.*

Proof: 1) It follows from (5) and (7) immediately (cf. Corollary 1.1 in [26]).
2) See [15]. ■

3. f-biharmonic maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. f-biharmonic maps between Riemannian manifolds which generalized biharmonic maps [20,21], were first studied by Ouakkas, Nasri and Djaa [27] recently. An f-biharmonic map $\psi : (M_1, g) \to (M_2, h)$ between Riemannian manifolds is the critical point of the f-bienergy functional

$$(E_2)_f(\psi) = \frac{1}{2}\int_{M_1} ||\tau_f(\psi)||^2 dv \tag{8}$$

where the f-tension field $\tau_f(\psi) = f\tau(\psi) + d\psi(\text{grad} f)$. In terms of Euler-Lagrange equation, ψ is f-biharmonic if and only if the **f-bitension field** of ψ

$$(\tau_2)_f(\psi) = \pm\triangle_2^f \tau_f(\psi) \pm fR'(\tau_f(\psi), d\psi)d\psi = 0 \tag{9}$$

where

$$\triangle_2^f \tau_f(\psi) = \bar{D} f \bar{D}\tau_f(\psi) - f\bar{D}_D\tau_f(\psi) = \sum_{i=1}^m (\bar{D}_{e_i} f\bar{D}_{e_i}\tau_f(\psi) - f\bar{D}_{D_{e_i}e_i}\tau_f(\psi)).$$

Here, D, $\bar{D}$ are the connections on TM_1, $\psi^{-1}TM_2$, respectively, $\{e_i\}_{1\le i\le m}$ is a local orthonormal frame at any point in M_1, and R' is the Riemannian curvature of M_2. There is a + or - sign convention in (9), and we take + sign in the context for simplicity. In particular, if $f = 1$, then $(\tau_2)_f(\psi) = \tau_2(\psi)$, the bitension field of ψ.

Theorem 2. *If $\psi : (M_1, g) \to (M_2, h)$ is a f-biharmonic map from a compact Riemannian manifold M_1 into a Riemannian manifold M_2 with non-positive curvature satisfying*

$$\langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) - \bar{D}_{e_i} f\bar{D}_{e_i}\tau_f(\psi), \tau_f(\psi)\rangle \ge 0 \tag{10}$$

then ψ is f-harmonic.

Proof: Since $\psi : M_1 \to M_2$ is f-biharmonic, it follows from (9) that

$$(\tau_2)_f(\psi) = \bar{D} f \bar{D} \tau_f(\psi) - f \bar{D}_D \tau_f(\psi) + f R'(\tau_f(\psi), \mathrm{d}\psi)\mathrm{d}\psi = 0. \qquad (11)$$

Suppose that the compact supports of the maps $\frac{\partial \psi_t}{\partial t}$ and $\bar{D}_{e_i} \frac{\partial \psi_t}{\partial t}$ ($\{\psi_t\} \in C^\infty(M_1 \times (-\epsilon, \epsilon), M_2)$ is a one parameter family of maps with $\psi_0 = \psi$) are contained in the interior of M. We compute

$$\begin{aligned}\frac{1}{2} f \triangle ||\tau_f(\psi)||^2 &= f\langle \bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + f\langle \bar{D}^*\bar{D}\tau_f(\psi), \tau_f(\psi)\rangle \\ &= f\langle \bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + f\langle \bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \\ &\quad -\bar{D}_{D_{e_ie_i}}\tau_f(\psi)), \tau_f(\psi)\rangle \\ &= f\langle \bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + \langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \qquad (12) \\ &\quad -\bar{D}_{e_i} f \bar{D}_{e_i}\tau_f(\psi) + \bar{D}_{e_i} f \bar{D}_{e_i}\tau_f(\psi) - f\bar{D}_{D_{e_ie_i}}\tau_f(\psi), \tau_f(\psi)\rangle \\ &= f\langle \bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + \langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \\ &\quad -\bar{D}_{e_i} f \bar{D}_{e_i}\tau_f(\psi), \tau_f(\psi)\rangle - \langle f(R'(\mathrm{d}\psi, \mathrm{d}\psi)\tau_f(\psi), \tau_f(\psi)\rangle \geq 0\end{aligned}$$

(where $\bar{D}^*\bar{D} = \bar{D}\bar{D} - \bar{D}_D$ [20]) by (10), (11), $f > 0$ and $R' \leq 0$. It implies that

$$\frac{1}{2}\triangle ||\tau_f(\psi)||^2 \geq 0.$$

By applying the Bochner's technique, we know that $||\tau_f(\psi)||^2$ is constant and that

$$\bar{D}_{e_i}\tau_f(\psi) = 0 \qquad \text{for all} \quad i = 1, 2, ...m.$$

It follows from Eells-Lemaire [15] results that $\tau_f(\psi)$=0, i.e., ψ is f-harmonic on M_1. ■

Corollary 3 ([20]). *If $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map from a compact Riemannian M_1 manifold into a Riemannian manifold M_2 with non-positive curvature, then ψ is harmonic.*

Proof: When $f = 1$ and $\psi : M_1 \to M_2$ is a biharmonic map from a compact Riemannian M_1 manifold into a Riemannian manifold M_2 with non-positive curvature, (11) becomes

$$\tau_2(\psi) = \bar{D}^*\bar{D}\tau(\psi) + R'(\tau(\psi), \mathrm{d}\psi)\mathrm{d}\psi = 0.$$

The identity (13) reduces to

$$\begin{aligned}\frac{1}{2}\triangle ||\tau(\psi)||^2 &= \langle \bar{D}_{e_i}\tau(\psi), \bar{D}_{e_i}\tau(\psi)\rangle + \langle \bar{D}^*\bar{D}\tau(\psi), \tau(\psi)\rangle \\ &= \langle \bar{D}_{e_i}\tau(\psi), \bar{D}_{e_i}\tau(\psi)\rangle - \langle R'(\mathrm{d}\psi, \mathrm{d}\psi)\tau(\psi), \tau(\psi)\rangle \geq 0\end{aligned}$$

since ψ is biharmonic, and M_2 is a Riemannian manifold with non-positive curvature R'. Note that (10) is automatically satisfied. It follows from the similar arguments as Theorem 3.1 that ψ is harmonic. ■

It is well-known from [18] that if $\psi : (M_1, g) \to (M_2, h)$ is a harmonic map of two Riemannian manifolds and $\phi : (M_2, h) \to (M_3, k)$ is totally geodesic of two Riemannian manifolds, then $\phi \circ \psi : (M_1, g) \to (M_3, k)$ is harmonic. However, if $\psi : (M_1, g) \to (M_2, h)$ is an f-biharmonic map, and $\phi : (M_2, h) \to (M_3, k)$ is totally geodesic, then $\phi \circ \psi : (M_1, g) \to (M_3, k)$ is not necessarily an f-biharmonic map. We obtain the following theorem instead.

Theorem 4. *If $\tau_f(\psi)$ is a Jacobi field for a smooth map $\psi : (M_1, g) \to (M_2, h)$ of two Riemannian manifolds, and $\phi : (M_2, h) \to (M_3, k)$ is a totally geodesic map of two Riemannian manifolds, then $\tau_f(\phi \circ \psi)$ is a Jacobi field.*

Proof: Let $D, D', \bar{D}, \bar{D}', \bar{D}'', \hat{D}, \hat{D}'$ and $\hat{D}''$ are the respective connections on TM_1, TM_2, $\psi^{-1}TM_2$, $\phi^{-1}TM_3$, $(\phi \circ \psi)^{-1}TM_3, T^*M_1 \otimes \psi^{-1}TM_2, T^*M_2 \otimes \phi^{-1}TM_3$ and $T^*M_1 \otimes (\phi \circ \psi)^{-1}TM_3$. Then we have

$$\bar{D}''_X \mathrm{d}(\phi \circ \psi)(Y) = (\hat{D}'_{\mathrm{d}\psi(X)} \mathrm{d}\phi)\mathrm{d}\psi(Y) + \mathrm{d}\phi \circ \bar{D}_X \mathrm{d}\psi(Y) \tag{13}$$

for all $X, Y \in \Gamma(TM_1)$. We have also

$$R^{M_3}(\mathrm{d}\phi(X'), \mathrm{d}\phi(Y'))\mathrm{d}\phi(Z') = R^{\phi^{-1}TM_3}(X', Y')\mathrm{d}\phi(Z') \tag{14}$$

for all $X', Y', Z' \in \Gamma(TM_2)$.
It is well-known from [18] that the tension field of the composition $\phi \circ \psi$ is given by

$$\tau(\phi \circ \psi) = \mathrm{d}\phi(\tau(\psi)) + \mathrm{Tr}_g\, D\mathrm{d}\phi(\mathrm{d}\psi, \mathrm{d}\psi) = \mathrm{d}\phi(\tau(\psi))$$

since ϕ is totally geodesic. Then the f-tension field of the composition of $\phi \circ \psi$ is

$$\tau_f(\psi \circ \phi) = \mathrm{d}\phi(\tau_f(\psi)) + f\, \mathrm{Tr}_g\, D\mathrm{d}\phi(\mathrm{d}\psi, \mathrm{d}\psi) = \mathrm{d}\phi(\tau_f(\psi))$$

since ϕ is totally geodesic. Recall that $\{e_i\}_{i=1}^m$ is a local orthonormal frame at any point in M_1, and let $\bar{D}^*\bar{D} = \bar{D}_{e_k}\bar{D}_{e_k} - \bar{D}_{D_{e_k}e_k}$ and $\bar{D}''^*\bar{D}'' = \bar{D}''_{e_k}\bar{D}''_{e_k} - \bar{D}''_{D_{e_k}e_k}$. Then we have

$$\begin{aligned} \bar{D}''^*\bar{D}''\tau_f(\phi \circ \psi) &= \bar{D}''^*\bar{D}''(\mathrm{d}\phi \circ \tau_f(\psi)) \\ &= \bar{D}''_{e_k}\bar{D}''_{e_k}(\mathrm{d}\phi \circ \tau_f(\psi)) - \bar{D}''_{D_{e_k}e_k}(\mathrm{d}\phi \circ \tau_f(\psi)). \end{aligned} \tag{15}$$

We derive from (13) that

$$\begin{aligned} \bar{D}''_{e_k}(\mathrm{d}\phi \circ \tau_f(\psi)) = (\hat{D}'_{\bar{D}_{e_j}\mathrm{d}\psi(e_k)}\mathrm{d}\phi)(\tau_f(\psi)) + \mathrm{d}\phi \circ \bar{D}_{e_k}(\tau_f(\psi)) \\ = \mathrm{d}\phi \circ \bar{D}_{e_k}\tau_f(\psi) \end{aligned}$$

since ϕ is totally geodesic. Therefore, we arrive at

$$\bar{D}''_{e_k}\bar{D}''_{e_k}(\mathrm{d}\phi\circ\tau_f(\psi)) = \bar{D}''_{e_k}(\mathrm{d}\phi\circ\bar{D}_{e_k}\tau_f(\psi)) = \mathrm{d}\phi\circ\bar{D}_{e_k}\bar{D}_{e_k}\tau_f(\psi) \quad (16)$$

and

$$\bar{D}''_{D_{e_k}e_k}(\mathrm{d}\phi\circ\tau(\psi)) = \mathrm{d}\phi\circ\bar{D}_{D_{e_k}e_k}\tau_f(\psi). \quad (17)$$

Substituting (16), (17) into (16), we deduce

$$\bar{D}''^{*}\bar{D}''\tau_f(\phi\circ\psi) = \mathrm{d}\phi\circ\bar{D}^{*}\bar{D}\tau_f(\psi). \quad (18)$$

On the other hand, it follows from (14) that

$$\begin{aligned} R^{M_3}&(\mathrm{d}(\phi\circ\psi)(e_i),\tau_f(\phi\circ\psi))\mathrm{d}(\phi\circ\psi)(e_i) \\ &= R^{\phi^{-1}TM_3}(\mathrm{d}\psi(e_i),\tau_f(\psi))\mathrm{d}\phi(\mathrm{d}\psi(e_i)) \\ &= \mathrm{d}\phi\circ R^{M_2}(\mathrm{d}\psi(e_i),\tau_f(\psi))\mathrm{d}\psi(e_i). \end{aligned} \quad (19)$$

By (18) and (19), we obtain

$$\begin{aligned} \bar{D}''^{*}\bar{D}''\tau_f(\phi\circ\psi) &+ R^{M_3}(\mathrm{d}(\phi\circ\psi)(e_i),\tau_f(\phi\circ\psi))\mathrm{d}(\phi\circ\psi)(e_i) \\ &= \mathrm{d}\phi\circ[\bar{D}^{*}\bar{D}\tau_f(\psi) + R^{M_2}(\mathrm{d}\psi(e_i),\tau_f(\psi))\mathrm{d}\psi(e_i)]. \end{aligned} \quad (20)$$

Consequently, if $\tau_f(\psi)$ is a Jacobi field, then $\tau_f(\phi\circ\psi)$ is a Jacobi field. ∎

Corollary 5 ([8]). *If $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map between two Riemannian manifolds and $\phi : (M_2, h) \to (M_3, k)$ is totally geodesic, then $\phi\circ\psi : (M_1, g) \to (M_3, k)$ is a biharmonic map.*

Proof: If $f = 1$ and $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map of two Riemannian manifolds, then $\tau_f(\psi) = \tau(\psi)$ is a Jacobi field. We can apply the analogous arguments as Theorem 3.3, and (20) becomes

$$\begin{aligned} \bar{D}''^{*}\bar{D}''\tau(\phi\circ\psi) &+ R^{M_3}(\mathrm{d}(\phi\circ\psi)(e_i),\tau(\phi\circ\psi))\mathrm{d}(\phi\circ\psi)(e_i) \\ &= \mathrm{d}\phi\circ[\bar{D}^{*}\bar{D}\tau(\psi) + R^{M_2}(\mathrm{d}\psi(e_i),\tau(\psi))\mathrm{d}\psi(e_i)] \end{aligned}$$

i.e., $\tau_2(\phi\circ\psi) = \mathrm{d}\phi\circ(\tau_2(\psi))$, where $\tau_2(\psi)$ is the bi-tension field of ψ. Hence, we can conclude the result. ∎

4. Stress f-bienergy Tensors

Let $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds. The stress energy tensor is defined by Baird and Eells [3] as

$$S(\psi) = e(\psi)g - \psi^{*}h$$

where $e(\psi) = \frac{|\mathrm{d}\psi|^2}{2}$. Thus we have $\mathrm{div}S(\psi) = -\langle\tau(\psi),\mathrm{d}\psi\rangle$. Hence, if ψ is harmonic, then ψ satisfies the conservation law for S (i.e., $\mathrm{div}\, S(\psi) = 0$). In

[27], the **stress f-energy tensor** of the smooth map $\psi : M_1 \to M_2$ was similarly defined as

$$S^f(\psi) = fe(\psi)g - f\psi^*h$$

and they obtained

$$\operatorname{div} S^f(\psi) = -\langle \tau_f(\psi), \mathrm{d}\psi \rangle + e(\psi)\mathrm{d}f.$$

In this case, an f-harmonic map usually does not satisfy the conservation law for S^f. In particular, by letting $f = F'(\frac{\mathrm{d}\psi|^2}{2})$, then $S^f(\psi) = F'(\frac{\mathrm{d}\psi|^2}{2})e(\psi)g - F'(\frac{\mathrm{d}\psi|^2}{2})\psi^*h$. It is different than following Ara's idea [1] to define $S^F(\psi) = F(\frac{|\mathrm{d}\psi|^2}{2})g - F'(\frac{\mathrm{d}\psi|^2}{2})\psi^*h$, and we have

$$\operatorname{div} S^F(\psi) = -\langle \tau_F(\psi), \mathrm{d}\psi \rangle.$$

It implies that if $\psi : M_1 \to M_2$ is an F-harmonic map between Riemannian manifolds, then it satisfies the conservation law for S^F.

The stress bienergy tensors and the conservation laws of biharmonic maps between Riemannian manifolds were first studied by Jiang [22] in 1987. Following his notions, we define the stress f-bienergy tensor of a smooth map as follows.

Definition 6. *Let $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds. The* ***stress f-bienergy tensor*** *of ψ is defined by*

$$\begin{aligned} S_2^f(X,Y) &= \frac{1}{2}|\tau_f(\psi)|^2\langle X, Y\rangle + \langle \mathrm{d}\psi, \bar{D}(\tau_f(\psi))\rangle\langle X, Y\rangle \\ &\quad -\langle \mathrm{d}\psi(X), \bar{D}_Y\tau_f(\psi)\rangle - \langle \mathrm{d}\psi(Y), \bar{D}_X\tau_f(\psi)\rangle \end{aligned} \tag{21}$$

for all $X, Y \in \Gamma(TM_1)$.

Remark that if $\psi : (M_1, g) \to (M_2, h)$ is an f-biharmonic map between two Riemannian manifolds, then ψ does not necessarily satisfy the conservation law for the stress f-bienergy tensor S_2^f. Instead, we obtain the following theorem.

Theorem 7. *If $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds, then we have*

$$\operatorname{div} S_2^f(Y) = \pm\langle J_{\tau_f(\psi)}(Y), \mathrm{d}\psi(Y)\rangle \qquad \textit{for all} \quad Y \in \Gamma(TM_1) \tag{22}$$

where $J_{\tau_f(\psi)}$ is the Jacobi field of $\tau_f(\psi)$.

Proof: For the map $\psi : M_1 \to M_2$ between two Riemannian manifolds, set $S_2^f = K_1 + K_2$, where K_1 and K_2 are (0, 2)-tensors defined by

$$\begin{aligned} K_1(X,Y) &= \frac{1}{2}|\tau_f(\psi)|^2\langle X,Y\rangle + \langle \mathrm{d}\psi, \bar{D}\tau_f(\psi)\rangle\langle X,Y\rangle \\ K_2(X,Y) &= -\langle \mathrm{d}\psi(X), \bar{D}_Y\tau_f(\psi)\rangle - \langle \mathrm{d}\psi, \bar{D}_X\tau_f(\psi)\rangle. \end{aligned}$$

Let $\{e_i\}$ be the geodesic frame at a point $a \in M_1$, and write $Y = Y^i e_i$ at the point a. We first compute

$$\begin{aligned}
\operatorname{div} K_1(Y) &= \sum_i (\bar{D}_{e_i} K_1)(e_i, Y) = \sum_i (e_i(K_1(e_i, Y) - K_1(e_i,\, \bar{D}_{e_i} Y)) \\
&= \sum_i (e_i(\frac{1}{2}|\tau_f(\psi)|^2 Y^i + \sum_k \langle \mathrm{d}\psi(e_k),\, \bar{D}_{e_k}\tau_f(\psi)\rangle Y^i) \\
&\quad -\frac{1}{2}|\tau_f(\psi)|^2 Y^i e_i - \sum_k \langle \mathrm{d}\psi(e_k),\, \bar{D}_{e_k}\tau_f(\psi)\rangle Y^i e_i)) \\
&= \langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi)\rangle + \sum_i \langle \mathrm{d}\psi(Y, e_i),\, \bar{D}_{e_i}\tau_f(\psi)\rangle \\
&\quad + \sum_i \langle \mathrm{d}\psi(e_i),\, \bar{D}_Y \bar{D}_{e_i}\tau_f(\psi)\rangle \\
&= \langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi)\rangle + \mathrm{Tr}\langle \bar{D}\mathrm{d}\psi(Y, .),\, \bar{D}.\tau_f(\psi)\rangle \\
&\quad + \mathrm{Tr}\langle \mathrm{d}\psi(.), \bar{D}^2\tau_f(\psi)(Y, .)\rangle.
\end{aligned} \tag{23}$$

We then compute

$$\begin{aligned}
\operatorname{div} K_2(Y) &= \sum_i (\bar{D}_{e_i} K_2)(e_i, Y) = \sum_i (e_i(K_2(e_i, Y) - K_2(e_i, \bar{D}_{e_i} Y)) \\
&= -\langle \bar{D}_Y \tau_f(\psi),\, \tau_f(\psi)\rangle - \sum_i \langle \bar{D}\mathrm{d}\psi(Y, e_i),\, \bar{D}_{e_i}\tau_f(\psi)\rangle \\
&\quad - \sum_i \langle \mathrm{d}\psi(e_i),\, \bar{D}_{e_i}\bar{D}_Y \tau_f(\psi) - \bar{D}_{D_{e_i}Y}\tau_f(\psi)\rangle \\
&\quad + \langle \mathrm{d}\psi(Y), \triangle\tau_f(\psi)\rangle = -\langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi)\rangle \\
&\quad - \mathrm{Tr}\langle \bar{D}\mathrm{d}\psi(Y, .),\, \bar{D}_.\tau_f(\psi)\rangle \\
&\quad - \mathrm{Tr}\langle \mathrm{d}\psi(.), \bar{D}^2\tau_f(\psi)(., Y)\rangle + \langle \mathrm{d}\psi(Y), \triangle\tau_f(\psi)\rangle.
\end{aligned} \tag{24}$$

Adding (24) and (25), we arrive at

$$\begin{aligned}
\mathrm{div} S_2^f(Y) &= \pm\langle \mathrm{d}\psi(Y),\ \triangle\tau_f(\psi) + \sum_i \langle \mathrm{d}\psi(e_i), R'(Y, e_i)\tau_f(\psi)\rangle \\
&= \pm\langle J_{\tau_f(\psi)}(Y),\, \mathrm{d}\psi(Y)\rangle
\end{aligned} \tag{25}$$

where $J_{\tau_f(\psi)}$ is the Jacobi field of $\tau_f(\psi)$. ■

Corollary 8. *If $\tau_f(\psi)$ is a Jacobi field (i.e., $J_{\tau_f(\psi)} = 0$) for a map $\psi : M_1 \to M_2$, then it satisfies the conservation law (i.e., $\operatorname{div} S_2^f = 0$) for the stress f-bienergy tensor S_2^f.*

Theorem 9 ([22]). *If $\psi : (M_1, g) \to (M_2, h)$ is biharmonic between two Riemannian manifolds, then it satisfies the conservation law for stress bienergy tensor S_2*

Proof: If $f = 1$ and $\psi : (M_1, g) \to (M_2, h)$ is biharmonic, then (26) yields to

$$\begin{aligned}\operatorname{div} S_2(Y) &= \pm\langle \mathrm{d}\psi,\, \triangle\tau(\psi) + \sum_i (\mathrm{d}\psi(e_i), R'(Y, X_i)\tau(\psi)\rangle \\ &= \pm\langle J_{\tau(\psi)}(Y),\, \mathrm{d}\psi(Y)\rangle = \pm\langle \tau_2(\psi),\, \mathrm{d}\psi(Y)\rangle\end{aligned}$$

where $\tau_2(\psi)$ is the bi-tension field of ψ (i.e., $\tau(\psi)$ is a Jacobi field). Hence, we can conclude the result. ■

Proposition 10. *Let $\psi : (M_1, g) \to (M_2, h)$ be a submersion such that $\tau_f(\psi)$ is basic, i.e., $\tau_f(\psi) = W \circ \psi$ for $W \in \Gamma(TM_2)$. Suppose that W is Killing and $|W|^2 = c^2$ is non-zero constant. If M_1 is non-compact, then $\tau_f(\psi)$ is a non-trivial Jacobi field.*

Proof: Since $\tau_f(\psi)$ is basic

$$\begin{aligned}S_2^f(X, Y) &= (\frac{c^2}{2} + \langle \mathrm{d}\psi, \bar{D}\tau_f(\psi)\rangle)(X, Y) - \langle \mathrm{d}\psi(X),\, \bar{D}_Y\tau_f(\psi)\rangle \\ &\quad -\langle \mathrm{d}\psi(Y),\, \bar{D}_X\tau_f(\psi)\rangle\end{aligned} \tag{26}$$

where $X, Y \in \Gamma(TM_1)$. Let a be a point in M_1 with the orthonormal frame $\{e_i\}_{i=1}^m$ such that $\{e_j\}_{j=1}^n$ are in $T_a^H M_1 = (T_a^V M_1)^\perp$ and $\{e_k\}_{k=n+1}^m$ are in $T_a^V M_1 = \operatorname{Ker} \mathrm{d}\psi(a)$. Because W is Killing, we have

$$\begin{aligned}\langle \mathrm{d}\psi,\, \bar{D}\tau_f(\psi)\rangle(a) &= \sum_j \langle \mathrm{d}\psi_a(e_j),\, \bar{D}_{e_j}\tau_f(\psi)\rangle + \sum_k \langle \mathrm{d}\psi_a(e_k),\, \bar{D}_{e_k}\tau_f(\psi)\rangle \\ &= \sum_j \langle \mathrm{d}\psi_a(e_j),\, D^{M_2}_{\mathrm{d}\psi_a(e_j)}W\rangle = 0.\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}S_2^f(a)(X, Y) &= \frac{c^2}{2}(X, Y) + \langle \mathrm{d}\psi_a(X),\, D^{M_2}_{\mathrm{d}\psi_a(Y)}W\rangle \\ &\quad -\langle \mathrm{d}\psi_a(Y),\, D^{M_2}_{\mathrm{d}\psi_a(X)}W\rangle = \frac{c^2}{2}(X, Y).\end{aligned}$$

If M_1 is not compact, $S_2^f = \frac{c^2}{2}g$ is divergence free and $\tau_f(\psi)$ is a non-trivial Jacobi field due to $c \neq 0$. ■

Proposition 11. *If $\psi : (M_1^2, g) \to (M_2, h)$ is a map from a surface with $S_2^f = 0$, then ψ is f-harmonic.*

Proof: Since $S_2^f = 0$, it implies

$$0 = \mathrm{Tr}\, S_2^f = |\tau_f(\psi)|^2 + 2\langle \bar{D}\tau_f(\psi),\, \mathrm{d}\psi\rangle - 2\langle \bar{D}\tau_f(\psi),\, \mathrm{d}\psi\rangle = |\tau_f(\psi)|^2.$$

■

Proposition 12. *If* $\psi : (M_1^m, g) \to (M_2, h)$ $(m \neq 2)$ *with* $S_2^f = 0$*, then*

$$\frac{1}{m-2}|\tau_f(\psi)|^2(X,Y) + \langle \bar{D}_X\tau_f(\psi), \mathrm{d}\psi(Y)\rangle + \langle \bar{D}_Y\tau_f(\psi), \mathrm{d}\psi(X)\rangle = 0 \quad (28)$$

for $X,\, Y \in \Gamma(T(M_1))$.

Proof: Suppose that $S_2^f = 0$, it implies $\mathrm{Tr}\, S_2^f = 0$. Therefore

$$\langle \bar{D}\tau_f(\psi), \mathrm{d}\psi\rangle \;=\; -\frac{m}{2(m-2)}|\tau_f(\psi)|^2, \qquad m \neq 2. \quad (29)$$

Substituting it into the definition of S_2^f, we arrive at

$$\begin{aligned} 0 \;=\; S_2^f(X,Y) &= -\frac{1}{m-2}|\tau_f(\psi)|^2(X,Y) \\ &-\langle \bar{D}_X\tau_f(\psi >, \mathrm{d}\psi(Y) > -\langle \bar{D}_Y\tau_f(\psi), \mathrm{d}\psi(X)\rangle. \end{aligned} \quad (30)$$

■

Corollary 13. *If* $\psi : (M_1, g) \to (M_2, h)$ $(m > 2)$ *with* $S_1^f = 0$ *and* $\mathrm{rank}\,\psi \leq m-1$*, then* ψ *is f-harmonic.*

Proof: Since $\mathrm{rank}\,\psi(a) \leq m-1$, for a point $a \in M_1$ there exists a unit vector $X_a \in \mathrm{Ker}\,\mathrm{d}\psi_a$. Letting $X = Y = X_a$, (28) gives to $\tau_f(\psi) = 0$. ■

Corollary 14. *If* $\psi : (M_1, g) \to (M_2, h)$ *is a submersion* $(m > n)$ *with* $S_2^f = 0$*, then* ψ *is f-harmonic.*

References

[1] Ara M., *Geometry of F-harmonic Maps*, Kodai Math. J. **22** (1999) 242-263.

[2] Ara M., *Stability of F-harmonic Maps into Pinched Manifolds*, Hiroshima Math. J. **31** (2001) 171-181.

[3] Baird P. and Eells J., *A Conservation Law for Harmonic Maps*, Lecture Notes in Math. vol. 894, Springer, Berlin 1981, pp. 1-15.

[4] Chiang Y.-J., *Harmonic Maps of V-manifolds*, Ann. Global Anal. Geom. **8** (1990) 315-344.

[5] Chiang Y.-J., *Spectral Geometry of V-manifolds and its Applications to Harmonic Maps*, Proc. Sympos. Pure Math., vol. 54, Part 1, Amer. Math. Soc., Providence 1993 pp. 93-99.

[6] Chiang Y.-J. and Ratto A., *Harmonic Maps on Spaces with Conical Singularities*, Bull. Soc. Math. France **120** (1992) 251-262.

[7] Chiang Y.-J. and Sun H., *2-harmonic Totally Real Submanifolds in a Complex Projective Space*, Bull. Inst. Math. Acad. Sinica **27** (1999) 99-107.

[8] Chiang Y.-J. and Sun H., *Biharmonic Maps on V-manifolds*, Int. J. Math. Math. Sci. **27** (2001) 477-484.

[9] Chiang Y.-J. and Wolak R., *Transversally Biharmonic Maps Between Foliated Riemannian Manifolds*, Internat. J. Math. **19** (2008) 981-996.

[10] Cieslinski J., Sym A. and Wesselius W., *On the Geometry of the Inhomogeneous Heisenberg Ferromagnet Model: Non-integrable Case*, J. Phys. A: Math. & Gen. **26** (1993) 1353-1364.

[11] Cieslinski J., Goldstein P. and Sym A., *On Integrability of the Inhomogeneous Heisenberg Ferromagnet: Examination of a New Test*, J. Phys. A: Math. & Gen. **27** (1994) 1645-1664.

[12] Course N., *f-harmonic Maps*, Thesis, University of Warwick, 2004.

[13] Course N., *f-harmonic Maps Which Map the Boundary of the Domain to One Point in the Target*, New York J. Math. **13** (2007) 423-435 (Electronic).

[14] Daniels M., Porsezian K. and Lakshmanan M., *On the Integrability of the Inhomogeneous Spherically Symmetric Heisenberg Ferromagnet in Arbitrary Dimension*, J. Math. Phys. **35** (1994) 6498-6510.

[15] Eells J. and Lemaire L., *A Report on Harmonic Maps*, London Math. Soc. **10** (1978) 1-68.

[16] Eells J. and Lemaire L., *Another Report on Harmonic Maps*, London Math. Soc. **20** (1988) 385-524.

[17] Eells J. and Lemaire L., *Selected Topics in Harmonic Maps*, C.B.M.S. Regional Conf. Series vol. 50, AMS, Providence 1983.

[18] Eells J. and Sampson J., *Harmonic Maps of Riemannian Manifolds*, Amer. J. Math. **86** (1964) 109-164.

[19] Huang P. and Tang H., *On the Heat Flow of f-harmonic Maps From D^2 into S^2*, Nonlinear Anal. **67** (2007) 2149-2156.

[20] Jiang G.-Y., *2-harmonic Maps Between Riemannian Manifolds* 1986, Annals Math. China A **7** (1986) 389-402.

[21] Jiang G.-Y., *2-harmonic Isometric Immersions Between Riemannian Manifolds*, Ann. Math. China A **7** (1986) 130-144.

[22] Jiang G.-Y., *The Conservation Law of Biharmonic Maps Between Riemannian Manifolds*, Acta Math. Sinica **30** (1987) 220-225.

[23] Lakshmanan M. and Bullough R., *Geometry of Generalized Nonlinear Schrödinger and Heisenberg Ferro-magnetic Spin Equations with x-dependent Coefficients*, Phys. Lett. A **80** (1980) 287-292.

[24] Li Y. X. and Wang Y. D., *Bubbling Location for f-harmonic Maps and Inhomogeneous Landau-Lifshitz Equations*, Comment. Math. Helv. **81** (2006) 433-448.

[25] Lichnerowicz A., *Applications Harmoniques et Varietes Kählеriennes*, Symposia Mathematica III, Academic Press, London 1970, pp. 341-402.

[26] Ou Y.-L., *On f-harmonic Morphisms Between Riemannian Manifolds*, arXiv:1103.5687v1 [math.DG] 29 Mar 2011.

[27] Ouakkas S., Basri R. and Djaa M., *On the f-harmonic and f-biharmonic Maps*, Int. E J. Geom. Topology **10** (2010) 11-27.

[28] Sacks J. and Uhlenbeck K., *The Existence of Minimal Immersions of 2-spheres*, Ann. Math. **113** (1981) 1-24.