GAUGE THEORIES WITH SPONTANEOUSLY BROKEN GAUGE SYMMETRY, CONNECTIONS AND DIRAC OPERATORS

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Abstract. In this paper we give a summary of the geometrical background of the idea of spontaneous symmetry breaking. For this purpose, we set out to discuss Yang–Mills–Higgs gauge theories from the perspective of reducible bundles. From this viewpoint, "elementary particles" are identified with vector bundles, and sections are considered to geometrically represent the states of the corresponding particle. Some physical background on the notion of "mass" is given in the introduction. Since the geometrical interpretation of a gauge boson is that of a connection, we proceed to discuss how, from a geometrical point of view, the Higgs boson can also be considered a connection. We start out with Connes' algebraic approach, where the "shifted Higgs boson" is considered a gauge potential on a non-commutative space. We summarize how a specific generalization of the notion of a Dirac operator can be used in order to define a generalization of de Rham's algebra. This generalization is used to define the non-commutative equivalent of the Yang–Mills action where its minima spontaneously break the gauge symmetry. In the last section we summarize how the Higgs boson can be considered a connection on a Clifford module bundle.

1. Introduction

The concept of a "spontaneously broken gauge theory" has been introduced in physics, for instance in solid state physics, within the phenomenon of superconductivity. This idea has also been adopted in elementary particle physics in order to describe the notion of *the mass of an elementary particle*, as such as of the electron.

To begin with, the notion of "mass" in the context of particle physics is quite different from that used in Newtonian mechanics. In the latter "mass" is a fundamental attribute of any (pointlike) particle; It can be mathematically described by a positve number ($m \in \mathbb{R}_+$), and it expresses its inertia against the action of some force. Since gravity is the most fundamental force we are all familiar with, "mass" and "weight" are widely considered as the same. However, from a physics point of view such an identification is not suitable. In particular, we cannot define the mass of a particle by its weight. There is even no strikt definition of mass at all, neither in Newtonian mechanics nor in elementary particle physics. Other than in Newtonian mechanics, however, in elementary particle physics the notion of mass is not considered a fundamental attribute of a particle. Instead, it is believed to be generated by the fundamental interaction of elementary particles with another one, called the **Higgs boson**.

At present we distinguish three ways of how elementary particles interact: the first results by the exchange of **gauge bosons**; the second way of how elementary particles interact results by the exchange of the above mentioned Higgs boson and which gives rise to the "mass of matter" that is in harmony with the gauge symmetry. Finally, the third kind of "communication" between elementary particles results by gravity. However, this interaction is usually believed to be too weak compared to the other two and is thus neglected within the phenomenology of particle physics.

How does "mass" manifest itself in the case of elementary particles? The answer to this simple question turns out to be tricky indeed, for various reasons. This holds true especially for particles from which matter is built of and which are called **fermions**[1] . Generally speaking, the notion of the "mass of a particle" makes sense only if the particle can be regarded as a **free particle**. The reason is that every kind of energy contributes to mass, according to Einsteins most famous formula $E = mc^2$. Thus, in general the masses of the fermions can be measured only indirectly. This is especially true for particles which do not have a "classical" counter part in nature as for instance the quarks. In contrast, the mass of an electron can be measured using methods of classical physics. For instance, the electron mass can be determined by measuring its deviation from a straight line when it moves in a magnetic field. On the

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[1] There are two kinds of particles known in nature: the fermions, which carry a 1/2 representation of the (double cover of the) rotational group. These kind of particles form the "basic building blocks of ordinary matter"; the second kind of particles known today are called bosons. They constitute an integer representation of the rotational group and build the carrier of forces. In particular, the spin-one representation is realized by gauge bosons. In contrast, the Higgs boson is assumed to carry a spin-zero representation. But this particle is not yet found in nature. Its existence, however, is highly expected because of the great success of the "standard model of particle physics", where the Higgs boson is a basic constituent.

is $U(1)$ gauge invariant. Assuming that $\Psi \equiv \mathrm{vac}$ is a *nonzero* and *constant* function we end up with

$$j = -2\langle \mathrm{vac}, \mathrm{vac} \rangle A\,. \tag{8}$$

Thus, we may identify the positive constant $2\|\mathrm{vac}\|^2$ with m^2 in (2)

$$m \equiv \sqrt{2}\|\mathrm{vac}\|\,. \tag{9}$$

Note that, in contrast to current (8), our definition of mass is now gauge invariant. But where does this "constant section" vac come from, and what is its geometrical significance? This and the corresponding geometrical description are summarized in the next paragraph.

2. The Geometry of the Bosonic Mass Matrices

In order to geometrically describe the idea of "spontaneous symmetry breaking" let us denote by $\mathcal{P}(\mathcal{M}, G)$ a G-principal bundle over a (compact) oriented, (pseudo-)Riemannian manifold $(\mathcal{M}, g_\mathrm{M})$ of dimension $\dim(\mathcal{M}) = n$. Here, G denotes a compact, real, semi-simple Lie group (typically some subgroup of $GL(N, \mathbb{C})$). Moreover, let ξ_E be some associated Hermitian vector bundle with typical fiber $\mathbb{C}^N$:

$$E := \mathcal{P} \times_\rho \mathbb{C}^N \xrightarrow{\pi_\mathrm{E}} \mathcal{M}\,. \tag{10}$$

Here, $G \xrightarrow{\rho} SU(N)$ denotes a unitary representation of the structure group G of $\mathcal{P}(\mathcal{M}, G)$. Let ξ_ad be the adjoint bundle associated to the G-principal bundle. We then denote by $\mathcal{G} := \Gamma(\xi_\mathrm{ad})$ the **gauge group** of $\mathcal{P}(\mathcal{M}, G)$. Any theory given by a (sufficiently smooth) functional

$$\begin{aligned}
&\mathcal{I}_\mathrm{YM+} \colon \Gamma(\xi_\mathrm{E}) \times \mathcal{A}(\xi_\mathrm{E}) \to \mathbb{R} \\
&(\Psi, \mathrm{d}^\mathrm{E}) \mapsto \mathcal{I}_\mathrm{YM}(\mathrm{d}^\mathrm{E}) + \mathcal{I}_+(\Psi, \mathrm{d}^\mathrm{E})
\end{aligned} \tag{11}$$

is referred to as a *gauge theory* if it is well-defined on the quotient space

$$(\Gamma(\xi_\mathrm{E}) \times \mathcal{A}(\xi_\mathrm{E}))/\mathcal{G}\,. \tag{12}$$

Here, $\Gamma(\xi_\mathrm{E})$ denotes the set of all sections of the vector bundle ξ_E. This is a module over the ring of smooth functions, denoted by $\mathfrak{A}$. The set $\mathcal{A}(\xi_\mathrm{E})$ is the affine space of (associated) smooth connections, represented by the corresponding (exterior) covariant derivatives on the respective vector bundle. The vector space of $\mathcal{A}(\xi_\mathrm{E})$ is $\Omega^1(\mathcal{M}, \rho'(\mathrm{Lie}_G))$, where Lie_G is the Lie algebra of

G and $\text{Lie}_G \xrightarrow{\rho'} su(N)$ is its "derived" representation. The functionals $\mathcal{I}_{\text{YM}}$ denotes the **Yang–Mills functional**

$$\mathcal{I}_{\text{YM}}(\, d^{\text{E}}) = \|F^{\text{E}}\|^2 \,, \tag{13}$$

where $F^{\text{E}} \in \Omega^2(\mathcal{M}, \rho'(\text{Lie}_G))$ is the *Yang-Mills curvature* defined by a connection on ξ_{E}. The $\mathcal{I}_+$ is some additional gauge invariant functional, which we will specify later.

A smooth G-invariant function

$$\begin{aligned} V_{\text{H}} : \mathbb{C}^N &\to \mathbb{R} \\ \mathbf{z} &\mapsto V_{\text{H}}(\mathbf{z}) \end{aligned} \tag{14}$$

which is also bounded from below is called a **general Higgs potential**. Clearly, such a function gives rise to a mapping from the $\mathfrak{A}$-module $\Gamma(\xi_{\text{E}})$ into $\mathfrak{A}$:

$$\begin{aligned} \Gamma(\xi_{\text{E}}) &\to \mathfrak{A} \\ \Psi &\mapsto \Psi^* V_{\text{H}} \,, \end{aligned} \tag{15}$$

which is defined for any $x \in \mathcal{M}$ by $\Psi^* V_{\text{H}}(x) := V_{\text{H}}(\psi(p))|_{p \in \pi^{-1}(x)}$. Here, we have used the canonical isomorphism $\Gamma(\xi_{\text{E}}) \simeq \mathcal{C}^{\infty}_{\text{eq}}(\mathcal{P}, \mathbb{C}^N)$, so that $\Psi(x) = [(p, \psi(p))]|_{p \in \pi^{-1}(x)} \in E$. Therefore, we obtain a functional

$$\begin{aligned} \mathcal{V}_{\text{H}} : \Gamma(\xi_{\text{E}}) &\to \mathbb{R} \\ \Psi &\mapsto \langle \Psi^* V_{\text{H}}, \mu_{\text{M}} \rangle \\ &= \int_{\mathcal{M}} \Psi^* V_{\text{H}} \, \mu_{\text{M}} \,. \end{aligned} \tag{16}$$

Here, $\mu_{\text{M}} \in \Omega^m(\mathcal{M})$ is the Riemannian volume form with respect to g_{M}.

Now let $\mathbf{z}_0 \in \mathbb{C}^N$ be a minimum of the Higgs potential. We denote by $I(\mathbf{z}_0) \subset G$ the corresponding stabilizer group. Up to equivalence, such a minimum determines a unique subgroup $H \subset G$ of the structure group of $\mathcal{P}(\mathcal{M}, G)$. The group H is referred to as the **little group**. Note that, for a given (G, ρ, V_{H}), there may or may not exist a nontrivial little group. In the case where (G, ρ, V_{H}) admits a nontrivial little group, more than one H may exist, depending on the orbit structure. We associate the appropriate **orbit bundle** $\xi_{\text{orbit}(H)}$ with typical fiber $\text{orbit}(H) \subset \mathbb{C}^N$ to a given little group H. Note that $\xi_{\text{orbit}(H)} \hookrightarrow \xi_{\text{E}}$ in a natural way. Then any section $\text{orb} \in \Gamma(\xi_{\text{orbit}(H)})$ gives rise to an H-reduction of $\mathcal{P}(\mathcal{M}, G)$. This follows from the fact that $\text{orbit}(H) \simeq G/H$. However, it can be proved that up to equivalence there is only one H-reduction of $\mathcal{P}(\mathcal{M}, G)$. Let us denote this reduction by $(\mathcal{Q}, \iota)$, where $\mathcal{Q}(\mathcal{M}, H)$ (up to equivalence) is a uniquely determined H-principal sub-bundle over $\mathcal{M}$ and $\iota \colon \mathcal{Q} \hookrightarrow \mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} \mathcal{M}$

the corresponding inclusion mapping (considered as a bundle homomorphism that induces the identity on $\mathcal{M}$).

Let $\mathcal{P}(\mathcal{M}, G)$ be H-reducible and $\mathbf{z}_0 \in \mathrm{orbit}(H)$ be a chosen minimum of the Higgs potential. We correspondingly identify the little group with the stabilizer group of this specified minimum and denote the orbit bundle by $\xi_{\mathrm{orbit}(\mathbf{z}_0)}$. Let also $(\mathbf{z}_0, \mathrm{orb})$ be a *specific reduction* of $\mathcal{P}(\mathcal{M}, G)$, with $\mathrm{orb} \in \Gamma(\xi_{\mathrm{orbit}(\mathbf{z}_0)})$. As we have previously mentioned, such a reduction defines a corresponding H-principal sub-bundle of $\mathcal{P}(\mathcal{M}, G)$ which is isomorphic to $\mathcal{Q}(\mathcal{M}, H)$. We therefore denote the chosen reduction $(\mathbf{z}_0, \mathrm{orb})$ by $(\mathcal{Q}, \iota, \mathbf{z}_0)$, where now $H = I(\mathbf{z}_0)$. For every specific reduction $(\mathcal{Q}, \iota, \mathbf{z}_0)$ of $\mathcal{P}(\mathcal{M}, G)$, there exists an associated **reduced vector bundle** $\xi_{\mathrm{E}, \mathbf{z}_0}$. It can be proved that $\xi_{\mathrm{E}, \mathbf{z}_0} \simeq \xi_{\mathrm{E}, \mathbf{z}_0'}$ if and only if $\mathrm{orbit}(\mathbf{z}_0) = \mathrm{orbit}(\mathbf{z}_0')$. Therefore, up to equivalence, there is a unique reduced vector bundle, $\xi_{\mathrm{E, red}}$, associated with an H-reduction of $\mathcal{P}(\mathcal{M}, G)$. It can be shown that the realification $r(\xi_{\mathrm{E, red}})$ of $\xi_{\mathrm{E, red}}$ decomposes into the Whitney sum of two real vector sub-bundles, called the **Higgs bundle** and the **Goldstone bundle**:

$$r(\xi_{\mathrm{E, red}}) = \xi_{\mathrm{Higgs}} \oplus \xi_{\mathrm{Goldstone}} \, . \tag{17}$$

Within the so-called "semiclassical approximation" of a full quantized theory, this Higgs bundle geometrically models what we previously have called the Higgs boson. The Goldstone bundle corresponds to "spurious" gauge degrees of freedom of the Higgs boson and might be "gauged to zero" using the *unitary gauge condition*. As it turns out, the rank of the Goldstone bundle corresponds to the number of "massive gauge bosons", whereas the dimension of the little group corresponds to the number of "massless gauge bosons". To see how all of this can be made precise geometrically, we note that there exists a canonical section of the appropriate reduced vector bundle that corresponds to a specific reduction. To be specific, let again $\mathbf{z}_0 \in \mathrm{orbit}(H)$ be a specific minimum of the Higgs potential. Correspondingly, let $\xi_{\mathrm{E}}, \mathbf{z}_0$ be the associated reduced vector bundle with respect to the reduction $(\mathcal{Q}, \iota, \mathbf{z}_0)$. Then we define

$$\begin{aligned} \mathrm{vac} \colon \mathcal{M} &\to E_{\mathbf{z}_0} \\ x &\mapsto [(q, \mathbf{z}_0)]|_{q \in \pi_{\mathcal{Q}}^{-1}(x)} \, . \end{aligned} \tag{18}$$

This constant (and covariantly constant) section is called the **vaccum section** corresponding to the reduction $(\mathcal{Q}, \iota, \mathbf{z}_0)$. Note that it explicitly refers to a chosen minimum $\mathbf{z}_0 \in \mathrm{orbit}(H)$ of the H-reduction of $\mathcal{P}(\mathcal{M}, G)$. Also note that the vacuum section may also be considered as a section in ξ_{E} using the definition $x \in \mathcal{M} \mapsto \mathrm{vac}(x) := [(p, \rho(g^{-1})\mathbf{z}_0)]|_{p \in \pi_P^{-1}(x)}$, where $p := \iota(q)g$ and $q \in \pi_{\mathcal{Q}}^{-1}(x)$. This section generalizes the physicists' notion of a **semiclassical**

vacuum which is usually identified with the chosen minimum itself. Indeed, if $\mathcal{P}(\mathcal{M}, G)$ denotes the trivial G-principal bundle $\mathcal{M} \times G \xrightarrow{\mathrm{pr}_1} \mathcal{M}$, then the vacuum section with respect to a chosen minimum $z_0 \in \mathrm{orbit}(H)$ of the Higgs potential corresponds to the canonical H-reduction $\mathcal{Q} := \mathcal{M} \times H \hookrightarrow \mathcal{P}$, $(x, h) \mapsto (x, h)$.

What does all this have to do with spontaneous symmetry breaking? The notion of spontaneous symmetry breaking usually refers to the assumption that the Euler-Lagrange equation of the "general Yang-Mills functional" $\mathcal{I}_{\mathrm{YM+}}$ admits a solution that is not $\mathcal{G}$ invariant. To make this geometrically more precise, let us specify the functional $\mathcal{I}_{\mathrm{YM+}}$ to $\mathcal{I}_{\mathrm{YMH}}$

$$\mathcal{I}_{\mathrm{YMH}} := \mathcal{I}_{\mathrm{YM}} + \mathcal{I}_{\mathrm{H}}\,, \tag{19}$$

where the **Yang-Mills-Higgs functional** reads

$$\mathcal{I}_{\mathrm{YMH}}(\Psi, \mathrm{d}^{\mathrm{E}}) := \|F^{\mathrm{E}}\|^2 + \|\mathrm{d}^{\mathrm{E}}\Psi\|^2 + \mathcal{V}_{\mathrm{H}}(\Psi)\,. \tag{20}$$

Note that the relative signs refer to a definite signature of the underlying metric structure of $\mathcal{M}$. Also, we have assumed that the base manifold $\mathcal{M}$ is compact. Otherwise we have to work with compactly supported sections or with sections fulfilling suitable boundary conditions. In what follows we will always assume that $(\mathcal{M}, g_{\mathrm{M}})$ denotes a compact Riemannian manifold (and for reasons that will become clear later we will also assume that $\dim \mathcal{M} = 2n$).

Definition 1. *The gauge theory built on the "Yang-Mills-Higgs functional" $\mathcal{I}_{YMH}$ is called "spontaneously broken" if there is a "vacuum pair" $(\mathrm{vac}, \partial^E) \in \Gamma(\xi_E) \times \mathcal{A}(\xi_E)$, consisting of a covariant derivative ∂^E that corresponds to a flat connection on ξ_E and a vacuum section $\mathrm{vac} \in \Gamma(\xi_E)$ defined by a specific H-reduction of $\mathcal{P}(\mathcal{M}, G)$.*

Note that a vacuum pair $(\mathrm{vac}, \partial^{\mathrm{E}})$ is a minimum of the Yang-Mills-Higgs functional and thus fulfills the Euler-Lagrange equations

$$\mathrm{d}^{\mathrm{E}} F^{\mathrm{E}} = 0$$
$$\mathrm{d}^{\mathrm{E}} * F^{\mathrm{E}} = *j_{\mathrm{YM}} \tag{21}$$
$$\delta^{\mathrm{E}} \mathrm{d}^{\mathrm{E}} \Psi = \Psi^* V_{\mathrm{H}}'\,.$$

Here, $*1 := \mu_{\mathrm{M}}$ is the Hodge map defined by g_{M}, and δ^{E} is the formal adjoint of the exterior covariant derivative d^{E}. The mapping $V_{\mathrm{H}}' \colon \mathbb{C}^N \to \mathbb{R}$ denotes the gradient of the Higgs potential, where the canonical identification $T_z \mathbb{C}^N = \mathbb{C}^N$ ($z \in \mathbb{C}^N$) has been taken into account.

Let us denote by $\mathrm{var}_\omega(\mathcal{I}_+)$ the variation of a general functional $\mathcal{I}_+$ with respect to a connection on ξ_{E}. This "Lie algebra valued" one-form is referred to as the

Yang-Mills current with respect to the functional $\mathcal{I}_+$ and usually denoted by $j_{\mathrm{YM}} \in \Omega^1(\mathcal{M}, \rho'(\mathrm{Lie}_G))$. In terms of the Higgs functional $\mathcal{I}_{\mathrm{H}}$ the corresponding Yang-Mills current reads

$$j_{\mathrm{YM}}(X) := 2\,\mathrm{Re}\langle \Psi,\ \mathrm{d}^{\mathrm{E}}\Psi(X)\rangle \tag{22}$$

for all tangent vector fields $X \in \Gamma(\tau_{\mathrm{M}})$ on $\mathcal{M}$. Here, $\langle\,,\rangle$ denotes the bilinear mapping on the $\mathfrak{A}$-module $\Gamma(\xi_{\mathrm{E}})$ that is induced by the Hermitian product on the vector bundle ξ_{E}. Thus, $j_{\mathrm{YM}}(X) \in \mathfrak{A} \otimes \mathrm{Lie}_G$.

In what way is this related to mass? Because of $\mathcal{A}(\xi_{\mathrm{E}}) \simeq \Omega^1(\mathcal{M}, \rho'(\mathrm{Lie}_G))$ we may consider any connection on ξ_{E} as a "disturbance" of a chosen flat connection and thus write ($t \in [0,1]$) $\mathrm{d}^{\mathrm{E}} = \partial^{\mathrm{E}} + tA$. Likewise, we may consider the "disturbance" of any section $\Psi \in \Gamma(\xi_{\mathrm{E}})$ with respect to a chosen vacuum section and write $\Psi = \mathrm{vac} + t\Psi_{\mathrm{H}}$. Note that $X \in \Gamma(\tau_{\mathrm{M}})$ for all $A(X) \in \Gamma(\xi_{\mathrm{ad}})$, where the vector bundle ξ_{ad} is the ad-bundle defined by the adjoint representation of the Lie algebra on itself. Physically, the pair (Ψ_{H}, A) is interpreted as representing the state of the Higgs and the gauge boson "against the chosen vacuum" $(\mathrm{vac}, \partial^{\mathrm{E}})$. Rewriting the above Euler-Lagrange equations with respect to these sections one obtains[1] up to $\mathcal{O}(t^2)$

$$\begin{aligned}
* \partial^{\mathrm{E}} * \partial^{\mathrm{E}} A + M_{\mathrm{YM}}^2 A &= 0\,, \\
* \partial^{\mathrm{E}} * \partial^{\mathrm{E}} \Psi_{\mathrm{H}} + M_{\mathrm{H}}^2 \Psi_{\mathrm{H}} &= 0\,.
\end{aligned} \tag{23}$$

Here, respectively, the **mass matrices** of the gauge boson and the Higgs boson are defined by

$$\begin{aligned}
(M_{\mathrm{YM}}^2)_{ab} &:= -2\langle \mathrm{vac}, \{\mathbf{T}_a, \mathbf{T}_b\}\,\mathrm{vac}\rangle \\
M_{\mathrm{H}}^2 &:= \mathrm{vac}^* \, V_{\mathrm{H}}''\,.
\end{aligned} \tag{24}$$

Here, $(\mathbf{T}_1, \ldots, \mathbf{T}_{\dim_G}) \subset \rho'(\mathrm{Lie}_G)$ is a basis, such that $A = \sum_{a=1}^{\dim_G} A^a \otimes \mathbf{T}_a$ and $\{\,,\}$ is the anticommutator in $\mathrm{End}(\mathbb{C}^N)$. The mapping V_{H}'' denotes the bilinear form induced by the Hessian of the Higgs potential. The number of zeros of M_{YM}^2 equals the dimension of the little group $H \subset G$ and is thus independent of the chosen specific H-reduction of $\mathcal{P}(\mathcal{M}, G)$ that gives rise to the vacuum section we actually work with. In other words, although the vacuum section is not gauge invariant the eigenvalues of the quadratic form M_{YM}^2 are nonethless gauge invariant. These eigenvalues are physically interpreted as the masses of the gauge bosons. Likewise, this holds true for the eigenvalues of the quadratic form M_{H}^2, which are considered the masses of the Higgs bosons.

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[1] Actually, this simple result holds true only when the above mentioned unitary gauge condition is used, where the Goldstone degrees of freedom become zero. This is analogous to the "Coulomb gauge condition" in ordinary electrodynamics, which is known to exhibit in the clearest way the physical degrees of freedom of the electromagetic field.

Note that so far we have talked about the Higgs boson, which is geometrically modeled by ξ_E, and about the gauge boson, which is geometrically modeled by ξ_{ad}. However, the quadradic forms defined by the respective mass matrices give rise to an additional structure in the case of a spontaneously broken gauge symmetry which does not exist in a "usual unbroken" gauge theory. Moreover, so far the chosen flat connection (if it exists at all!) has been assumed to be arbitrary. However, because of the additional structure introduced by the mass matrices it is reasonable to choose only flat connections that are "compatible" with the extra structure. We call a flat connection on ξ_E (resp. ξ_{ad}) to be compatible with the mass matrix M_H^2 (resp. M_{YM}^2), if the eigenbasis of the corresponding quadratic form is also an eigenbasis of the connection form defined by the flat connection. In this case the exterior covariant derivatives become "diagonal" with respect to the eigenbasis of the corresponding mass matrices. Therefore, in this eigenbasis the Euler–Lagrange equations (up to $\mathcal{O}(t^2)$) decompose into a set of decoupled equations, each of which physically represents the dynamics of a state of a *free boson*. Formally, these free bosons correspond to Hermitian line bundles and we say that the vector bundle ξ_E (ξ_{ad}) decomposes into the Whitney sum of Hermitian line bundles up to order $\mathcal{O}(t^2)$. Note that in the case where $\mathcal{P}(\mathcal{M}, G)$ is trivial, we can simply use (vac, d), where d is the covariant derivative that corresponds to the trivial connection on $\mathcal{M} \times \mathbb{C}^N \xrightarrow{\mathrm{pr}_1} \mathcal{M}$ (resp. $\mathcal{M} \times \mathrm{End}(\mathbb{C}^N) \xrightarrow{\mathrm{pr}_1} \mathcal{M}$), and vac corresponds to the canonical H-reduction of $\mathcal{P}(\mathcal{M}, G)$. The triviality of $\mathcal{P}(\mathcal{M}, G)$ becomes necessary if all of the free gauge bosons are massive.

So far, we have indicated how the gauge bosons may aquire mass using the mechanism of spontaneous symmetry breaking. However, one may ask why there are two kinds of bosons: the gauge boson and the Higgs boson. As is well-known, the geometry of the gauge boson is that of a connection on some G-principal bundle. However, what is the geometrical origin of the Higgs boson? Correspondingly, one may ask what the geometrical significance of the Higgs potential is. A huge amount of work has been done over the last decade to answer these questions. In what follows we will summarize Connes' idea to regard the Higgs boson as a gauge potential on a non-commutative space. From this point of view the Higgs potential becomes a particular Yang–Mills functional.

3. The Higgs Boson as a Connection

In this section, we want to discuss Connes' idea to consider the Higgs boson as a connection on a non-commutative space. For this, we will (very) briefly summarize the basic construction of a non-commutative differential algebra that generalizes the well-known de Rham algebra of "commutative" differential

geometry. Before doing so, however, we discuss the "mass matrix of the fermions" as a motivion for what follows.

In the former section we have discussed how the notion of mass of the bosons can be brought into harmony with the dogma of gauge invariance. For this purpose, one postulates the existence of a new particle — the Higgs boson. The interaction of this Higgs boson with the gauge boson gives rise to the bosonic mass matrices. The eigenstates of the corresponding quadratic forms associated with the mass matrices are physically interpreted as the states of "free bosons". But what about the masses of the fermions, known to be the basic building blocks of matter? How does the Higgs boson act with the fermion in order for the latter to aquire mass? This turns out to be more subtle than in the case of the bosons. The reason is that in the case of fermions not only the "inner degrees" are involed but also the degrees of freedom that are connected with space time. In the last section, we discussed only the inner degrees of freedom of the bosons. For instance, we considered the gauge bosons to be represented by the vector bundle ξ_{ad}. However, the gauge bosons define a spin one representation of the rotational group $SO(3)$ and thus are also represented by the cotangent bundle τ_{M}^* of the base manifold $\mathcal{M}$. Consequently, with respect to a chosen vacuum pair, the gauge boson is represented by

$$\xi_{\mathrm{gauge}} := \tau_{\mathrm{M}}^* \otimes \xi_{\mathrm{ad}} .\tag{25}$$

In contrast, the Higgs boson is believed to be in the trivial representation of the rotational group and thus is geometrically represented only by the vector bundle ξ_{E} of the inner degrees of freedom. The free bosons have to fulfill a second order differential equation since the "exterior bosonic degrees of freedom" form an integer representation of the rotational group.

In the case of a fermion, one has to take into account that it forms a one-half representation of the (double cover of the) rotational group. As a consequence, the admissible states of a "free fermion" have to obey a first order differential equation — the **Dirac equation**. Therefore, a fermion is represented by a specific **Clifford module bundle**

$$\xi_{\mathcal{E}} := \xi_S \otimes \xi_{\mathrm{E}_f} .\tag{26}$$

Here, we assume that $(\mathcal{M}, g_{\mathrm{M}}, \varsigma_{\mathrm{M}})$ is an oriented Riemannian spin-manifold with spin-structure ς_{M} and ξ_S denotes the appropriate **spinor bundle**. This geometrically represents the "exterior fermionic degrees of freedom". The "interior fermionic degrees of freedom" are represented by yet another Hermitian vector bundle ξ_{E_f}. If the dimension of $\mathcal{M}$ is even, the spinor bundle ξ_S is $\mathbb{Z}_2$-graded with respect to the canonical involution $\gamma_5 \in \mathrm{End}(S)$

$$\xi_S = \xi_{S_L} \oplus \xi_{S_R} ,\tag{27}$$

representing the left handed and the right handed part of the fermion in question. Concerning the fermion there is still another subtle point one has to take into account. This point is tied to the experimentally well established fact that a specific interaction of the fermions, called the **weak interaction**[1] and represented by the gauge group $SU(2)$-differentiates between left and right handed fermions (actually, this is how the above mentioned $\mathbb{Z}_2$-grading of the spinor bundle is realized in nature). As a consequence, also the inner fermionic degrees of freedom become $\mathbb{Z}_2$-graded

$$\xi_{\mathrm{E}_f} = \xi_{\mathrm{E}_{f,L}} \oplus \xi_{\mathrm{E}_{f,R}} . \tag{28}$$

Here, the Hermitian subvector bundles $\xi_{\mathrm{E}_{f,L}}$ and $\xi_{\mathrm{E}_{f,R}}$ are defined with respect to the fundamental representation and the trivial representation of $SU(2)$, respectively. It is exactly this $\mathbb{Z}_2$-grading which offers us an understanding of the Higgs boson as a connection! To clarify this, let us consider the following example.

Let $(\mathcal{M}, g_\mathrm{M})$ again be Minkowski's space time. The corresponding (complexified) Clifford algebra can then be identified with $\mathrm{End}(\mathbb{C}^4)$. Correspondingly, the spinor bundle ξ_S can be identified with the trivial Hermitian vector bundle

$$\mathcal{M} \times (\mathbb{C}^2_L \oplus \mathbb{C}^2_R) \xrightarrow{\mathrm{pr}_1} \mathcal{M} . \tag{29}$$

The inner fermionic degrees of freedom are assumed to be represented by the Hermitian vector bundle ξ_{E_f}

$$\mathcal{M} \times (\mathbb{C}^2_L \oplus \mathbb{C}_R) \xrightarrow{\mathrm{pr}_1} \mathcal{M} . \tag{30}$$

Moreover, we assume that

$$\xi_\mathrm{E} \simeq \xi_{E_{f,L}} . \tag{31}$$

We write a state which represents the inner fermionic degrees as $\psi = (\psi_L, \psi_R) \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{C}^2_L \oplus \mathbb{C}_R)$. Now let us assume that such a state represents a free fermion of mass m and thus obeys Dirac's original equation[2]

$$\mathrm{i}\slashed{\partial}\Psi = m\Psi , \tag{32}$$

where $\Psi = (\Psi_L, \Psi_R)$ represents the total degrees of freedom, i.e. $\Psi = \sum_{j=1}^4 \mathsf{s}_j \otimes \psi_j$ and (s_j) a frame of the spinor bundle. The first order differential operator $\slashed{\partial} := \sum_{\mu=0}^3 \gamma^\mu \partial_\mu$ is the Dirac operator, where $\gamma^\mu \in \mathrm{End}(\mathbb{C}^4)$

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[1] This kind of interaction, e. g., is responsible for the decay of a nucleus.

[2] This equation was introduced by P. A. M. Dirac around 1928 in order to relativistically generalize Schrödinger's equation of ordinary quantum mechanics. Note that the Dirac equations is the "square root" of the equation that an admissible state of a free Higgs boson has to fulfill.

generates the Clifford algebra. As a consequence of the canonical involution $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, the Dirac equation (32) decomposes into the system

$$\begin{aligned} i\not{\partial}\Psi_L &= m\Psi_R\,, \\ i\not{\partial}\Psi_R &= m\Psi_L\,. \end{aligned} \tag{33}$$

This decomposition, however, is not $SU(2)$ gauge invariant because of the presence of the mass parameter $m \in \mathbb{R}_+$. This "mess of the mass" is analogous to the problem encountered with London's equation (2). To remedy this flaw, one uses again the Higgs boson and postulates a new kind of interaction, besides the gauge interaction of the fermions, called the **Yukawa interaction**. For this purpose, let us denote by $\varphi \in \mathcal{C}^\infty(\mathcal{M},\mathbb{C}^2)$ a state of the Higgs boson. We then introduce the *odd* endomorphism $\phi \in \mathcal{C}^\infty(\mathcal{M},\mathrm{End}^-(E))$ by

$$\phi := \begin{pmatrix} 0 & g_{\mathrm{Yuk}}\varphi \\ g_{\mathrm{Yuk}}\varphi* & 0 \end{pmatrix}\,, \tag{34}$$

where the new parameter $g_{\mathrm{Yuk}} \in \mathbb{R}_+$ is referred to as the *Yukawa coupling constant*. The action of the above endomorphism is defined as follows ($x \in \mathcal{M}$):

$$\phi(x)\psi(x) := \left(g_{\mathrm{Yuk}}\varphi(x)\psi_R(x),\ g_{\mathrm{Yuk}}\langle\varphi(x),\psi_L(x)\rangle\right) \in \mathbb{C}_L^2 \oplus \mathbb{C}_R\,. \tag{35}$$

Note that this action is indeed $SU(2)$ invariant. Therefore, we may rewrite the Dirac equation (32) in a $SU(2)$ gauge invariant manner

$$i\not{\partial}_A\Psi = \Phi\Psi\,, \tag{36}$$

where, respectively, A is a $SU(2)$ gauge potential, such that $\not{\partial}_A = \sum_\mu \gamma^\mu(\partial_\mu + \mathbb{1}\otimes A_\mu)$ and $\Phi := \mathbb{1}\otimes\phi$.

In the case where the state of the Higgs field is identified with some chosen vacuum state $\varphi = \mathrm{vac}$, we may identify the mass of the fermion with[1]

$$m := g_{\mathrm{Yuk}}\|\mathrm{vac}\|\,, \tag{37}$$

which is $SU(2)$ gauge invariant.

All this can also be worked out for the case of non-trivial bundles (not obvious, but true). The crucial point here is that we have to introduce two new operators in order to describe fermions: a first-order differential operator (a Dirac operator) and the odd zero-order differential operator (34) defined by the Higgs

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[1] That this is indeed a reasonable definition is again most obvious with respect to the unitary gauge condition, mentioned in the previous section. Note that in our specific example the rank of the real subbundle $\xi_{\mathrm{Higgs}} \subset r(\xi_{E,\,\mathrm{red}})$ equals one. As a result, the vacuum section is defined by a single real number, which in the context of a full quantum theory is referred to as the "vacuum expectation value" of the Higgs boson.

boson. In other words, besides the gauge coupling the fermions also interact with the Higgs boson via the Yukawa coupling. The main mathematical feature of the Yukawa coupling is that it exchanges left with right handed fermions, i. e. it is an odd operator (in contrast to the covariant derivative, which is an even operator). Note that this holds also true for the **twisted spin Dirac operator** $\displaystyle{\not{\partial}}_A$. This will be crucial for all that is following.

Regarded as an endomorphism, the gauge group $\mathcal{G}$ acts by conjungation on ϕ, that is $\phi^\gamma = \rho(\gamma^{-1})\, \phi\, \rho(\gamma)$, for all $\gamma \in \mathcal{G}$. With respect to some chosen vacuum section $\mathrm{vac} \in \Gamma(\xi_\mathrm{E})$ (where again we assume that $\xi_\mathrm{Higgs} \simeq \xi_{\mathrm{E}_{f,L}}$) we consider the endomorphism that corrsponds to the "shifted state" of the Higgs boson $\varphi_0 \equiv \varphi - \mathrm{vac}$

$$\phi_0 \equiv \phi - \mathcal{D}\,. \tag{38}$$

Here, the endomorphism $\mathcal{D}$ is defined by the vacuum section vac in the same way than ϕ is defined by φ. The reason why we interpret the Higgs boson as a gauge boson results from the following observation: The reduced gauge group $\mathcal{H}$ acts on the shifted endomorphism ϕ_0 as

$$\phi_0^\gamma = \rho(\gamma^{-1})\phi_0\rho(\gamma) + \rho(\gamma^{-1})[\mathcal{D}, \rho(\gamma)]\,, \tag{39}$$

which indeed looks very much the same as the well-known transformation law of a gauge potential under a gauge transformation. For this to really make sense, however, one has to ensure that the derivative $[\mathcal{D}, \cdot]$ on $\mathrm{End}(E_f)$ actually defines an exterior derivative. How this can be achieved will be explained in the next paragraph.

3.1. Connes' Differential Algebra

To get started, let again $(\mathcal{M}, or_\mathrm{M}, g_\mathrm{M}, \varsigma)$ be a compact, oriented Riemannian spin manifold. Also, let $\dim \mathcal{M} = 2n$. As a vector bundle we may identify the Clifford bundle $Cl(\mathcal{M}, g_\mathrm{M})$ with the Grassmann bundle $\xi_{\Lambda\mathcal{M}}$. We denote by $\not{\partial}$ the **spin Dirac operator** on the associated spinor bundle ξ_S. This operator is uniquely determined by the following two conditions:

$$\begin{aligned} [\not{\partial}, f] &= \gamma(\,\mathrm{d}f)\,, & f \in \mathfrak{A} \\ [\not{\partial}, a] &= \gamma(\partial^\mathrm{Cl}a)\,, & a \in \Gamma(Cl(\mathcal{M}, g_\mathrm{M}))\,, \end{aligned} \tag{40}$$

where $T^*\mathcal{M} \xrightarrow{\gamma} \mathrm{End}(S)$ denotes the induced Clifford action, ∂ the covariant derivative defined by the Riemannian connection on the cotangent bundle τ_M^* and ∂^Cl the appropriate lift of ∂ to the Clifford bundle. From the first condition and the above mentioned identification between the Clifford and the Grassmann

bundle one recognizes that de Rham's exterior differential can be expressed in terms of the spin Dirac operator

$$d = [\not{\partial}, \cdot] .$$ (41)

To make this point more precise, let $(\mathcal{B}, \Delta)$ be an involutive differential algebra over an associative, involutive, unital algebra $\mathfrak{B}$. Also, let $\mathfrak{A}$ be an associative, involutive, unital algebra. The differential algebra $(\hat{\Omega}\mathfrak{A}, \hat{\delta})$ is called the *universal differential envelope* of $\mathfrak{A}$, provided it fulfills the following universal property: For every involutive and injective homomorphism $\mathfrak{A} \xrightarrow{h} \mathcal{B}^0 = \mathfrak{B}$ there exists exactly one homomorphism $\tilde{h}$, so that the diagram commutes.

Let a k-form in $\hat{\omega} \in \hat{\Omega}^k \mathfrak{A}$ be written as $\hat{\omega} := a_0 \hat{\delta} a_1 \cdots \hat{\delta} a_k$ with $a_0, a_1, \ldots, a_k \in \hat{\Omega}^0 \mathfrak{A} := \mathfrak{A}$. The existence of the universal algebra associated to $\mathfrak{A}$ can be proved, for instance, by an explicit construction of a free algebra consisting of "words" like $\hat{\omega}$, subject to appropriate relations. The uniqueness of $(\hat{\Omega}\mathfrak{A}, \hat{\delta})$ follows as usual from the universal property. Note that the universal differential envelope is cohomologically trivial, that is every closed form is actually exact. In order to construct out of $\mathfrak{A}$ non-trivial differential algebras we follow Connes' construction using spectral triples, see [5] and [7]. Let $(\mathcal{H}, \pi, \mathcal{D})$ be a **spectral triple** consisting of a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, a faithful and involutive representation $\mathfrak{A} \xrightarrow{\pi} \mathrm{End}(\mathcal{H})$, and a "generalized Dirac operator" $\mathcal{D} \in \mathrm{End}(\mathcal{H})$. This means an unbounded linear operator on the Hilbert space $\mathcal{H}$, such that the resolvent and the operators $[\mathcal{D}, \pi(a)]$ are bounded for all $a \in \mathfrak{A}$. Then, it can be verfied that the mapping

$$\hat{\pi}: \hat{\Omega}(\mathfrak{A}) \to \mathrm{End}(\mathcal{H})$$
$$\hat{\omega} := a_0 \hat{\delta} a_1 \cdots \hat{\delta} a_k \mapsto \hat{\pi}(\hat{\omega}) := \pi(a_0)[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_k)]$$ (42)

defines an algebra homomorphism. However, to define a differential δ on the subalgebra $\hat{\pi}(\hat{\Omega}\mathfrak{A})$ by the relation $\delta\hat{\pi}(\hat{\omega}) := \hat{\pi}(\hat{\delta}\hat{\omega})$ generally fails. For this reason we consider the quotient differential algebra $\hat{\Omega}(\mathfrak{A})/\mathfrak{J}$, where $\mathfrak{J} \in \hat{\Omega}(\mathfrak{A})$

is the two-sided ideal generated by

$$\bigoplus_{k \in \mathbb{Z}} [\mathrm{ker}^k(\hat{\pi}) + \delta(\mathrm{ker}^{k-1}(\hat{\pi}))] . \tag{43}$$

Resulting from the above construction

$$\hat{\pi} : \hat{\Omega}(\mathfrak{A})/\mathfrak{J} \to \mathrm{End}(\mathcal{H}) \tag{44}$$

is now a faithful homomorphism of differential algebras, where the differential δ_D on

$$\Omega_D\mathfrak{A} := \hat{\pi}(\hat{\Omega}\mathfrak{A}/\mathfrak{J}) \simeq \hat{\pi}(\hat{\Omega}\mathfrak{A})/\hat{\pi}(\mathfrak{J}) \subset \mathrm{End}(\mathcal{H}) \tag{45}$$

is defined by $\delta_D[\hat{\omega}] := [\hat{\delta}\hat{\omega}]$. We denote the equivalence class of a k-form $[\hat{\omega}]$ by $\omega \in \Omega_D^k\mathfrak{A}$.

A very remarkable thing to be noted is that in the case of $\mathfrak{A} := C^\infty(\mathcal{M}, \mathbb{C})$ and $(L^2(\xi_S), \pi, \partial\!\!\!/)$ and denoting the **Dirac triple**, one obtains the following isomorphism:

$$(\Omega_D\mathfrak{A}, \delta_D) \simeq (\Omega(\mathcal{M}), \mathrm{d}) . \tag{46}$$

Here, the representation π is simply defined by multiplication, i. e. $\pi(f)\Psi := f\Psi$ for all square integrable sections Ψ of the spinor bundle ξ_S. Note that any information contained in the metric structure on $\mathcal{M}$ is lost. This is due to the "junk" $\mathfrak{J}$, which contains the entire "even part of the Clifford multiplication". For instance, let (ω_1, ω_2) be one-forms. Then, the multiplication in the Clifford algebra yields: $\omega_1\omega_2 = -g_{\mathrm{M}}(\omega_1, \omega_2) + \omega_1 \wedge \omega_2$. Here, the even part $g_{\mathrm{M}}(\omega_1, \omega_2)$ is an element of $\mathcal{J}$ (not obvious, but true). Moreover, the above given construction involves commutators, only and therefore is independent of the chosen spin structure. Actually, there is no dependence at all. Therefore, the construction of the de Rham algebra out of a Dirac operator works also for general Clifford modul bundles.

According to Gelfand's theorem a certain class of topological spaces X can be fully recovered from the commutative algebra of continous functions on X (so-called "normal spaces"). The same holds true in the case of a smooth manifold $\mathcal{M}$. Its structure is encoded in the commutative algebra of smooth functions on $\mathcal{M}$. Correspondingly, in Connes' non-commutative geometry a non-commutative space is given by a non-commutative algebra $\mathfrak{A}$. The above summarized construction allows us to costruct a Yang–Mills gauge theory also on non-commutative spaces. To do so, however, we have to generalize the scalar product in the de Rham algebra to the non-commutative case. This is one of the most subtle points encountered in non-commutative geometry ("integration" being always more subtle than "differentiation"!). Again, Connes

has found the equivalent of integration — the so-called **Dixmier trace**. We only mention that this kind of trace indeed generalizes the usual inner product in de Rham's algebra, see again [5].

3.2. The Non-Commutative Yang–Mills Action

As we have already mentioned the set of sections $\mathcal{E} := \Gamma(\xi_E)$ of a vector bundle ξ_E over a smooth manifold $\mathcal{M}$ is an $\mathfrak{A}$-module, where in this case $\mathfrak{A} = C^\infty(\mathcal{M})$. According to Swan's theorem, a vector bundle of finite rank over a smooth manifold can thus be regarded as a *finitely generated, projective* $\mathfrak{A}$-*module*. This terminology just means that for every $\mathcal{E}$ there is a number $N \in \mathbb{N}$ and an $\mathfrak{A}$-module $\mathcal{E}'$, so that

$$\mathfrak{A}^N \simeq \mathcal{E} \oplus \mathcal{E}' . \tag{47}$$

In other words, for every vector bundle ξ_E there exists a vector bundle ξ_E', so that the Whitney sum of both is equivalent to the corresponding trivial vector bundle. Note that the direct complement $\mathcal{E}'$ is by no means unique, and a choice of it is in one-to-one correspondence to a choice of a finite rank projector $\wp \in \mathrm{End}_{\mathfrak{A}}(\mathfrak{A}^N)$, so that $\mathcal{E} \simeq \mathrm{Im}(\wp)$. As a consequence, the covariant derivative of any linear connection on the vector bundle ξ_E reads

$$\mathrm{d}^E = \wp \circ \mathrm{d} , \tag{48}$$

with the curvature[1]

$$F^E \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathfrak{A}} \Omega^2(\mathcal{M})$$
$$u \mapsto (\wp \circ \mathrm{d}\wp \wedge \mathrm{d}\wp)u . \tag{49}$$

A gauge potential A can be defined by

$$\mathbf{e} \otimes A := \mathrm{d}\wp(\mathbf{e}) , \tag{50}$$

where $\mathbf{e} \equiv (\mathbf{e}_1, \dots, \mathbf{e}_N) \subset \mathfrak{A}^N$ denotes the standard basis of the free module $\mathfrak{A}^N$. Note that this is in fact an $\mathfrak{A}$-bimodule. If we write $u = \wp(\mathbf{u})$ with $\mathbf{u} \in \mathfrak{A}^N$ then

$$\mathrm{d}^E u = \wp(\mathrm{d}\mathbf{u} + A\mathbf{u}) =: \wp(\nabla\mathbf{u}) . \tag{51}$$

The corresponding curvature reads

$$F^E u = \wp\big((\mathrm{d}A + A \wedge A)\mathbf{u})\big) =: \wp(F\mathbf{u}) . \tag{52}$$

The up-shot of all of this is the simple message that in order to define the Yang–Mills curvature one only needs the *differential algebra* $(\Omega\mathfrak{A}, \mathrm{d})$. The

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[1] Note that any projector $\wp$ fulfills $\wp \circ \mathrm{d}\wp \circ \wp \equiv 0$.

connection form is defined by an anti-Hermitian one form $A \in \Omega^1 \mathfrak{A}$ $(A^* = -A)$ on which the *group of unitaries*

$$\mathcal{G} := \{g \in \mathfrak{A}; \, g^*g = gg^* = e\} \tag{53}$$

acts on via the usual transformation (please, compare this also with formula (39))

$$A \mapsto A^g := g^{-1}Ag + g^{-1}\, \mathrm{d}g \,. \tag{54}$$

Note that $\mathcal{G} \subset \mathfrak{A} = \Omega^0 \mathfrak{A}$, so that $\mathrm{d}g \in \Omega^1 \mathfrak{A}$.

Let us again denote by $\langle\,,\,\rangle$ the inner product on $\Omega \mathfrak{A}$ (i. e. the "Dixmier trace" in the case of $\mathfrak{A}$ equals the commutative algebra of smooth functions, or just the usual trace if $\mathfrak{A}$ is some matrix algebra). The (non-)commutative Yang–Mills functional is then defined in the analogues manner as in the commutative case

$$\mathcal{I}_{\mathrm{YM}} := \langle F, F \rangle \,. \tag{55}$$

However, this definition does not depend on whether or not the algebra $\mathfrak{A}$ is commutative or not. All the above constructions are at their very heart "algebraic". In particular, one may consider the algebra of **quaternions**, which we denote again by $\mathfrak{A}$. Note that the corresponding group of unitarities can be identified with $SU(2)$. Then, as a kind of miracle one obtains

$$\mathcal{I}_{\mathrm{YM}} \sim \mathrm{trace}(\mathbb{1} - \phi^*\phi)^2 \,. \tag{56}$$

Here, $\phi := \phi_0 + \mathcal{D}$ where $\phi_0 \in \Omega_D^1 \mathfrak{A}$ denotes a gauge potential and $\mathcal{D} \in \mathrm{End}(\mathbb{C}^2)$ is the corresponding fermionic mass matrix, we have discussed in the above example. But now it is considered a generalized Dirac operator in the sense of Connes. In these terms a state of the Higgs boson appears as a shifted gauge potential and thus transforms homogeneously with respect to the "gauge group" $SU(2)$. So, in some sense this point of view is the flipside of the viewpoint we have started out with in the second section. For a fine reference of the details see, e. g., [9, 14] and [12]. In particular, we recommend the latter reference for a pedagogical treatment of the notion of the tensor product of a spectral triple in the case of Yang–Mills–Higgs theory. Note that the Yang–Mills functional (56) is positive semidefinite and each minimum of this functional necessarily breaks the $SU(2)$ gauge symmetry. Finally, we should mention that the above mentioned constructions can be generalized to what is nowadays called a *real geometry*, see cf. [7, 8] and [15] and the appropriate references therein.

We finish this work with some remarks concerning the relation between Dirac operators and connections within the framework of commutative geometry. We

also mention how the Higgs boson might be considered as a connection also within this frame.

4. Clifford Modules and the Higgs Boson

In the previous section we have considered the states of the Higgs boson as odd endomorphisms on the twisted spinor bundle ξ_ε. Moreover, we have seen how Dirac's original equation can be made $SU(2)$ gauge invariant by use of the Higgs boson. As it turns out, the sum of the twisted spin Dirac operator ∂_A and the odd endomorphism Φ defines again a Dirac operator in a mathematically reasonable sense. This general first operator

$$D := \slashed{\partial}_A + \Phi \tag{57}$$

is referred to as the **Dirac-Yukawa operator** in the case that Φ defines the Yukawa coupling. This, however, can be generalized to arbitrary $\mathbb{Z}_2$-graded Clifford module bundles, also denoted by ξ_ε, see, e. g., [2] and Chapter 3 in [3]. A Dirac operator in this general setting is then defined by any first order differential operator that acts on $\Gamma(\xi_\varepsilon)$ and satisfies the basic relation

$$[D, f] = \gamma(\mathrm{d}f) \tag{58}$$

for all $f \in \mathfrak{A}$. Here, $\mathcal{C}(\mathcal{M}, g_\mathrm{M}) \xrightarrow{\gamma} \mathrm{End}(\xi_\varepsilon)$ denotes the given Clifford action. Let us denote again by $\mathcal{A}(\xi_\varepsilon) \simeq \Omega^1(\mathcal{M}, \mathrm{End}^+(\mathcal{E}))$ the affine set of all linear connections on the Clifford module bundle ξ_ε and by $\mathcal{D}(\xi_\varepsilon)$ the set of all Dirac operators compatible with the given Clifford action. It is not hard to check that the set $\mathcal{D}(\mathcal{E})$ is an affine set, such that $\mathcal{D}(\mathcal{E}) \simeq \Omega^0(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$. For this reason, the difference of two Dirac operators is an odd endomorphism. Moreover, it can be shown that (see, e. g., [13])

$$\mathcal{D}(\xi_\varepsilon) \simeq \mathcal{A}(\xi_\varepsilon)/\ker\gamma. \tag{59}$$

Thus each Dirac operator is represented by a class of connections on the Clifford module bundle.

It exists a distinguished class of connections, called **Clifford connections**, on each Clifford module bundle over an even dimensional manifold $\mathcal{M}$. On a twisted spinor bundle these Clifford connections correspond to twisted spin connections. We denote by ∂_A the covariant deriviative of a Clifford connection. These connections are fully characterized by the relation

$$[\partial_A, a] = \gamma(\partial^{\mathrm{Cl}}a) \tag{60}$$

for all $a \in \Gamma(\mathcal{Cl}(\mathcal{M}, g_{\mathrm{M}}))$. Moreover, on every Clifford module bundle there exists a distiguished one-form $\xi \in \Omega^1(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$ subject to the relations

$$\partial_A \xi \equiv 0,$$
$$\gamma(\xi) = \mathrm{Id}_{\mathcal{E}}.$$

(61)

This one-form ξ gives rise to a linear mapping

$$\delta_\xi : \Omega^k \left(\mathcal{M}, \mathrm{End}^{\pm}(\mathcal{E}) \right) \rightarrow \Omega^{k+1} \left(\mathcal{M}, \mathrm{End}^{\mp}(\mathcal{E}) \right)$$
$$\Psi \mapsto \xi \Psi.$$

(62)

In particular, we obtain a one-form $\omega_\Phi \in \Omega^1(\mathcal{M}, \mathrm{End}^+(\mathcal{E}))$ to every $\Phi \in \Omega^0(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$. We call this form a **Dirac form**. If $D \in \mathcal{D}(\mathcal{E})$ is given, we can associate a covariant derivative to this Dirac operator

$$\nabla := \partial_A + \omega_\Phi,$$

(63)

where $\Phi := D - \partial\!\!\!/_A$.

From this point of view, every state ϕ of the Higgs boson also defines a connection on the twisted spinor bundle. However, note that the mapping (62) is not a differential and thus in the present approach the algebra $\Omega(\mathcal{M}, \mathrm{End}(E))$ can only be considered a bi-graded algebra. This is in contrast to the previously discussed approach where a certain subalgebra of $\mathrm{End}(E)$ was constructed as a differential algebra.

Since $(\mathcal{M}, g_{\mathrm{M}})$ is assumed to be a Riemannian manifold every Dirac operator on a Clifford module $\xi_{\mathcal{E}}$ is an elliptic operator. Therefore, one can define the heat trace of the square of the Dirac operator at hand. Each coefficient in the asymptotic expansion of the heat trace is known to encode geometric information. In particular, the subleading term can be expressed by a specific trace evaluated by a certain power of the corresponding Dirac operator. This trace is called the **Wodzickis' residue** (see [16] and [15]). Actually, it is the trace on the algebra of classical pseudo differential operators. Note that in four dimension the subleading coefficient is the only term in the asymptotic expansion that can be expressed in terms of the Wodzicki residue. Moreover, this coefficient is the only which is linear in the scalar curvature of the base manifold $\mathcal{M}$.

As it turns our there exists a generalization of the Dirac-Yukawa operator such that the Wodzicki residue evaluated with respect to this **Pauli-Dirac-Yukawa operator** is but the Yang-Mills-Higgs functional with the Higgs potential used in the standard model of particle physics (see cf. the second part in [13]). However, in this approach a non-trivial condition on the metric of the base

manifold $\mathcal{M}$ is also involved. This condition is given by the well-known Einstein–Hilbert functional

$$\int_{\mathcal{M}} r_M \, \mu_M \,, \tag{64}$$

which is but the Wodzicki residue evaluated with respect to any Dirac operator that is defined by some Clifford connection (see [10], or [11]). Since this is independent of the chosen Clifford connection one has to deal with more general Dirac operators, i. e. those that are not defineable by Clifford connections — in order to describe the Yang–Mills action in terms of Dirac operators (see [1] and the given references therein). Therefore, the metric involved in the coupled Euler–Lagrange equation of the gauge and Higgs bosons can no longer be chosen at will but has to satisfy the Einstein equation. From this perspective the Einstein equations occur as a kind of "constraint". This is quite different from the approach mentioned in the previous section (see, however, [4]). As a final remark we mention that the Yang–Mills functional, when derived from a Dirac operator, only depends on the equivalence class defining the corresponding Dirac operator. Since two representatives of a connection class defining a Dirac operator are not gauge related in general, the symmetry of the Yang–Mills action seems in fact bigger than in the usual appproach discussed in our first section.

Acknowledgements

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