Fourth International Conference on
Geometry, Integrability and Quantization
June 6–15, 2002, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
**Coral Press**, Sofia 2003, pp 11–41

# DEFORMATION QUANTIZATION IN QUANTUM MECHANICS AND QUANTUM FIELD THEORY

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**Abstract**. We discuss deformation quantization in quantum mechanics and quantum field theory. We begin with a discussion of the mathematical question of deforming the commutative algebra of functions on a manifold into a non-commutative algebra by use of an associative product. We then apply these considerations to the commutative algebra of observables of a classical dynamical system, which may be deformed to the non-commutative algebra of quantum observables. This is the process of deformation quantization, which provides a canonical procedure for finding the measurable quantities of a quantum system. The deformation quantization approach is illustrated, first for the case of a simple harmonic oscillator, then for an oscillator coupled to an external source, and finally for a quantum field theory of scalar bosons, where the well-known formula for the number of quanta emitted by a given external source in terms of the Poisson distribution is reproduced.

The relation of the star product method to the better-known methods involving the representation of observables as linear operators on a Hilbert space, or the representation of expectation values as functional integrals, is analyzed. The final lecture deals with a remarkable formula of Cattaneo and Felder, which relates Kontsevich's star product to an expectation value of a product of functions on a Poisson space, and indicates how this formula may be interpreted.

## 1. Introduction

One may distinguish three main approaches to understanding quantum mechanics (for a more detailed analysis see Styer *et al* [41]). In chronological order the first is the operator formalism, in which physical states are represented as vectors in a Hilbert space, and observables as linear operators on the states. The measurable quantities are the matrix elements of the operators

between states. The second is the Feynman's path integral approach. Here the measurable quantities are represented as expectation values which involve the functional integration of the classical observables evaluated on all potential trajectories in phase space, and weighted by an exponential factor involving the classical action. Finally, in deformation quantization the measurable quantities are given as expectation values involving the ordinary integration of the star product of the classical observables with phase space distributions which represent the physical states. We now present a more detailed description of these three approaches.

The operator formalism goes back to Dirac [15] and von Neumann [36]. The mathematical apparatus involving linear operators in Hilbert spaces has been extensively studied in the intervening years, and the treatment of non-relativistic systems is well understood. This is not the case for relativistic systems, where one must go over to the second-quantized field theory, and where the perturbation series exhibits divergencies whose interpretation is problematical. It is nevertheless possible to do precision calculations in quantum electodynamics, which show an excellent agreement with experiment [30]. The main limitation of this approach is that it has not proved possible to adapt it to a covariant description of the non-abelian gauge theories which describe the other fundamental interactions of elementary particles, the strong and weak interactions. For non-covariant quantum treatments of these theories see [10, 20].

The path integral formalism was developed by Feynman in connection with his calculations in quantum electrodynamics, but he later extended his considerations to give a fundamentally new approach to all quantum mechanical phenomena [21]. His approach has proved remarkably well-suited to getting an intuitive grasp of a very wide scope of problems in theoretical physics [37]. The first breakthrough in the quantum treatment of non-abelian gauge theories also used this method [19]. However, it has proved intractable to exact mathematical analysis for realistic field theories, although for some quantum mechanical systems and lower-dimensional field theories such analyses are possible [24].

The most recent approach to quantum physics is deformation quantization. It is based on phase space techniques developed by the pioneers of quantum mechanics; Weyl, Wigner, and von Neumann [45, 47, 36]. The star product was discovered in this context by Groenewold [25], and developed by Moyal [35]. The mathematical fundament was laid by Gerstenhaber [23]. But it was only recognized as an autonomous program for treating quantum mechanical problems in the papers of Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [5]. At that time the problems which could be treated by this method were relatively restricted, but that has changed over the years [3, 7, 16, 18, 22, 39, 40], so that today it may well be considered as a rival of the other two main approaches to quantum mechanics.

Why might we be interested in yet another approach to quantum mechanics? Aside from the potential convenience of the calculational techniques involved, deformation quantization has clear conceptual advantages. In retrospect, we can see that the meaning of quantization was not really understood in the earlier approaches. Indeed, the difficulties with Dirac's quantization postulates in the operator formalism were already noted by Groenewold [25], and later formalized by van Hove [1, 42]. The relationship between classical and quantum mechanical systems is also clarified. While in the oprator approach quantum systems can in principle be treated without any reference to their classical counterparts, the path integral formalism neccessarily has as its starting point the classical action. In the semi-classical approximation the appearance of the classical action in the quantum-mechanical treatment can be made plausible, but in the general case it remains somewhat unmotivated. In deformation quantization, on the other hand, the non-commutative algebra of observables emerges naturally as a deformation of the classical commutative algebra of functions on phase space.

The explication of this last statement will form the starting point of our present review. We shall then proceed to illustate how the method works for a large range of physical problems. We shall finally discuss some recent results which highlight the power of this method in a case where the other approaches are not instructive. To the extent that original results are described here, they are based on work done in collaboration with Peter Henselder in Dortmund [26, 27].

Alternative aspects of the program of deformation quantization are discussed in other recent reviews [17, 43, 48].

## 2. Deformations of Algebras

In this section we describe the mathematical setting for deformation quantization theory. The results are essentially due to Gerstenhaber [23].

### 2.1. Associative Algebras

Let $V$ be a vector space. For $k = 0, 1, 2, \ldots$ define the space of $k$-multilinear mappings

$$M^k(V) = \{m\colon \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \to V\,;\, m \text{ multilinear}\}\,. \tag{2.1}$$

Now let $a \in M^k(V)$, $b \in M^l(V)$, and take vectors $x_1, \ldots, x_{k+l-1} \in V$. Then define a mapping

$$\circ_i\colon M^k(V) \times M^l(V) \to M^{k+l-1}(V) \tag{2.2}$$

given by

$$(a \circ_i b)(x_1, \ldots, x_{k+l-1}) = a(x_1, \ldots, x_{i-1}, b(x_i, \ldots, x_{i+l-1}), \ldots, x_{k+l-1}). \quad (2.3)$$

With this we can define a composition law

$$a \diamond b = \sum_i (-1)^{(i-1)(l-1)} a \circ_i b \quad (2.4)$$

and a bracket

$$[a, b]_{\mathcal{G}} = a \diamond b - (-1)^{(k-1)(l-1)} b \diamond a \,. \quad (2.5)$$

Gerstenhaber [23] has shown that $[\,,]_{\mathcal{G}}$ satisfies a super-Jacobi identity:

$$[a, [b, c]_{\mathcal{G}}]_{\mathcal{G}} + (-1)^{(|c|-1)(|a|+|b|)} [c, [a, b]_{\mathcal{G}}]_{\mathcal{G}} + (-1)^{(|a|-1)(|b|+|c|)} [b, [c, a]_{\mathcal{G}}]_{\mathcal{G}} = 0 \,. \quad (2.6)$$

An element $m \in M^2(V)$ defines a product on $V$, since $m\colon V \times V \to V$. Let $a, b \in M^2(V)$, and $x, y, z \in V$. Then

$$(a \diamond b)(x, y, z) = a(b(x, y), z) - a(x, b(y, z)) \quad (2.7)$$

$$\begin{aligned} [a, b]_{\mathcal{G}}(x, y, z) = {} & a(b(x, y), z) - a(x, b(y, z)) \\ & + b(a(x, y), z) - b(x, a(y, z)) \end{aligned} \quad (2.8)$$

and

$$\frac{1}{2}[a, a]_{\mathcal{G}}(x, y, z) = a(a(x, y), z) - a(x, a(y, z)) \,. \quad (2.9)$$

Define $x \cdot y = a(x, y)$. Then the last equation may be written as

$$\frac{1}{2}[a, a]_{\mathcal{G}}(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \,. \quad (2.10)$$

We see from this that an element $a \in M^2(V)$ which satisfies

$$[a, a]_{\mathcal{G}} = 0 \quad (2.11)$$

determines an associative algebra structure on $V$.
Let $m \in M^2(V)$. Define the mapping $\delta_m\colon M^i(V) \to M^{i+1}(V)$ by

$$\delta_m n = [m, n]_{\mathcal{G}} \quad (2.12)$$

for $n \in M^i(V)$. The super-Jacobi identity for this case is

$$\begin{aligned} & [m, [m, n]_{\mathcal{G}}]_{\mathcal{G}} + (-1)^{(|n|-1)(2|m|)} [n, [m, m]_{\mathcal{G}}]_{\mathcal{G}} \\ & \quad + (-1)^{(|m|-1)(|m|+|n|)} [m, [n, m]_{\mathcal{G}}]_{\mathcal{G}} \\ & \quad = 2[m, [m, n]_{\mathcal{G}}]_{\mathcal{G}} - [[m, m]_{\mathcal{G}}, n]_{\mathcal{G}} = 0 \,. \end{aligned} \quad (2.13)$$

If $m$ is an associative product, $[m,m]_{\mathcal{G}} = 0$, then this becomes

$$\delta_m^2 n = 0 \quad \text{for all} \quad n\,. \tag{2.14}$$

Define the total space $M(V) = \oplus_i M^i(V)$. Then $(M(V), \delta_m)$ is the Hochschild complex of $(V,m)$. The cohomology of this complex is the **Hochschild cohomology**.

## 2.2. Deformations of Associative Algebras

Let $m_0 \in M^2(V)$ be an associative product. A deformation of $m_0$ is an element $m(\epsilon) \in M^2(V)$ such that

$$m(\epsilon) = m_0 + \epsilon m_1 + \epsilon^2 m_2 + \cdots \tag{2.15}$$

is a formal power series in the parameter $\epsilon$. The product determined by $m(\epsilon)$ is associative if

$$\begin{aligned}[m(\epsilon), m(\epsilon)]_{\mathcal{G}} = {} & [m_0, m_0]_{\mathcal{G}} + 2\epsilon[m_0, m_1]_{\mathcal{G}} \\ & + \epsilon^2(2[m_0, m_2]_{\mathcal{G}} + [m_1, m_1]_{\mathcal{G}}) + \cdots = 0\,.\end{aligned} \tag{2.16}$$

Remarks:

- $[m_0, m_0]_{\mathcal{G}} = 0$ by the assumption that $m_0$ is associative.
- $[m_0, m_1]_{\mathcal{G}} = \delta_{m_0} m_1 = 0$ means that $m_1$ is a $\delta_{m_0}$-cocyle. A symmetric mapping $m_0$ signifies a commutative product. Assume that $m_1$ is antisymmetric. Then define $\{x,y\} = m_1(x,y)$. Use now Eq. (2.8) with $a = m_0$ and $b = m_1$. This yields

  $$\delta_{m_0} m_1(x,y,z) = z \cdot \{x,y\} - x \cdot \{y,z\} + \{x \cdot y, z\} - \{x, y \cdot z\}. \tag{2.17}$$

  We then have for the antisymmetrized sum

  $$\begin{aligned}\frac{1}{2}\left(\delta_{m_0} m_1(x,y,z) - \delta_{m_0} m_1(x,z,y) + \delta_{m_0} m_1(z,x,y)\right) \\ = x \cdot \{y,z\} + \{x,z\} \cdot y - \{x \cdot y, z\}\,.\end{aligned} \tag{2.18}$$

  Hence $\delta_{m_0} m_1 = 0$ means that $m_1$ is a derivation.
- $\frac{1}{2}[m_1, m_1]_{\mathcal{G}} = -\delta_{m_0} m_2$ means that the cocycle $[m_1, m_1]_{\mathcal{G}}$ is a coboundary.
- Assume now that $m_2$ is symmetric. Then

  $$\sum_{\text{cyclic}} \left(\delta_{m_0} m_2 + \frac{1}{2}[m_1, m_1]_{\mathcal{G}}\right) = \{\{x,y\},z\} + \{\{y,z\},x\} + \{\{z,x\},y\}. \tag{2.19}$$

We see that the $\epsilon^2$-term in the expansion vanishes if $m_1$ satisfies the Jacobi identity.

Conclusions: With the above (anti-)symmetry conditions the deformation of the commutative product $m_0$ can be extended to second order if the coefficient $m_1$ satisfies

- antisymmetry,
- the Leibnitz rule,
- the Jacobi identity.

These are the defining conditions for $m_1(x,y) = \{x,y\}$ to be a **Poisson bracket**. We see that if we have a space with a commutative product and a Poisson structure we can construct a deformed associative product at least to second order. Possible obstructions to further extension of the deformation series lie in the third Hochschild cohomology class.

We shall not pursue these formal arguments further; our purpose here was only to indicate how the question of the existence of such deformed products can be formulated in a way that makes it accessible to mathematical analysis. As a matter of fact many important results concerning the existence of such products have been achieved, see Ref. [44].

## 2.3. Deformations of Algebras of Functions

We now take for the vector space $V$ the space of smooth functions on a manifold $M$, that is $V = C^\infty(M)$. For functions $f, g \in C^\infty(M)$ and $x \in M$ we take for the commutative product $m_0$ just the usual pointwise product of functions:

$$(m_0(f,g))(x) = f(x)g(x)\,. \tag{2.20}$$

As we have discussed, $m_1$ can be identified with a Poisson bracket structure:

$$m_1(f,g) = \{f,g\} = \alpha^{ij} f \overleftarrow{\partial}_i \overrightarrow{\partial}_j g\,. \tag{2.21}$$

Here $\alpha^{ij}$ is the Poisson tensor which characterizes the Poisson structure. $(M,\alpha)$ is a Poisson manifold. If the Poisson tensor is invertible then $M$ must be even dimensional, and $(M,\alpha)$ is a symplectic manifold.

We are now ready to define the central concept of a star product. A star product on a Poisson manifold is a deformation of the commutative pointwise product of Eq. (2.20):

$$f * g = m(\epsilon)(f,g), \tag{2.22}$$

with the parameter $\epsilon = \mathrm{i}\hbar/2$, and such that

$$\lim_{\hbar\to 0}\left(\frac{1}{\mathrm{i}\hbar}\right)[f,g]_* = \{f,g\}\,. \tag{2.23}$$

where $[f,g]_* = (f*g - g*f)$ is the **star commutator**.

We shall initially restrict our considerations to symplectic manifolds. An important role in the analysis of these manifolds is played by *Darboux's Theorem* [34]: there exist canonical coordinates $(q,p)$ for a symplectic manifold $M$ for which the coefficients $\alpha^{ij}$ are constants, and in these coordinates the Poisson bracket may be written as

$$\{f,g\} = f(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)g\,. \tag{2.24}$$

We actually restrict ourselves at the start to the flat manifold $M = \mathbb{R}^2$. Here there exists a star product already found by Goenewold [25], the **Moyal star product** [35], given explicitly by

$$\begin{aligned} f *_M g &= f\,\mathrm{e}^{(\frac{\mathrm{i}\hbar}{2})\alpha^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j}g = fg + \frac{\mathrm{i}\hbar}{2}\{f,g\} + \cdots \\ &= \sum_{m,n=0}^{\infty}\left(\frac{\mathrm{i}\hbar}{2}\right)^{m+n}\frac{(-1)^m}{m!n!}(\partial_p^m\partial_q^n f)(\partial_p^n\partial_q^m g)\,. \end{aligned} \tag{2.25}$$

Another star product is the **normal star product** given by

$$f *_N g = f\,\mathrm{e}^{\hbar\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}}g\,, \tag{2.26}$$

which is expressed in terms of the holomorphic coordinates $a$ and $\bar{a}$, which are related to the canonical coordinates by

$$a = \sqrt{\frac{\omega}{2}}\left(q + \mathrm{i}\frac{p}{\omega}\right), \qquad \bar{a} = \sqrt{\frac{\omega}{2}}\left(q - \mathrm{i}\frac{p}{\omega}\right) \tag{2.27}$$

where $\omega$ is a frequency parameter.

Two star products $*$ and $*'$ are *c-equivalent* if there exists an invertible transition operator

$$T = 1 + \hbar T_1 + \cdots = \sum_{n=0}^{\infty}\hbar^n T_n \tag{2.28}$$

where the $T_n$ are bidifferential operators, null on constants, such that

$$T(f *' g) = Tf * Tg\,. \tag{2.29}$$

In terms of the holomorphic coordinates the Moyal star product is

$$f *_M g = f \, \mathrm{e}^{\frac{\hbar}{2}(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)} g \tag{2.30}$$

and the transition operator connecting this to the normal star product is

$$T = \mathrm{e}^{-\frac{\hbar}{2} \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}} . \tag{2.31}$$

For the $2m$-dimensional phase space $M = \mathbb{R}^{2m}$, whose points are parametrized by the canonical coordinates $x = \{q_1, \ldots, q_m, p_1, \ldots, p_m\}$, the Moyal star product is just

$$f *_M g = f \, \mathrm{e}^{\frac{i\hbar}{2} \sum_i (\overleftarrow{\partial}_{q_i} \overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q_i})} g \, . \tag{2.32}$$

The generalization of the other expressions above to the phase space $M = \mathbb{R}^{2m}$ is equally straightforward. In this case all possible star products are $c$-equivalent to the Moyal product [14].

Besides these representations of star products using differential operators, one can also consider integral representations. The first such representation was given for the Moyal star product by von Neumann [36], it is (for a two-dimensional phase space)

$$(f *_M g)(\vec{r}) = \int \mathrm{d}\vec{r}_1 \, \mathrm{d}\vec{r}_2 \, f(\vec{r}_1) g(\vec{r}_2) \, \mathrm{e}^{\frac{i}{\hbar} S(\vec{r}_1, \vec{r}_2, \vec{r})} \tag{2.33}$$

where $\vec{r} = (q, p)$, $\vec{r}_i = (q_i, p_i)$ for $i = 1, 2$, and $S(\vec{r}_1, \vec{r}_2, \vec{r})$ is four times the area of the triangle in phase space with vertices $(\vec{r}_1, \vec{r}_2, \vec{r})$, see Fig. 1. The associativity of the star product is easy to see in this representation [49].

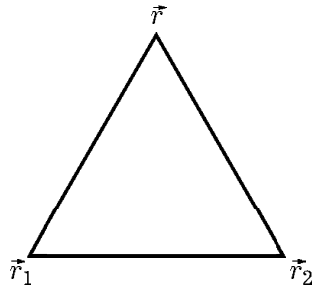


**Figure 1.** A triangle in phase space

## 3. Quantum Mechanical Systems

In classical mechanics the state of a physical system with one degree of freedom is represented by a point $\vec{r} = (q, p)$ in a two-dimensional phase space $M$. The system evolves in time from an initial state $\vec{r}_0 = (q_0, p_0)$ by moving along a

curve in $M$ which is determined by the equations of motion, here the Hamilton equations. Physical observables are functions on the phase space $f \in C^\infty(M)$. The measurable quantities are the values of the observables in specific states $f(\vec{r}_0)$.

For our present purposes it is convenient to describe physical states as distributions in phase space $\pi_{\vec{r}_0}(\vec{r}) = \delta(\vec{r} - \vec{r}_0)$. The measurable quantities are obtained as

$$f(\vec{r}_0) = \int f(\vec{r})\pi_{\vec{r}_0}(\vec{r})\,\mathrm{d}\vec{r} = \int f(\vec{r})\delta(\vec{r} - \vec{r}_0)\,\mathrm{d}\vec{r}. \tag{3.1}$$

In classical mechanics two observables are multiplied by using the pointwise multiplication of functions:

$$(f \cdot g)(\vec{r}) = f(\vec{r})g(\vec{r})\,. \tag{3.2}$$

Hence the algebra of **classical observables** is associative and commutative.

To go over to quantum mechanics we replace the pointwise multiplication of functions by the **star product** of these functions:

$$f * g = \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\hbar}{2}\right)^n m_n(f,g)\,. \tag{3.3}$$

The algebra of observables is now associative but non-commutative. In the **classical limit** $\hbar \to 0$ we have

$$f * g \to f \cdot g = m_0(f,g)\,. \tag{3.4}$$

In the **semi-classical limit** we have

$$\lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}[f,g]_* = \{f,g\} = m_1(f,g)\,. \tag{3.5}$$

This replaces **Dirac's quantization condition** $[\hat{f},\hat{g}] = \mathrm{i}\hbar\widehat{\{f,g\}}$, where $\hat{f}$ and $\hat{g}$ are the operators corresponding to the phase space functions $f$ and $g$. The star product thus reproduces Heisenberg's **uncertainty relations**, and obviously incorporates the characteristic quantum mechanical non-locality, as can be seen directly in either the differential or integral representations, Eqs (2.25) and (2.33).

A physical system is specified by its Hamilton function $H(q,p)$. The **time-evolution function** is a solution of the differential equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Exp}_*(Ht) = H * \mathrm{Exp}_*(Ht)\,, \tag{3.6}$$

which is just telling us that the Hamilton function is the generator of the time evolution of the system. When the Hamilton function is time-independent this equation has the solution

$$\mathrm{Exp}_*(Ht) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\mathrm{i}t}{\hbar}\right)^n (H*)^n \tag{3.7}$$

with

$$(H*)^n = \underbrace{H * H * \cdots * H}_{n \text{ times}} . \tag{3.8}$$

The **Fourier–Dirichlet** expansion of the time-evolution function is

$$\mathrm{Exp}_*(Ht) = \sum_E \pi_E \, \mathrm{e}^{-\mathrm{i}Et/\hbar} \tag{3.9}$$

where the projectors $\pi_E(q,p)$ describe the states of energy $E$. Inserting this into the time-evolution equation yields the $*$-eigenvalue equation

$$H * \pi_E = E\pi_E . \tag{3.10}$$

The projectors are normalized and idempotent:

$$\frac{1}{2\pi\hbar} \int \pi_E(q,p) \, \mathrm{d}q \, \mathrm{d}p = 1, \qquad \pi_E * \pi_E = \delta_{EE'} \pi_E . \tag{3.11}$$

The **spectral decomposition** of $H$ is

$$H = \sum_E E\pi_E . \tag{3.12}$$

### 3.1. The Simple Harmonic Oscillator

In this case the Hamilton function of the system is

$$H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 = \omega a \bar{a} \tag{3.13}$$

where $a$ and $\bar{a}$ are the holomorphic coordinates of Eq. (2.27). These variables have the Poisson brackets $\{a, \bar{a}\} = 1$. We then have from Eq. (2.26) the normal star products

$$\bar{a} *_N a = a\bar{a}, \qquad a *_N \bar{a} = a\bar{a} + \hbar, \tag{3.14}$$

and the star commutator is $[a, \bar{a}]_* = \hbar$, as required. The time-evolution equation in terms of the normal star product is

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Exp}_N(Ht) = (H + \hbar\omega\bar{a}\partial_{\bar{a}}) \, \mathrm{Exp}_N(Ht) , \tag{3.15}$$

with the solution

$$\mathrm{Exp}_N(Ht) = \mathrm{e}^{a\bar{a}/\hbar} \exp\left(\mathrm{e}^{-\mathrm{i}\omega t} a\bar{a}/\hbar\right). \tag{3.16}$$

Expanding the exponential yields the Fourier-Dirichlet expansion:

$$\mathrm{Exp}_N(Ht) = \mathrm{e}^{-a\bar{a}/\hbar} \sum_{n=0}^{\infty} \frac{1}{\hbar^n n!} \bar{a}^n a^n \, \mathrm{e}^{-\mathrm{i}n\omega t}. \tag{3.17}$$

We read off

$$\pi_0^{(N)} = \mathrm{e}^{-a\bar{a}/\hbar}, \quad \pi_n^{(N)} = \frac{1}{\hbar^n n!} \pi_0^{(N)} \bar{a}^n a^n, \quad E_n = n\hbar\omega. \tag{3.18}$$

To go over to the Moyal product scheme apply the transition operator

$$T = \mathrm{e}^{-\frac{\hbar}{2}\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}}. \tag{3.19}$$

The result is

$$\begin{aligned} T\pi_0^{(N)} &= \pi_0^{(M)} = 2\,\mathrm{e}^{-2a\bar{a}/\hbar} \\ T\pi_n^{(N)} &= \pi_n^{(M)} = \frac{1}{\hbar^n n!} \bar{a}^n *_M \pi_0^{(M)} *_M a^n \end{aligned} \tag{3.20}$$

and

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,. \tag{3.21}$$

The projectors may also be written as

$$\pi_n^{(M)} = 2\,\mathrm{e}^{-2H/\hbar\omega} L_n\left(\frac{4H}{\hbar\omega}\right) \tag{3.22}$$

where $L_n(x)$ are the Laguerre polynomials, related to the Hermite polynomials by

$$\mathrm{e}^{-b^2} L_n(a^2 + b^2) = \int \mathrm{d}x\, \mathrm{e}^{-x^2} H_n(x-a) H_n(x+a)\, \mathrm{e}^{-2bx} \tag{3.23}$$

compare Eq. (3.25) below.

### 3.2. The Operator Formalism

A **quantization prescription** is a map $\Theta$: $C^\infty(M) \to \mathcal{A}$ from the smooth functions on phase space to linear operators on a Hilbert space. Groenewold [25] showed that

$$\Theta(f)\Theta(g) = \Theta(f * g). \tag{3.24}$$

This means that the operator algebra is a **representation** of the star product algebra. Indeed, results obtained in the star product formalism are intimately related to results in the operator formalism. For example, the projectors (often called **Wigner functions for pure states**) are related to the Schrödinger wave functions by

$$\pi_E(q,p) = \int \psi_E^*(q+\xi/2)\psi_E(q-\xi/2)\,\mathrm{e}^{-\mathrm{i}\xi p}\,\mathrm{d}\xi\,. \tag{3.25}$$

From this we find

$$\begin{aligned} \frac{1}{2\pi\hbar}\int \pi_E(q,p)\,\mathrm{d}p &= |\psi_E(q)|^2 \\ \frac{1}{2\pi\hbar}\int \pi_E(q,p)\,\mathrm{d}q &= |\tilde{\psi}_E(p)|^2 \end{aligned} \tag{3.26}$$

where $\tilde{\psi}_E(p)$ is the Fourier transform of $\psi_E(q)$. The expectation value of the Hamilton function in the state characterized by $\pi_E$ is

$$E = \int \mathrm{d}p\,\mathrm{d}q H(q,p) * \pi_E(q,p)\,. \tag{3.27}$$

In Eqs (3.24) and (3.27) we may use different star products and their corresponding projectors. Different choices for the star product correspond to different choices for the operator ordering in Eq. (3.24) [11, 2].

The relation of the operator formalism to the path integral approach in a general ordering scheme has been studied by Cohen [12]. A direct relation of the deformation quantization procedure to quantization procedures involving the path integral has been worked out by Sharan [39] and Dito [16].

### 3.3. The Forced Harmonic Oscillator

In this subsection we follow [27], which can be consulted for further details. The Hamilton function for an oscillator acted on by the external source $J(t)$ is

$$H = \omega a\bar{a} - J(t)\bar{a} - \bar{J}(t)a\,. \tag{3.28}$$

The time evolution of this system is determined by the differential equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_J(t,t_i) = H *_N U_J(t,t_i) = [H + \hbar(\omega\bar{a} - J(t))\partial_{\bar{a}}]U_J(t,t_i) \quad (3.29)$$

where we are working in the normal product scheme. The solution is

$$U_J(t_f,t_i) = \mathrm{e}^{-a\bar{a}/\hbar}\exp\left[\frac{1}{\hbar}a\bar{a}\,\mathrm{e}^{\mathrm{i}\omega(t_f-t_i)} + \frac{\mathrm{i}}{\hbar}a\,\mathrm{e}^{\mathrm{i}\omega t_f}\int_{t_i}^{t_f}\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\omega s}\bar{J}(s)\right.$$
$$\left. + \frac{\mathrm{i}}{\hbar}\bar{a}\,\mathrm{e}^{-\mathrm{i}\omega t_f}\int_{t_i}^{t_f}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}\omega s}J(s) - \frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\mathrm{d}s\int_{s}^{t_f}\mathrm{d}u\,\mathrm{e}^{\mathrm{i}\omega(u-s)}J(s)J(u)\right]. \quad (3.30)$$

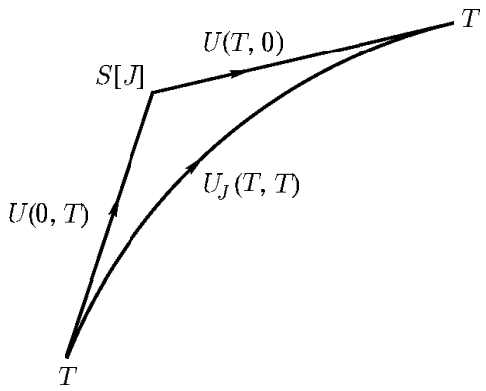


**Figure 2.** The scattering function

In the **scattering situation** the source acts only in the time-interval $[-T,T]$. The asymptotic dynamics is governed by the time-evolution function $U = U_J(J=0)$. The **scattering function** relates asymptotic in- and out-states, see Fig. 2. The formula is:

$$S[J] = \lim_{T\to\infty} U(0,T) *_N U_J(T,-T) *_N U(-T,0). \quad (3.31)$$

One can show in general [13] that the free time-development of a phase space function is given by

$$U(0,T) *_N f(a,\bar{a}) *_N U(-T,0) = f(a\,\mathrm{e}^{-\mathrm{i}\omega T}, \bar{a}\,\mathrm{e}^{\mathrm{i}\omega T}). \quad (3.32)$$

For our case this yields

$$S[J] = \exp\left[\frac{\mathrm{i}}{\hbar}a\bar{j}(\omega) + \frac{\mathrm{i}}{\hbar}\bar{a}j(\omega) - \frac{1}{2\hbar}\iint \mathrm{d}s\,\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\omega|s-u|}J(s)J(u)\right] \quad (3.33)$$

where

$$j(\omega) = \int_{\infty}^{\infty} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\omega s} J(s) \tag{3.34}$$

is the Fourier transform of $J(s)$. Define now a **field function**

$$\phi(t) = a\, \mathrm{e}^{-\mathrm{i}\omega t} + \bar{a}\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{3.35}$$

We may then write

$$S[J] = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t J(t)\phi(t)} \exp\left[-\frac{1}{2\hbar^2}\iint \mathrm{d}t\, \mathrm{d}t' J(t) D_F(t-t') J(t')\right] \tag{3.36}$$

with

$$D_F(t) = \hbar\left[\theta(t)\, \mathrm{e}^{-\mathrm{i}\omega t} + \theta(-t)\, \mathrm{e}^{\mathrm{i}\omega t}\right]. \tag{3.37}$$

Here $\theta(t)$ is the Heaviside function

$$\theta(t) = \begin{cases} 0 & \text{for} t \leq 0 \\ 1 & \text{for} t > 0\,. \end{cases} \tag{3.38}$$

$S[J]$ corresponds to the scattering operator in quantum field theory, $D_F(t)$ to the *Feynman propagator*, see Eqs (3.69) and (3.74) below.

The **generating functional** is the vacuum expectation value of the scattering operator:

$$Z_0[J] = \frac{1}{2\pi\hbar}\int \mathrm{d}a^2 S[J] *_N \pi_0^N = \mathrm{e}^{-\frac{1}{2\hbar^2}\int\int \mathrm{d}t\, \mathrm{d}t' J(t) D_F(t-t') J(t')}. \tag{3.39}$$

To calculate off-diagonal matrix elements use the Wigner functions

$$\pi_{m,n}^{(N)} = \frac{1}{\sqrt{\hbar^{m+n} m! n!}} \pi_0^{(N)} \bar{a}^m a^n \tag{3.40}$$

with $\pi_{n,n}^{(N)} = \pi_n^{(N)}$. The **transition amplitudes** are then

$$\mathrm{Amp}(0 \to n) = \frac{1}{2\pi\hbar}\int \mathrm{d}a^2 \pi_{0,n} *_N S[J] *_N \pi_0^{(N)} = \frac{(\mathrm{i}j(\omega))^n}{\hbar^{n/2}\sqrt{n!}}\, \mathrm{e}^{-|j(\omega)|^2/2\hbar}. \tag{3.41}$$

The **probability** for the transition is

$$P_n = |\,\mathrm{Amp}(0 \to n)|^2 = \frac{|j(\omega)|^{2n}}{\hbar^n n!}\, \mathrm{e}^{-|j(\omega)|^2/\hbar}. \tag{3.42}$$

In quantum field theory this gives the **Poisson distribution** for the number of emitted quanta [29]:

$$P_n = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!} \tag{3.43}$$

with

$$\bar{n} = \sum_{n=0}^{\infty} n P_n = |j(\omega)|^2/\hbar\,. \tag{3.44}$$

### 3.4. Multiple Star Products and Wick's Theorem

In this subsection we again follow [27]. Related work was done earlier by Lesche [32]. Define

$$M_{12} = \left(\frac{\mathrm{i}\hbar}{2}\right) \sum_{i,j=1}^{2m} \alpha^{ij} \frac{\partial}{\partial x_1^i} \frac{\partial}{\partial x_2^j} \tag{3.45}$$

where $x_\alpha^i$ $(i = 1, \ldots, 2m)$ is the $i$-th component of phase space point $x_\alpha$, and $\alpha^{ij}$ are the coefficients of the Poisson structure on $M$. The star product of two phase space functions may then be written as

$$(f *_M g)(x) = \mathrm{e}^{M_{12}} f(x_1) g(x_2)|_{x_1 = x_2 = x}\,. \tag{3.46}$$

For the star product of $r$ functions of the holomorphic coordinates $a, \bar{a}$ we obtain

$$(f_1 * f_2 * \cdots * f_r)(a, \bar{a}) = \mathrm{e}^{\left(\sum_{i<j} M_{ij}\right)} f_1(a_1, \bar{a}_1) \cdots f_r(a_r, \bar{a}_r)|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}}\,. \tag{3.47}$$

Consider functions $f_i$ which are linear in $a$ and $\bar{a}$:

$$f_i(a, \bar{a}) = A_i a + B_i \bar{a}\,. \tag{3.48}$$

For such functions the star product may be written in the form of a **Wick theorem** by expanding the exponential: for example for $r = 4$

$$\begin{aligned} f_1 *_M f_2 *_M f_3 *_M f_4 = {} & f_1 f_2 f_3 f_4 + G_{12}(f_3 f_4) + G_{13}(f_2 f_4) + G_{14}(f_3 f_3) \\ & + G_{23}(f_1 f_4) + G_{24}(f_1 f_3) + G_{34}(f_1 f_2) \\ & + G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23} \end{aligned} \tag{3.49}$$

where the **contractions**

$$G_{ij} = M_{ij} f_i f_j = \frac{\hbar}{2}(A_i B_j - A_j B_i) \tag{3.50}$$

are constants. We may also write

$$M_{ij} = G_{ij} \frac{\partial}{\partial f_i} \frac{\partial}{\partial f_j}, \tag{3.51}$$

and Eq. (3.47) then becomes

$$f_1 *_M f_2 *_M \cdots *_M f_r = \exp\left(\sum_{i<j} G_{ij} \frac{\partial}{\partial f_i} \frac{\partial}{\partial f_j}\right) \prod_{m=1}^{r} f_m \,. \tag{3.52}$$

It should be clear from the above that not only the original form [46], but also the various **generalized Wick theorems** which have been discussed in the literature [2, 33] are direct consequences of the structure of the relevant star products.

A product of operators is the Weyl transform of the star product of the corresponding phase space functions [25]. For example, for the Moyal product scheme:

$$\begin{aligned} \hat{f}_1 \cdots \hat{f}_r &= \Theta_M\{(f_1 *_M \cdots *_M f_r)(a, \bar{a})\} \\ &= \Theta_M \left\{ \exp\left(\sum_{i<j} M_{ij}\right) \prod_{m=1}^{r} f_m(a_m, \bar{a}_m)|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}} \right\}. \end{aligned} \tag{3.53}$$

For a quantization scheme which is $c$-equivalent to the Moyal scheme we use the corresponding contraction factors $X_{ij}$ instead of the Moyal contraction factors $M_{ij}$. We may write $X_{ij} = X_{\{ij\}} + M_{ij}$, where $X_{\{ij\}} = \frac{1}{2}(X_{ij} + X_{ji})$ is the symmetric part of $X_{ij}$, since the antisymmetric part is fixed for all $c$-equivalent star products by the definition, see Eq. (2.23).

The **time-ordered product** of $r$ time-dependent operators is given by the prescription

$$\begin{aligned} &\mathcal{T}\{\hat{f}_1(t_1) \cdots \hat{f}_r(t_r)\} \\ &= \Theta_X \left[ \exp\left( \sum_{i<j} (X_{\{ij\}} + \epsilon(t_i - t_j) M_{ij}) \right) \prod_{m=1}^{r} f_m(a_m, \bar{a}_m, t_m)|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}} \right] \end{aligned} \tag{3.54}$$

since the transposition of two operators leaves $X_{\{ij\}}$ invariant, while the signs of $\epsilon(t_i - t_j)$ and of $M_{ij}$ reverse. For the case of normal ordering we may write

the exponent in Eq. (3.54) as

$$\begin{aligned} T_{ij} &= N_{\{ij\}} + \epsilon(t_i - t_j)M_{ij} \\ &= \frac{\hbar}{2}[(\partial_{a_i}\partial_{\bar{a}_j} + \partial_{\bar{a}_i}\partial_{a_j}) + \epsilon(t_i - t_j)(\partial_{a_i}\partial_{\bar{a}_j} - \partial_{\bar{a}_i}\partial_{a_j})] \\ &= \frac{\hbar}{2}[(1 + \epsilon(t_i - t_j))\partial_{a_i}\partial_{\bar{a}_j} + (1 - \epsilon(t_i - t_j))\partial_{\bar{a}_i}\partial_{a_j}] \\ &= \hbar[\theta(t_i - t_j)\partial_{a_i}\partial_{\bar{a}_j} + \theta(t_j - t_i)\partial_{\bar{a}_i}\partial_{a_j}]\,. \end{aligned} \tag{3.55}$$

Suppose now that the functions $f_m$ are linear in $a_m$ and $\bar{a}_m$, and have a periodic time dependence:

$$f_m(t) = A_m a_m \,\mathrm{e}^{-\mathrm{i}\omega t} + B_m \bar{a}_m \,\mathrm{e}^{\mathrm{i}\omega t}. \tag{3.56}$$

By Eq. (3.50) the relevant contractions are

$$\begin{aligned} D_{ij}(t - t') &= T_{ij} f_i(t) f_j(t') \\ &= \hbar\left[A_i B_j \theta(t - t')\,\mathrm{e}^{-\mathrm{i}\omega(t-t')} + A_j B_i \theta(t' - t)\,\mathrm{e}^{\mathrm{i}\omega(t-t')}\right], \end{aligned} \tag{3.57}$$

which is a generalization of the expression in Eq. (3.37). We write, in analogy to Eq. (3.51),

$$T_{ij} = \iint \mathrm{d}t\,\mathrm{d}t' \frac{\delta}{\delta f_i(t)} D_{ij}(t - t') \frac{\delta}{\delta f_j(t')} \tag{3.58}$$

where the $\delta/\delta f(t)$ are functional derivatives. For the operators

$$\hat{f}_m(t) = A_m \hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t} + B_m \hat{a}^\dagger \,\mathrm{e}^{\mathrm{i}\omega t} \tag{3.59}$$

we get a quantum mechanical form of Wick's theorem by inserting these expressions into Eq. (3.54):

$$\begin{aligned} &\mathcal{T}\{\hat{f}_1(t_1)\cdots\hat{f}_r(t_r)\} \\ &= \Theta_N\Big[\exp\Big(\sum_{i<j}\iint \mathrm{d}t\,\mathrm{d}t' \frac{\delta}{\delta f_i(t)} D_{ij}(t - t') \frac{\delta}{\delta f_j(t')}\Big) f_1(t_1)\cdots f_r(t_r)\big|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}}\Big]. \end{aligned} \tag{3.60}$$

Since we have modified the star product contractions in Eq. (3.54) by the insertion of the $\epsilon(t_i - t_j)$ factors, the time-ordered product is *not* the Weyl transform of a star product. This can be seen from the fact that the time-ordered product is symmetric in its arguments, whereas the star products have an antisymmetric part fixed by their definition, Eq. (2.23).

### 3.5. Quantum Field Theory

A free scalar field may be written as

$$\phi_m(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a_m(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx} + \bar{a}_m(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\right] \tag{3.61}$$

where $\hbar\omega_{\mathbf{k}} = \sqrt{\hbar^2\mathbf{k}^2 + m^2}$. This is the infinite-dimensional generalization of the formula (3.56) for the finite-dimensional case. Corresponding to the formulae

$$M_{12} = \frac{\hbar}{2}\left(\frac{\partial}{\partial a_1}\frac{\partial}{\partial \bar{a}_2} - \frac{\partial}{\partial \bar{a}_1}\frac{\partial}{\partial a_2}\right) \tag{3.62}$$

and

$$G_{ij} = M_{ij} f_i f_j \tag{3.63}$$

we now have

$$\begin{aligned}\frac{1}{2}D(x_1 - x_2) = \frac{\hbar}{2}\iiint \mathrm{d}^3k \frac{d^3k_1}{(2\pi)^{\frac{3}{2}}} \frac{\mathrm{d}^3k_2}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_2}}} \\ \times \left[\frac{\delta}{\delta a_1(\mathbf{k})}\frac{\delta}{\delta \bar{a}_2(\mathbf{k})} - \frac{\delta}{\delta \bar{a}_1(\mathbf{k})}\frac{\delta}{\delta a_2(\mathbf{k})}\right] \\ \times \left(a_1(\mathbf{k}_1)\,\mathrm{e}^{-\mathrm{i}k_1x_1} + \bar{a}_1(\mathbf{k}_1)\,\mathrm{e}^{\mathrm{i}k_1x_1}\right) \\ \times \left(a_2(\mathbf{k}_2)\,\mathrm{e}^{-\mathrm{i}k_2x_2} + \bar{a}_2(\mathbf{k}_2)\,\mathrm{e}^{\mathrm{i}k_2x_2}\right) \\ = \frac{1}{2}\left[D^+(x_1 - x_2) + D^-(x_1 - x_2)\right]\end{aligned} \tag{3.64}$$

where

$$D^{\pm}(x) = \pm\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\hbar}{2\omega_{\mathbf{k}}}\,\mathrm{e}^{\mp\mathrm{i}kx} \tag{3.65}$$

are the propagators for the components of positive and negative frequencies, and $D(x)$ is the Schwinger function [29].

For the quantum field operators

$$\hat{\Phi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[\hat{a}(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx} + \hat{a}^{\dagger}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\right] \tag{3.66}$$

we obtain, in analogy to Eq. (3.53),

$$\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_r) \tag{3.67}$$

$$= \Theta_M \left\{ \exp\left[ \frac{1}{2} \sum_{i<j} \iint \mathrm{d}^4x\, \mathrm{d}^4y \frac{\delta}{\delta\phi_i(x)} D(x-y) \frac{\delta}{\delta\phi_j(y)} \right] \prod_{m=1}^{r} \phi_m(x_m)|_{\phi_m=\phi} \right\}$$

and, in analogy to Eq. (3.60),

$$\mathcal{T}\{\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_r)\} \tag{3.68}$$
$$= \Theta_N \left\{ \exp\left[ \sum_{i<j} \iint \mathrm{d}^4x\, \mathrm{d}^4y \frac{\delta}{\delta\phi_i(x)} D_F(x-y) \frac{\delta}{\delta\phi_j(y)} \right] \prod_{m=1}^{r} \phi_m(x_m)|_{\phi_m=\phi} \right\}.$$

Here $D_F$, the **Feynman propagator**, is given by the infinite dimensional generalization of Eq. (3.37):

$$\begin{aligned} D_F(x_1-x_2) = \iiint \mathrm{d}^3k \frac{\mathrm{d}^3k_1}{(2\pi)^{\frac{3}{2}}} \frac{\mathrm{d}^3k_2}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_2}}} \\ \times \hbar \left[ \theta(t_1-t_2) \frac{\delta}{\delta a_1(\mathbf{k})} \frac{\delta}{\delta \bar{a}_2(\mathbf{k})} + \theta(t_2-t_1) \frac{\delta}{\delta \bar{a}_1(\mathbf{k})} \frac{\delta}{\delta a_2(\mathbf{k})} \right] \\ \times \left( a_1(\mathbf{k}_1)\, \mathrm{e}^{-\mathrm{i}k_1x_1} + \bar{a}_1(\mathbf{k}_1)\, \mathrm{e}^{\mathrm{i}k_1x_1} \right) \\ \times \left( a_2(\mathbf{k}_2)\, \mathrm{e}^{-\mathrm{i}k_2x_2} + \bar{a}_2(\mathbf{k}_2)\, \mathrm{e}^{\mathrm{i}k_2x_2} \right) \\ = \theta(t_1-t_2) D^+(x_1-x_2) - \theta(t_2-t_1) D^-(x_1-x_2)) \,. \end{aligned} \tag{3.69}$$

We may simplify Eq. (3.68) by using the symmetry of the Feynman propagator, $D_F(x_1-x_2) = D_F(x_2-x_1)$. It becomes:

$$\mathcal{T}\{\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_r)\} \tag{3.70}$$
$$= \Theta_N \left\{ \exp\left[ \frac{1}{2} \iint \mathrm{d}^4x\, \mathrm{d}^4y \frac{\delta}{\delta\phi(x)} D_F(x-y) \frac{\delta}{\delta\phi(y)} \right] \phi(x_1)\cdots\phi(x_r) \right\}.$$

Note that in this case it is no longer necessary to use different fields which are set equal only after the differentiation; because of the symmetry the correct combinatorics are guaranteed by the Leibnitz rule for differentiation. Eq. (3.70) is the field-theoretic version of Wick's theorem.

For $n = 2$ Wick's theorem is

$$\mathcal{T}\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\} = \Theta_N\{\phi(x_1)\phi(x_2)\} + D_F(x_1-x_2) \,. \tag{3.71}$$

Since the vacuum expectation value of the normal product vanishes, this yields the familiar relation

$$D_F(x_1-x_2) = \langle 0|\mathcal{T}\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\}|0\rangle \,. \tag{3.72}$$

Wick's theorem may also be written in the form of a generating function:

$$\mathcal{T}\{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x J(x)\hat{\Phi}(x)}\} \tag{3.73}$$
$$= \Theta_N\{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x J(x)\phi(x)}\}\exp\left[-\frac{1}{2\hbar^2}\iint \mathrm{d}^4x\,\mathrm{d}^4y J(x)D_F(x-y)J(y)\right]$$

where $J(x)$ is an external source. Eq. (3.70) then results by expanding both sides of Eq. (3.73) in powers of $J$ and comparing coefficients. Note that

$$\hat{S}[J] = \mathcal{T}\{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x J(x)\hat{\Phi}(x)}\} = \mathcal{T}\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x \hat{H}_{\mathrm{int}}(x)}\} \tag{3.74}$$

is the **scattering operator** of quantum field theory [29], so that Eq. (3.73) corresponds to the perturbation expansion of the scattering operator. This is just the operator form of our previous result, Eq. (3.36), which was derived completely within the phase space formalism of deformation quantization theory. The generating functional for the perturbation series is, by Eq. (3.73),

$$Z_0[J] = \langle 0|\hat{S}[J]|0\rangle = \exp\left[-\frac{1}{2\hbar^2}\iint \mathrm{d}^4x\,\mathrm{d}^4y J(x)D_F(x-y)J(y)\right] \tag{3.75}$$

in agreement with our previous result, Eq. (3.39). When a self-interaction term is included in the interaction Hamiltonian, $\hat{H}_{\mathrm{int}} = -J\phi + V(\phi)$, the generating functional for the interacting theory becomes

$$Z[J] = \frac{1}{N}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x V\left(\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} Z_0[J] \tag{3.76}$$

where the normalization constant is $N = Z[J=0]$.

## 4. Star Products on Poisson Manifolds

### 4.1. The Kontsevich star product

We may write the Moyal product of two phase space functions as

$$f * g = fg + \left(\frac{\mathrm{i}\hbar}{2}\right)\alpha^{ij}(\partial_i f)(\partial_j g) + \frac{1}{2}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \alpha^{ij}\alpha^{lm}(\partial_i\partial_l f)(\partial_j\partial_m g) + \cdots \tag{4.1}$$

In a graphical notation we represent a vertex $\{f,g\} = \alpha^{ij}(\partial_i f)(\partial_j g)$ as in Fig. 3. The graphical representation for the Moyal product then takes the form given in Fig. 4. Here the phase space $M$ is a symplectic manifold and the coefficients of the Poisson structure $\alpha^{ij}$ are constants. If we consider functions on a Poisson manifold $M$ then the coefficients $\alpha^{ij}(x)$ are in general functions of $x \in M$.

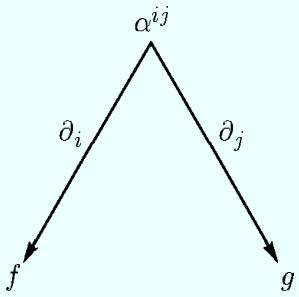


**Figure 3.** Graphical representation of a vertex

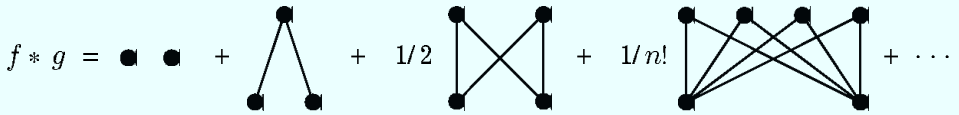


**Figure 4.** Graphical representation of the Moyal product

The **Kontsevich star product** [31] of two functions $f, g \in C^\infty(M)$, where $M$ is a Poisson manifold, is represented graphically as in Fig. 5. The series includes the graphs which appear in the representation of the Moyal product, plus graphs such as the the last one in the figure, which stands for the expression

$$\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \alpha^{lm}(\partial_l \alpha^{ij})(\partial_f)(\partial_j \partial_m g)\,. \tag{4.2}$$

In this way the Kontsevich product may be seen as a natural extension of the Moyal product on symplectic manifolds to the more general framework of Poisson manifolds.

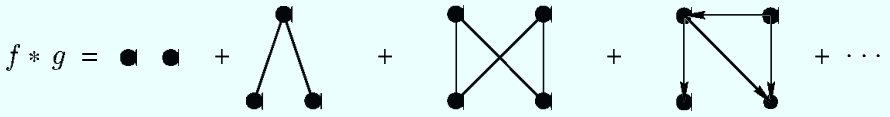


**Figure 5.** Graphical representation of the Kontsevich product

Kontsevich tells us in his paper which graphs are admissible; e. g. graphs with closed loops such as Fig. 6 are forbidden. He also provides the numerical coefficients for the various graphs in terms of certain angular integrals; e. g. the coefficient of the graph in Fig. 3 is

$$\frac{1}{(2\pi)^2} \iint\limits_{\phi_1 < \phi_2} \mathrm{d}\phi_1\, \mathrm{d}\phi_2 = \frac{1}{2}\,. \tag{4.3}$$

We shall see in the following how these coefficients can be understood in a field theoretic framework.

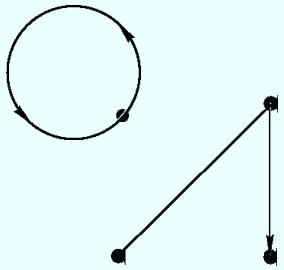

**Figure 6.** Graphical representation of a forbidden graph

Note that when the Poisson manifold is symplectic the coefficients $\alpha^{ij}$ are constants, so that terms like that corresponding to the last graph in Fig. 5 vanish, see Eq. (4.2), and the Kontsevich product reduces to the Moyal product.

## 4.2. The Poisson-Sigma Model

This is a two-dimensional topological field theoretic model defined on the disc $D_2 \in \mathbb{R}^2$ and involving a set of scalar fields $X^i \colon D_2 \to M$ and gauge fields $A_i \colon D_2 \to T^*M$, where $M$ is a Poisson manifold [38]. The classical action for the model is

$$S[X, A] = \int_{D_2} (A_i \, \mathrm{d}X^i + \alpha^{ij}(X) A_i A_j) \, . \tag{4.4}$$

Here the $\alpha^{ij}(X)$ are the coefficients of the Poisson structure on $M$, and the $X^i$ can be thought of as coordinates on $M$.

Cattaneo and Felder [9] give a remarkable formula for the Kontsevich product of two functions on a Poisson manifold in terms of the expectation value of the ordinary product in the Poisson-sigma model:

$$(f * g)(x) = \int \mathcal{D}X \mathcal{D}A f(X(1)) g(X(2)) \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[X,A]} . \tag{4.5}$$

One has to integrate over all the field configurations $X$ which satisfy the boundary condition $X(\infty) = x \in M$. Here $1, 2, \infty$ are three points on the boundary of $D_2$, in anti-clockwise ordering, as in Fig. 7. The Kontsevich expression for the star product results from the perturbative expansion of the above expectation value in terms of Feynman graphs, as we shall explain below.

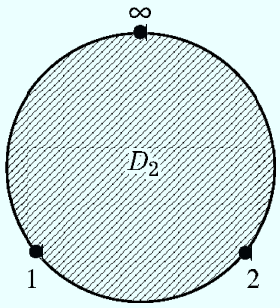


**Figure 7.** The disc and its boundary

When the manifold $M$ is symplectic the Poisson structure is non-degenerate, and the matrix $[\alpha^{ij}]$ is invertible: the coefficients of the inverse matrix are $\Omega_{ij} = [\alpha^{ij}]^{-1}$. In this case we may perform the Gaussian integration over the $A$-fields in Eq. (4.5), with the result

$$(f * g)(x) = \int \mathcal{D}X f(X(1))g(X(2))\, \mathrm{e}^{\frac{1}{\hbar}\int \Omega_{ij}\, \mathrm{d}X^i\, \mathrm{d}X^j} \tag{4.6}$$

where $\int \Omega_{ij}\, \mathrm{d}X^i\, \mathrm{d}X^j$ is the symplectic area of the image of the disc $D_2$ in $M$. At first sight it would seem as if we are dealing here with an infinite-dimensional functional integration. As a matter of fact, topological field theories involve only a finite number of degress of freedom, so the above integration must actually be finite-dimensional. In this case the reduction may be described as follows. We have to integrate only over field configurations which are not topologically equivalent, so that the integration reduces to a sum over representatives of the various homotopy classes [28]. Because we are doing perturbation theory about a trivial solution of the equations of motion (see Eq. (4.11) below), we can restrict ourselves to the trivial topological sector. Hence the expression in Eq. (4.6) is actually a single integration over the phase space $M$, and the formula (4.6) for the star product is the same as von Neumann's expression for the Moyal product, Eq. (2.33).

### 4.3. The Superfield Formalism

Before we can quantize a gauge theory using path integral techniques we must replace the gauge invariant classical action by an **effective action**, which is no longer gauge invariant, but which satisfies instead the BRST-symmetry [6]. For general gauge theories this is done by using the Batalin–Vilkovisky formalism [4]. This involves an extended phase space constructed by first including the Faddeev–Popov ghost fields, one for each gauge degree of freedom, and then doubling the number of degrees of freedom by including for each field an **antifield** of opposite Grassman parity. Since we arrive in this way at a theory

with the same number of bosonic and fermionic fields, it is plausible that the effective action is the **supersymmetric extension** of the original classical action [8]. In the model considered here this is indeed the case, and we shall construct the supersymmetric extension of the action by using the **superfield formalism**. We therefore replace the original fields of the Poisson-sigma model, $X^i$ and $A_i$, by

$$\tilde{X}^i(w,\zeta) = X^i(w) + \zeta^\mu A^*_{\mu i}(w) - \frac{1}{2}\zeta^\mu\zeta^\nu C^*_{i\mu\nu}(w) \tag{4.7}$$

$$\tilde{A}_i(z,\theta) = C_i(z) + \theta^\mu A_{\mu i}(z) + \frac{1}{2}\theta^\mu\theta^\nu X^*_{i\mu\nu}(z)\,. \tag{4.8}$$

Here $z, w \in D_2$ and the $\theta^\mu, \zeta^\mu$ $(\mu = 1,2)$ are Grassman variables. The $C_i(z)$ are the Faddeev–Popov ghosts. $A^*_{\mu i}(z), C^*_{i\mu\nu}(z), X^*_{i\mu\nu}(z)$ are the antifields. The Lagrangian of the theory is $L = \int \mathrm{d}^2\theta\mathcal{L}$, with $\mathcal{L}$ the supersymmetric Lagrangian density

$$\mathcal{L} = \tilde{A}D\tilde{X}^i + \alpha^{ij}(\tilde{X})\tilde{A}_i\tilde{A}_j\,. \tag{4.9}$$

Here $D\tilde{X}$ denotes the supersymmetric covariant derivative,

$$D = \theta^\mu \frac{\partial}{\partial u_\mu} \tag{4.10}$$

where $(u^1, u^2)$ parametrize the points of $D_2$. In the following we shall perform a perturbation expansion about the trivial classical solution

$$X^i(u) = x^i, \qquad A_i(u) = 0 \tag{4.11}$$

where the $x^i$ are constants. We then write $\tilde{X} = x + \tilde{\xi}$, with $\tilde{\xi}(\infty) = 0$, and

$$\mathcal{L} = \tilde{A}_i D\tilde{\xi}^i + \alpha^{ij}(\tilde{X})\tilde{A}_i\tilde{A}_j \tag{4.12}$$

where the first term gives rise to the kinetic term in the action, the second to the interaction term. From the Taylor expansion $\alpha^{ij}(x+\tilde{\xi}) = \alpha(x) + \tilde{\xi}^k\partial_k\alpha^{ij} + \cdots$ we get derivatives of $\alpha^{ij}$, from $f(\tilde{X}(1)) = f(x+\tilde{\xi}(1)) = f(x) + \tilde{\xi}^i(1)\partial_i f + \cdots$ we get derivatives of $f$. From these derivatives we shall form the Moyal, respectively the Kontsevich star products, see Eq. (4.26) below.

### 4.4. The Propagator

In field theory the key ingredient in the perturbation expansion is the propagator. In the superfield formalism the propagator arises from the kinetic term in the Lagrangian density $\tilde{A}_i D\tilde{\xi}^i$. It is

$$\langle \tilde{\xi}^i(w,\zeta)\tilde{A}_j(z,\theta)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^i_j D\phi(z,w) \tag{4.13}$$

where

$$D = \theta^\mu \frac{\partial}{\partial z^\mu} + \zeta^\mu \frac{\partial}{\partial w^\mu} \tag{4.14}$$

and $\phi(z,w)$ is an eigenfunction of the Laplacian $D^2\phi(z,w) = 2\pi\delta(z-w)$, so that

$$D\langle \tilde{\xi}^i(w)\tilde{A}_j(z)\rangle = \mathrm{i}\hbar\delta^i_j\delta(z-w)\,. \tag{4.15}$$

We see that the propagator is the Green's function corresponding to the differential operator in the kinetic term of the Lagrangian, as expected.
To determine the function $\phi(z,w)$ we consider the differential equation

$$\mathrm{d}_w \wedge \mathrm{d}_w\phi(z,w) = 2\pi\delta(z-w)\,\mathrm{d}^2w \tag{4.16}$$

where the derivative is

$$\mathrm{d}_w = \mathrm{d}w\frac{\partial}{\partial w} + \mathrm{d}\bar{w}\frac{\partial}{\partial \bar{w}}\,. \tag{4.17}$$

The solution which satisfies the correct boundary conditions is

$$\phi(z,w) = \frac{1}{2\mathrm{i}}\ln\frac{(z-w)(z-\bar{w})}{(\bar{z}-\bar{w})(\bar{z}-w)} \tag{4.18}$$

since $\phi(z,w) = 0$ for $z$ real, i. e. for $z$ on the boundary of the disc.

### 4.5. Hyperbolic Geometry

The two-dimensional disc $D_2$ can be conformally mapped to the Poincaré half-plane as depicted in Fig. 8. The geodesics in the Poincaré half-plane are vertical lines (these are the geodesics connecting interior points to $\infty$), and semi-circles (these are the geodesics connecting two interior points).

**Proposition 4.1.** *The function $\phi(z,w)$ of Eq. (4.18) is the angle between the geodesic through the points $(z,w)$ and the geodesic through the points $(z,\infty)$, see Fig. 9.*

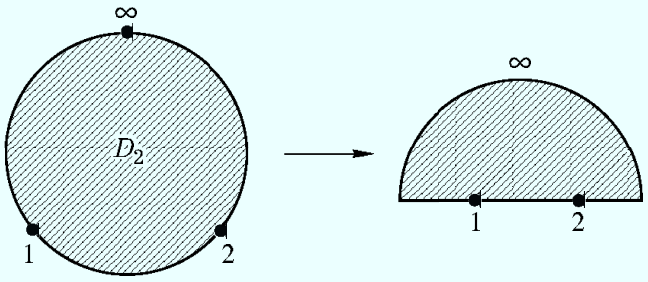


**Figure 8.** The Poincaré half-plane

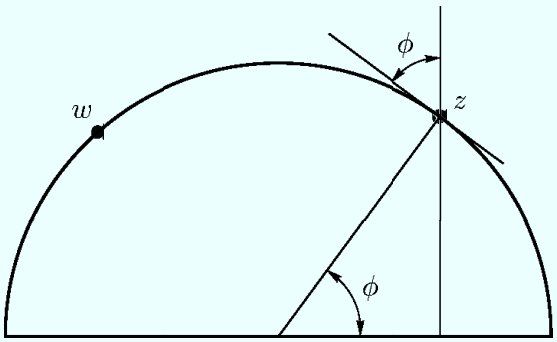


**Figure 9.** Angles between geodesics

Indeed, define the function

$$T(z,w) = \frac{(z-w)(z-\bar{w})}{(\bar{z}-\bar{w})(\bar{z}-w)} \,. \tag{4.19}$$

Now scale the diagram in the above figure to the unit circle, and choose for the origin of coordinates the center of the semi-circle which contains the points $(z,w)$. Neither of these choices affects the value of $T(z,w)$. Then we may calculate

$$T(z,w) = \frac{z^2+1-z(w+\bar{w})}{\bar{z}^2+1-\bar{z}(w+\bar{w})} = \frac{z(z+\frac{1}{z}-w-\bar{w})}{\bar{z}(\bar{z}+\frac{1}{\bar{z}}-w-\bar{w})} = \frac{z}{\bar{z}} \,. \tag{4.20}$$

In the last equation we have used the fact that $z\bar{z}=1$ implies $\bar{z}=\frac{1}{z}$ and $\frac{1}{\bar{z}}=z$. From the geometry of the figure $T(z,w)=z/\bar{z}=\mathrm{e}^{2\mathrm{i}\phi}$, or

$$\ln T(z,w) = 2\mathrm{i}\phi(z,w) \tag{4.21}$$

which agrees with Eq. (4.18).

### 4.6. The Perturbative Expansion

We now have all the tools assembled which we need in order to evaluate the perturbative expansion of the expression (4.5). In the superfield formalism

$$(f * g)(x) = \int \mathcal{D}\tilde{X}\mathcal{D}\tilde{A} f(\tilde{X}(1)) g(\tilde{X}(2)) \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\tilde{S}_{\mathrm{kin}} + \tilde{S}_{\mathrm{int}})}. \tag{4.22}$$

The vacuum expectation value of a function $\mathcal{O}(\tilde{X}, \tilde{A})$ is

$$\langle \mathcal{O}(\tilde{X}, \tilde{A}) \rangle = \int \mathcal{D}\tilde{X}\mathcal{D}\tilde{A} \mathcal{O}(\tilde{X}, \tilde{A}) \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tilde{S}_{\mathrm{kin}}}. \tag{4.23}$$

Hence by expanding the last exponent we may write

$$\langle f(\tilde{X}(1)) g(\tilde{X}(2)) \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int \alpha^{ij}\tilde{A}_i\tilde{A}_j} \rangle \tag{4.24}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{i}\hbar}{2}\right)^n \left\langle f(\tilde{X}(1)) g(\tilde{X}(2)) \left(\int \alpha^{ij}\tilde{A}_i\tilde{A}_j\right)^n \right\rangle. \tag{4.25}$$

It turns out that the terms involving non-physical fields (Faddeev–Popov ghosts and antifields) do not contribute to the expectation value [9]. The first relevant non-trivial term, which corresponds to the graph of Fig. 3, comes from using the Taylor expansions mentioned after Eq. (4.12), and is

$$\begin{aligned} &(\partial_i f)(\partial_j g)\alpha^{lm}(x)\langle \xi^i(1)\xi^j(2)A_l(u)A_m(u)\rangle \\ &\quad = \alpha^{lm}(x)(\partial_i f)(\partial_j g) \iint [\langle \xi^i(1)a_l(u)\rangle\langle \xi^j(2)A_m(u)\rangle \\ &\qquad - \langle \xi^i(1)A_m(u)\rangle\langle \xi^j(2)A_l(u)\rangle] \end{aligned} \tag{4.26}$$

where in the last line we have used Wick's theorem. Insert the values of the propagators, and use the antisymmetry of $\alpha^{ij}$, to obtain

$$\begin{aligned} &2\left(\frac{\mathrm{i}\hbar}{2}\right)\frac{1}{(2\pi)^2}\alpha^{ij}(x)(\partial_i f)(\partial_j g)\iint \mathrm{d}\phi(1,u)\,\mathrm{d}\phi(2,u) \\ &\qquad = \frac{\mathrm{i}\hbar}{2}\alpha^{ij}(\partial_i f)(\partial_j g) = \frac{\mathrm{i}\hbar}{2}\{f,g\}. \end{aligned} \tag{4.27}$$

We have used here

$$\iint \mathrm{d}\phi(1,u)\,\mathrm{d}\phi(2,u) = \frac{1}{2}(2\pi)^2 \tag{4.28}$$

since the angles range from 0 to $2\pi$ with the restriction $\phi(1,u) < \phi(2,u)$, as can be seen from Fig. 10.

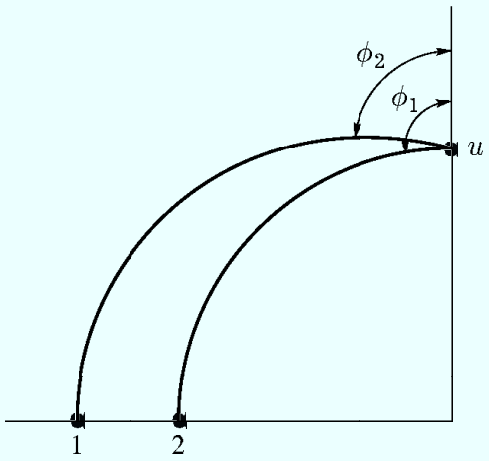


**Figure 10.** Comparison of angles

Some of the vanishing terms in the expansion of Eq. (4.24) are shown in Fig. 11. In quantum field theory one uses a renormalization scheme in which the contributions of the tadpole graphs vanish. This coincides with Kontsevich's rule excluding graphs involving closed loops [31].

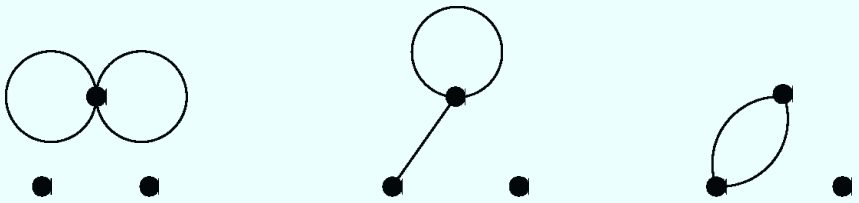

**Figure 11.** Some vanishing graphs

## 5. Summary

We hope to have convinced the reader of the following points:

1. Deformation quantization provides a unified conceptual framework for classical and quantum physics.
2. The passage from a classical system to its quantum counterpart is clarified. Dirac's quantization rule is generalized in a way which avoids the no-go theorems affecting previous treatments. The admissible quantization schemes are classified.
3. One has a viable alternative to operator methods and path integrals for treating problems in relativistic quantum field theory.
4. Star products provide an important bridge between mathematics and physics. Methods from quantum field theory can be used to gain insight into modern

mathematical developments. This last aspect was illustrated here for the case of Kontsevich's star product defined on Poisson manifolds.

## Acknowledgement

We want to thank C. Zachos for drawing our attention to the paper by B. Lesche [32].

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