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# NETS OF ASYMPTOTIC LINES IN A RIEMANNIAN HYPERSURFACE WITH NON-SYMMETRIC METRIC CONNECTION

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**Abstract**. If $M$ is a Riemannian space we denote by $g$ its Riemannian metric and by $\nabla$ its Riemannian connection. Additionally we assume that $M$ admits a non-symmetric metric connection $\nabla^*$. Let $C\colon x^i = x^i(s)$ be a curve in $M_n(\nabla, g)$ parametrized by the arc length. This curve can also be considered to be in $M_n(\nabla^*, g)$. Using the normal curvature of $C$ in $M_n(\nabla^*, g)$, one can obtain the asymptotic lines of $M_n(\nabla^*, g)$.

A curve in a hypersurface is defined to be an asymptotic line if the normal curvature along the curve vanishes identically. Here we investigate the special nets which are formed by the tangent fields of asymptotic lines in $M_n(\nabla^*, g)$ from that of the nets in $M_n(\nabla, g)$ and by using the coefficients of $\dot{\nabla}^*$ and $\dot{\nabla}$.

## 1. Introduction

Let $M$ be an $n$-dimensional Riemannian space having a symmetric connection $\nabla$ and let us denote by $g_{ij}$ and $\left\{ {i \atop jk} \right\}$ the metric connection and the Christoffel symbols formed with the help of $g_{ij}$. Such a Riemannian space will be denoted by $M_n(\nabla, g)$.

Let $\underset{r}{v}^i$ $(r = 1, 2, \ldots, n)$ be the contravariant components of the n independent vector fields $\underset{r}{\vec{v}}$ in $M_n$ which satisfy the condition $g_{ij}\underset{r}{v}^i\underset{r}{v}^j = 1$.

Following [1], we defined the covector fields $\overset{r}{v}$ satisfying the equalities

$$\underset{r}{v}^i \overset{r}{v}_j = \delta^i_j, \qquad \underset{r}{v}^i \overset{p}{v}_j = \delta^p_r \qquad i, j, r, p = 1, 2, \ldots, n. \tag{1}$$

A set of $n$ linearly independent vector fields $\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}$ defined on an $n$-dimensional Riemannian space is called a net and will be denoted by $\theta(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ [2].

The net $\theta$ in $M_n$ is said to be a Chebyshev net of the first kind if every vector field belonging to $\theta$ is translated parallely along the integral curves of the remaining fields of the net [2].

If for a fixed $r$, the condition $\dot{\nabla}_{[k}\overset{r}{v}_{j]} = 0$ holds, $\theta$ is said to be a metrically $r$-Chebyshev net where the bracket indicates the antisymmetrization. If this condition holds for all $r = 1, 2, \ldots, n$ the net is named to be strongly metrically Chebyshev net. If the net $\theta$ satisfies the condition $\sum_r \dot{\nabla}_{[i}\overset{r}{v}_{j]} = 0$, such a net is called a equidistant Chebyshev net [2].

## 2. On a Type of Non-Symmetric Metric Connection on a Riemannian Space

Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$ with metric tensor $g$. Let $\nabla$ be the Riemannian connection of $M$. This space is denoted by $M_n(\nabla, g)$.

In [3], the author refers to connection which is not symmetric with respect to its subscripts as non-symmetric connection. In this paper, we study this non-symmetric connection. Let $\nabla^*$ be any other linear connection defined on $M$ $(n > 3)$.

A non-symmetric connection $\nabla^*$ is called a non-symmetric metric connection, if it satisfies further the equation $\nabla^* g = 0$. In a coordinate neighborhood $\{U; (x^1, x^2, \ldots, x^n)\}$, where $g_{ij}$, $\left\{ {i \atop jk} \right\}$ and $\Gamma^i_{jk}$ denote the local representations of g, $\nabla$ and $\nabla^*$, respectively. By using the condition $\nabla^* g = 0$, we have

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} = 0 \tag{2}$$

where $\Gamma^i_{jk}$ denote the coefficients of the connection $\nabla^*$. Introducing the indices $i$, $j$, $k$ in Eq. (2) in a cyclic way we get the following equation, [4]

$$\Omega^m_{ij} = T^m_{ij} + g^{mk}(g_{hj}T^h_{ik} + g_{hi}T^h_{jk}) \tag{3}$$

where

$$\Omega^m_{(ij)} = g^{mk}(g_{hj}T^h_{ik} + g_{hi}T^h_{jk}) \qquad \Omega^m_{ij} \neq \Omega^m_{(ij)} \tag{4}$$

and $T_{ij}^m = \frac{\Gamma_{ij}^m - \Gamma_{ji}^m}{2}$, $\Omega_{ij}^m \neq \Omega_{[ij]}^m$. Thus, we have

$$\Gamma_{ij}^m = \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} + \Omega_{ij}^m \,. \tag{5}$$

This space will be denoted by $M_n(\nabla^*, g)$.

The properties of Riemannian space admitting a semi-symmetric metric connection were studied by many authors [5, 6]. As far as we know Riemannian spaces admitting a non-symmetric metric connection have not been studied intensively.

Let $M_n(\nabla^*, g)$ be a hypersurface with coordinates $x^i$ $(i = 1, 2, \ldots, n)$ in a Riemannian space $M_{n+1}(\nabla^*, g)$ with coordinates $y^a$ $(a = 1, 2, \ldots, n+1)$ and equipped with a non-symmetric metric connection. Suppose that the metrics of $M_n(\nabla^*, g)$ and $M_{n+1}(\nabla^*, g)$ are positive definite and that they are given, respectively, by $g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$ and $g_{ab}\, \mathrm{d}y^a\, \mathrm{d}y^b$ which are connected by the relation

$$g_{ij} = g_{ab} y_i^a y_j^b \qquad i, j = 1, 2, \ldots, n; \quad a, b = 1, 2, \ldots, n+1 \tag{6}$$

and where $y_i^a$ denotes the covariant derivative of $y^a$ with respect to $x^i$.

Let $n^a$ be the contravariant components of the vector field in $M_{n+1}(\nabla^*, g)$ normal to $M_n(\nabla^*, g)$ and let the relation $g_{ab} n^a n^b = 1$ be satisfied. The moving frame $\{y_a^i, n_a\}$ on $M_n(\nabla^*, g)$, reciprocal to the moving frame $\{y_i^a, n^a\}$ is defined by the relation, [1]

$$n^a n_a = 1 \qquad n_a y_i^a = 0 \qquad n^a y_a^i = 0 \qquad y_i^a y_a^j = \delta_i^j \,. \tag{7}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be, respectively, the contravariant components of the vector fields $\underset{r}{\vec{v}}$ in $M_n(\nabla^*, g)$ relative to $M_{n+1}(\nabla^*, g)$ and $M_n(\nabla^*, g)$. Denote the contravariant components relative to $M_{n+1}(\nabla^*, g)$ and $M_n(\nabla^*, g)$ of the covector fields $\overset{r}{v}$ by $\overset{r}{v}_a$ and $\overset{r}{v}_i$, respectively. We then have

$$\underset{r}{v}^a = y_i^a \underset{r}{v}^i \overset{r}{v}_a = y_a^i \overset{r}{v}_i \,. \tag{8}$$

The covariant derivatives of $A$ relative to $\nabla$ and $\nabla^*$ denoted, respectively, by $\nabla A$ and $\nabla^* A$ are defined by

$$\nabla_k A^i = \partial_k A^i + \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} A^h \tag{9}$$

and

$$\nabla_k^* A^i = \partial_k A^i + \Gamma_{hk}^i A^h \tag{10}$$

where $\nabla^*$ and $\partial$ are the operators of covariant and partial derivatives. It is easy to see that the tensorial derivatives of $A$ relative to $M_n(\nabla^*, g)$ and $M_{n+1}(\nabla^*, g)$ are related by

$$\dot{\nabla}_k^* A = y_k^c \dot{\nabla}_c^* A \tag{11}$$

Taking the tensorial derivative of $y_i^a$ with respect to $x$ using (5) and (10), we find

$$\dot{\nabla}_j^* y_i^a = \dot{\nabla}_j y_i^a - \Omega_{ij}^h y_h^a + \Omega_{bc}^a y_i^b y_j^c \tag{12}$$

where

$$\dot{\nabla}_j y_i^a = \frac{\partial^2 y^a}{\partial x^i \partial x^j} - \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} y_h^a + \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} y_i^b y_j^c$$

and $\dot{\nabla}^*$ is the operator of the tensorial derivative of $M_n(\nabla^*, g)$ and $\dot{\nabla}$ is the operator of tensorial derivative of $M_n(\nabla, g)$.

Differentiating covariantly of each side of the fourth relation in (6) with respect to $u^i$, we obtain [4]

$$\dot{\nabla}_j^* y_i^a = \Omega_{ij}^* n^a + B_{ij}^k y_k^a \tag{13}$$

and

$$\dot{\nabla}_j^* y_a^i = \Omega_j^{i*} n_a + E_{jk}^i, \qquad y_a^k E_{jk}^i = -B_{jk}^i \tag{14}$$

where $\Omega_j^{i*} = \Omega_{jm}^* g^{mi}$.

The unit normal $n^a$ is a contravariant vector in the $y$'s whose tensorial derivative with respect to the $x$'s is

$$\dot{\nabla}_k^* n^a = -\Omega_{km}^* g^{mj} y_j^a \,. \tag{15}$$

Concerning the Chebyshev nets of the first kind and geodesic nets belonging to $M_n^*(\nabla^*, g)$, we have the propositions [4]. If $\theta(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ is a Chebyshev net of the first kind with respect to $M_{n+1}(\nabla^*, g)$, then the condition

$$\begin{gathered} \underset{rs}{\bar{a}}^{a*} = \underset{s}{v}^d \dot{\nabla}_d^* \underset{r}{v}^a = 0\,, \qquad B_{ij}^m \underset{r}{v}^i \underset{s}{v}^j + \underset{rs}{a}^{m*} = 0 \\ r, s = 1, 2, \ldots, n\,, \quad r \neq s \end{gathered} \tag{16}$$

is satisfied.

If $\theta(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ is a geodesic net with respect to $M_{n+1}(\nabla^*, g)$, then the conditions holds [4]

$$\begin{gathered} \underset{r}{\bar{c}}^{a*} = \underset{r}{v}^d \dot{\nabla}_d^* \underset{r}{v}^a = 0, \qquad B_{ij}^m \underset{r}{v}^i \underset{r}{v}^j + \underset{r}{c}^{m*} = 0 \\ r = 1, 2, \ldots, n\,. \end{gathered} \tag{17}$$

If a curve in the hypersurface $M_n(\nabla^*, g)$ is a geodesic in the enveloping space $M_{n+1}$, then it satisfies in the following equation

$$\underset{rr}{\kappa}^* = 0 \tag{18}$$

where $\underset{rr}{\kappa}^*$ is the normal curvature of $M_n(\nabla^*, g)$.

**Definition 1.** *If the tensorial derivative of the unit normal vector field* $n^a$ *in the direction of* $\underset{r}{v}^d$ *is orthogonal to* $\underset{r}{v}^b$, *i. e.,*

$$(\underset{r}{v}^d \dot{\nabla}^*_d n^a) g_{ab} \underset{r}{v}^b = (\underset{r}{v}^k \dot{\nabla}^*_k n^a) g_{ab} \underset{r}{v}^b = 0 \tag{19}$$

*then a curve* $\underset{r}{C}$ *belonging to* $M_n(\nabla^*, g)$ *is called an asymptotic line of* $M_n(\nabla^*, g)$.

A direction in $M_n(\nabla^*, g)$ which is self-conjugate is said to be asymptotic. An asymptotic line in a hypersurface is a curve whose direction at every point is asymptotic. From (1), (15) and (19), it is evident that the normal curvature of a hypersurface in an asymptotic direction is zero, i. e.,

$$\underset{rr}{\kappa}^* = \Omega^*_{ij} \underset{r}{v}^i \underset{r}{v}^j = 0\,. \tag{20}$$

In this paper we study if the nets formed by $n$-families of asymptotic lines are Chebyshev nets of the first kind or geodesic nets or strongly-metrically Chebyshev nets relative to $M_{n+1}(\nabla^*, g)$, and the differential equation which they satisfy is obtained.

**Theorem 1.** *If the net* $\eta(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *in* $M_n(\nabla^*, g)$ *is a Chebyshev net of the first kind with respect to* $M_{n+1}(\nabla^*, g)$, *then the following conditions hold*

$$(\dot{\nabla}^*_m \Omega^*_{kh} + g_{ht} g^{jl} B^t_{jm} \Omega^*_{kl} - \Omega^*_{ih} B^i_{km}) \underset{r}{v}^k \underset{r}{v}^h \underset{s}{v}^m = 0$$
$$r, s = 1, 2, \ldots, n\,, \quad r \neq s\,.$$

**Proof:** If we take the tensorial derivative of (19) in the direction $\underset{s}{v}^m$ and by means of (16), we get

$$g_{ab} \underset{r}{v}^b \underset{r}{v}^k \underset{s}{v}^m \dot{\nabla}^*_m \dot{\nabla}^*_k n^a + \underset{rs}{a}^{k*} g_{ab} \underset{r}{v}^b \dot{\nabla}^*_k n^a = 0$$
$$r, s = 1, 2, \ldots, n, \quad r \neq s\,. \tag{21}$$

□

By virtue of (15) and (16), (21) can be expressed in the form

$$B^k_{ij}\Omega^*_{kh}\underset{r}{v}^h\underset{r}{v}^i\underset{s}{v}^j + g_{ab}\underset{r}{v}^b\underset{r}{v}^k\underset{s}{v}^m\dot{\nabla}^*_m\dot{\nabla}^*_k n^a = 0 \qquad r,s = 1,2,\dots,n \quad r \neq s\,. \tag{22}$$

On the other hand, using (7), (8), (13) and (15), we find that

$$g_{ab}\underset{r}{v}^b\underset{s}{v}^m\underset{r}{v}^k\dot{\nabla}^*_m\dot{\nabla}^*_k n^a = -g_{nt}g^{ij}\Omega^*_{ki}B^t_{jm}\underset{r}{v}^n\underset{r}{v}^k\underset{s}{v}^m - (\underset{s}{v}^m\dot{\nabla}^*_m\Omega^*_{kh})\underset{r}{v}^h\underset{r}{v}^k \qquad r,s = 1,2,\dots,n\,, \quad r \neq s\,. \tag{23}$$

From (22) and (23), we get

$$\Omega^*_{kh}B^k_{ij}\underset{r}{v}^i\underset{r}{v}^h\underset{s}{v}^j - \Omega^*_{kl}g^{lj}g_{nt}B^t_{jm}\underset{r}{v}^n\underset{r}{v}^k\underset{s}{v}^m - (\underset{s}{v}^m\dot{\nabla}^*_m\Omega^*_{kh})\underset{r}{v}^h\underset{r}{v}^k = 0 \qquad r,s = 1,2,\dots,n\,, \quad r \neq s \tag{24}$$

or

$$(\dot{\nabla}^*_m\Omega^*_{kh} + g_{ht}g^{jl}\Omega^*_{kl}B^t_{jm} - \Omega^*_{ih}B^i_{km})\underset{r}{v}^h\underset{r}{v}^k\underset{s}{v}^m = 0 \qquad r,s = 1,2,\dots,n\,, \quad r \neq s\,. \tag{25}$$

**Theorem 2.** *If the net $\eta(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ in $M_n(\nabla^*,g)$ is a geodesic net with respect to $M_{n+1}(\nabla^*,g)$, then the following relations hold*

$$(\dot{\nabla}^*_s\Omega^*_{km} - B^h_{ks}\Omega^*_{hm} + g_{nm}g^{hj}\Omega^*_{kh}B^n_{js})\underset{r}{v}^k\underset{r}{v}^m\underset{r}{v}^s = 0\,, \qquad r = 1,2,\dots,n\,.$$

**Proof:** Let $\underset{r}{\bar{c}}^{a*}$ and $\underset{r}{c}^{i*}$ be the components of the geodesic vector fields of the geodesic with respect to $M_{n+1}(\nabla^*,g)$ and $M_n(\nabla^*,g)$. □

If we take the tensorial derivative of (19) in the direction $\underset{r}{v}^s$ and using (17) we get

$$g_{ab}\underset{r}{v}^b\underset{r}{v}^k\underset{r}{v}^s\dot{\nabla}^*_s\dot{\nabla}^*_k n^a + \underset{r}{c}^{k*}g_{ab}\underset{r}{v}^b\dot{\nabla}^*_k n^a = 0 \qquad r,s = 1,2,\dots,n\,, \quad a,b = 1,2,\dots,n+1\,. \tag{26}$$

In virtue of (15), (17) and (26), we can write

$$B^k_{ij}\Omega^*_{km}\underset{r}{v}^i\underset{r}{v}^j\underset{r}{v}^m + g_{ab}\underset{r}{v}^b\underset{r}{v}^s\underset{r}{v}^k\dot{\nabla}^*_s\dot{\nabla}^*_k n^a = 0 \tag{27}$$

from which by using (7), (8), (13) and (15) ($r,s = 1,2,\dots,n$, $a,b = 1,2,\dots,n+1$), we have

$$g_{ab}\underset{r}{v}^b\underset{r}{v}^s\underset{r}{v}^k\dot{\nabla}^*_s\dot{\nabla}^*_k n^a = -(\underset{r}{v}^s\dot{\nabla}^*_s\Omega^*_{km})\underset{r}{v}^k\underset{r}{v}^m - g_{nt}B^n_{js}\Omega^*_{km}g^{mj}\underset{r}{v}^k\underset{r}{v}^s\underset{r}{v}^t\,. \tag{28}$$

Using the equations (27) and (28), we get

$$(\dot{\nabla}^*_s \Omega^*_{km} - B^h_{ks}\Omega^*_{hm} + g_{nm}g^{hj}\Omega^*_{kh}B^n_{js})\underset{r}{v}^k \underset{r}{v}^s \underset{r}{v}^m = 0\,. \tag{29}$$

**Theorem 3.** *Let any net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ *belonging to the hypersurface* $M_n(\nabla^*,g)$ *of space* $M_{n+1}(\nabla^*,g)$ *be a strongly-metrically Chebyshev net with respect to* $M_{n+1}(\nabla^*,g)$. *A necessary and sufficient condition that the net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ *be a strongly-metrically Chebyshev net with respect to* $W_n(\nabla^*,g)$ *is that the tensor* $B^m_{hj}$ *must be symmetric.*

**Proof:** From (7) and (13) we have

$$(\dot{\nabla}^*_j y^t_a) y^a_h = -B^t_{hj} \tag{30}$$

and

$$(\dot{\nabla}^*_h y^t_c) y^c_j = -B^t_{jh}\,. \tag{31}$$

On the other hand, using (8), (11), (30), (31) and the expression $\dot{\nabla}^*_c \overset{r}{v}_a - \dot{\nabla}^*_a \overset{r}{v}_c = 0$, we obtain

$$\dot{\nabla}^*_j \overset{r}{v}_h - \dot{\nabla}^*_h \overset{r}{v}_j = \overset{r}{v}_t(B^t_{hj} - B^t_{jh})$$

or

$$\dot{\nabla}^*_{[j} \overset{r}{v}_{h]} = \overset{r}{v}_t B^t_{[hj]} \tag{32}$$

where $\dot{\nabla}^*_j \overset{r}{v}_h - \dot{\nabla}^*_h \overset{r}{v}_j = 2\dot{\nabla}^*_{[j} \overset{r}{v}_{h]}$ and $B^t_{jh} - B^t_{hj} = 2B^t_{[jh]}$. Here $B^t_{[jh]}$ denotes the antisymmetric part of tensor $B^t_{jh}$.

We assume that $B^t_{jh}$ is symmetric. Using (2.31), we get $\dot{\nabla}^*_{[j} \overset{r}{v}_{h]} = 0$. And conversely, from (32), we get $\dot{\nabla}^*_{[j} \overset{r}{v}_{h]} = 0$ which is the required result. □

**Corollary 1.** *Let any net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ *belonging to the hypersurface* $M_n(\nabla^*,g)$ *of the space* $M_{n+1}(\nabla^*,g)$ *be a strongly-metrically Chebyshev net with respect to* $M_{n+1}(\nabla^*,g)$. *A necessary and sufficient condition that the net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$ *be a strongly-metrically Chebyshev net with respect to* $W_n(\nabla^*,g)$ *is that the tensor* $E^m_{hj}$ *should be symmetric.*

From Theorem 3 and the expression (14) the proof is clear.

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