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MATTER, FIELDS, AND REPARAMETRIZATION-INVARIANT SYSTEMS

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Abstract. We study reparametrization-invariant systems, mainly the relativistic particle and its D-dimensional extended object generalization d-brane. The corresponding matter Lagrangians naturally contain background interactions, like electromagnetism and gravity. For a d-brane that doesn't alter the background fields, we define non-relativistic equations assuming integral sub-manifold embedding of the d-brane. The mass-shell constraint and the Klein–Gordon equation are shown to be universal when gravity-like interaction is present. Our approach to the Dirac equation follows Rund's technique for the algebra of the γ-matrices that doesn't rely on the Klein–Gordon equation.

1. Introduction

There are two very useful methods in classical mechanics: the Hamiltonian and the Lagrangian approach [14, 10, 11, 16, 4]. The Hamiltonian formalism gives rise to the canonical quantization, while the Lagrangian approach is used in the path-integral quantization. Usually, in classical mechanics, there is a transformation that relates these two approaches. However, for a reparametrization-invariant systems there are problems when changing from the Lagrangian to the Hamiltonian approach [10, 11, 16, 20, 15]. Classical mechanics of a reparametrization-invariant system and its quantization is the topic of the current study.

Fiber bundles provide the mathematical framework for classical mechanics, field theory, and even quantum mechanics if viewed as a classical field theory. When studying the structures that are important to physics, we should also understand why one fiber bundle should be more "physical" than another, why the

"physical" base manifold seems to be a four-dimensional Lorentzian manifold [3, 26, 21], how one should construct an action integral for a given fiber bundle [14, 4, 8, 9, 19], and how a given system may be quantized. Starting with the tangent or cotangent bundle is natural because these bundles are related to the notion of a classical mechanics of a point-like matter. Since our knowledge comes from experiments that involve classical apparatus, the physically accessible fields should be generated by matter and should couple with matter as well. Therefore, understanding the matter Lagrangian for a classical system is very important.

In this paper we outline some aspects of the non-relativistic, relativistic, and a la Dirac-equation quantization of reparametrization-invariant classical mechanics systems. In its canonical form, the matter Lagrangian for reparametrization-invariant systems contains well known interaction terms, such as electromagnetism and gravity. For a reparametrization-invariant systems there are constraints among the equations of motion which is a problem. Nevertheless, there are procedures for quantizing such theories [16, 5, 25, 13, 24]. For example, changing coordinates $(x, v) \leftrightarrow (x, p)$ is one problem, when $h = pv - L \equiv 0$ is another. This $h \equiv 0$ problem is usually overcome either by using a gauge fixing to remove the reparametrization-invariance or by using some of the constraint equations available instead of h [16]. Here, we will demonstrate another approach $(v \to \gamma)$ which takes advantage of $h \equiv 0$.

In section two we briefly review the classical mechanics of reparametrization-invariant d-branes. In the third section we argue for a one-time-physics as an essential ingredient for a non-relativistic limit. The fourth section is concerned with the relativistic Klein–Gordon equation, relativistic mass-shell equation, and Dirac equation. Our conclusions and discussions are in section five.

2. Classical Mechanics of d-branes

In this section, we briefly review our study of the geometric structures in the classical mechanics of reparametrization-invariant systems [12] by focusing on the relativistic charged particle and its D-dimensional extended object generalization (d-brane). In reference [12] we have discussed the question: "What is the matter Lagrangian for a classical system?" Starting from the assumption that there should not be any preferred trajectory parameterization in a smooth space-time, we have arrived at the well known and very important reparametrization-invariant system: the charged relativistic particle. Imposing reparametrization invariance of the action $S = \int L(x, v)\, \mathrm{d}\tau$ naturally leads to a first order homogeneous functions [12].

The Lagrangian for the charged relativistic particle corresponds to the first two terms in a series expression of a first order homogeneous function. When this

expression is considered as a Lagrangian, we call it **canonical form** of the first order homogeneous Lagrangian:

$$L\left(\vec{x},\vec{v}\right)=\sum_{n=1}^{\infty}Q_n\sqrt[n]{S_n\left(\vec{v},\ldots,\vec{v}\right)}=qA_\alpha v^\alpha+m\sqrt{g_{\alpha\beta}v^\alpha v^\beta}+\ldots. \quad (1)$$

The choice of the canonical form is based on the assumption of one-to-one correspondence between interaction fields S_n and their sources [12]. Finding a procedure, similar to the Taylor series expansion, to extract the components of each symmetric tensor ($S_{\alpha_1\alpha_2\ldots\alpha_n}$), for a given homogeneous function of first order, would be a significant step in our understanding of the fundamental interactions.

Encouraged by our results, we have continued our study of reparametrization-invariant systems by generalizing the idea to a D-dimensional extended objects (d-branes). In doing so we have arrived at the string theory Lagrangian (1-brane extended object) [16] and the Dirac–Nambu–Goto Lagrangian for a d-brane [18].

The classical mechanics of a point-like particle is concerned with the embedding $\phi\colon \mathbb{R}^1\to M$. The map ϕ provides the trajectory (the world line) of the particle in the target space M. In this sense, we are dealing with a 0-brane that is a one dimensional object. If we think of an extended object as a manifold D with dimension denoted also by D ($\dim D = D = d+1$ where $d=0,1,2,\ldots$), then we should seek $\phi\colon D\to M$ such that some action integral is minimized. From this point of view, we are dealing with embedding of a D-dimensional object in to a target space M. If x^α denote coordinate functions on M and z^i coordinate functions on D, then we can introduce a **generalized velocity vector** $\vec{\omega}$ with components ω^Γ:

$$\omega^\Gamma=\frac{\Omega^\Gamma}{\mathrm{d}z}=\frac{\partial\left(x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_D}\right)}{\partial(z^1z^2\ldots z^D)},$$

$$\mathrm{d}z=\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^D,\quad \Gamma=1,\ldots,\binom{\dim M}{\dim D}.$$

In the above expression, $\dfrac{\partial\left(x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_D}\right)}{\partial(z^1z^2\ldots z^D)}$ represents the Jacobian of the transformation from coordinates $\{x^\alpha\}$ over the manifold M to coordinates $\{z^a\}$ over the d-brane [7]. In this notation the canonical expression for the homogeneous Lagrangian of first order is:

$$L\left(\vec{\phi},\vec{\omega}\right)=\sum_{n=1}^{\infty}\sqrt[n]{S_n\left(\vec{\omega},\ldots,\vec{\omega}\right)}=A_\Gamma\omega^\Gamma+\sqrt{g_{\Gamma_1\Gamma_2}\omega^{\Gamma_1}\omega^{\Gamma_2}}+\ldots. \quad (2)$$

Notice that $\vec{x}, \vec{v}$, and $\vec{\phi} = \vec{x} \circ \phi$ have the same number of components while the generalized velocity $\vec{\omega}$ has $\binom{\dim M}{\dim D}$.

From the above expressions (1) and (2), one can see that the corresponding matter Lagrangians L, in their canonical form, contain electromagnetic (A) and gravitational g interactions, as well as interactions that are not clearly identified yet (S_n, $n > 2$). At this stage, we have a theory with background fields since we don't know the equations for the interaction fields A, g, and S_n. To complete the theory, we need to introduce actions for these interaction fields. If one is going to study the new interaction fields S_n, $n > 2$, then some guiding principles for writing field Lagrangians are needed.

One such principle uses the external derivative d, multiplication $\wedge$, and Hodge dual $*$ operations in the external algebra $\Lambda(T^*M)$ over M to construct objects proportional to the volume form over M. For example, for any n-form (A) the expressions $A \wedge *A$ and $\mathrm{d}A \wedge * \mathrm{d}A$ are forms proportional to the volume form.

The next important principle comes from the symmetry in the matter equation. That is, if there is a transformation $A \to A'$ that leaves the matter equations unchanged, then there is no way to distinguish A and A'. Thus the action for the field A should obey the same gauge symmetry. For the electromagnetic field $(A \to A' = A + \mathrm{d}f)$ this leads to the field Lagrangian $\mathcal{L} = \mathrm{d}A \wedge * \mathrm{d}A = F \wedge *F$, when for gravity it leads to the Cartan–Einstein action $S[R] = \int R_{\alpha\beta} \wedge *(\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta)$ [12, 1]. In our study we have also found an extra term $R^\wedge$ that exists only in four dimensional theories. This term $R^\wedge$ comes from fully anti-symmetrized $R_{\alpha[\beta,\gamma]\rho)}$ Ricci tensor R.

3. Non-relativistic Limit

Here, we briefly argue that a one-time-physics is needed to assure causality via finite propagational speed in case of point particles. For d-branes the one-time-physics reflects separation of the internal from the external coordinates when the d-brane is considered as a sub-manifold of the target space manifold M. The non-relativistic limit is considered to be the case when the d-brane is embedded as a sub-manifold of M.

3.1. Causality and Space-Time Metric Signature

It is well known that the Einstein general relativity occurs more degree of freedom in four and higher dimensions. We have already mentioned the $R^\wedge$ term which is only possible in a four-dimensional space-time. Another argument for 4D space-time is based on geometric and differential structure of various brane and target spaces [12, 21]. All these are reasons why the spacetime seems

to be four dimensional. Why the space-time seems to be $1+3$ have been recently discussed by using arguments *a la* Wigner [3, 26]. However, these arguments are deducing that the space-time is $1+3$ because only this signature is consistent with particles with finite spin. In our opinion one should turn this argument backwards claiming that one should observe only particles with finite spin because the signature is $1+3$.

Here we present an argument that only one-time-physics is consistent with a finite propagational speed. Our main assumptions are: a gravity-like term $\sqrt{g(\vec{\omega},\vec{\omega})}$ is always present in the matter Lagrangian, and the matter Lagrangian is a real-valued; thus $g(\vec{v},\vec{v}) \geq 0$. For simplicity, we consider the 0-brane mechanics first.

The use of a covariant formulation allows one to select a local coordinate system so that the metric is diagonal $(+,+,\ldots,+,-,\cdots-)$. If we denote the plus coordinates as time coordinates and the minus ones as space coordinates, then there are three essential cases:

1. No time coordinates. Thus $g(\vec{v},\vec{v}) = -\sum_{\alpha}(v^{\alpha})^2 < 0$, which contradicts $(g(\vec{v},\vec{v}) \geq 0)$.
2. Two or more time coordinates. Thus $g(\vec{v},\vec{v}) = (v^0)^2 + (v^1)^2 - \sum_{\alpha=2}^{n}(v^{\alpha})^2 \Rightarrow 1+\dot{\lambda} \geq \vec{v}^2_{\text{space}}$.
3. Only one time coordinate. Thus $g(\vec{v},\vec{v}) = (v^0)^2 - \sum_{\alpha=1}^{n}(v^{\alpha})^2 \Rightarrow 1 \geq \vec{v}^2_{\text{space}}$.

Clearly for two or more time coordinates we do not have finite coordinate velocity $\mathrm{d}x/\mathrm{d}t)$ when the coordinate time t is chosen so that $t = x^0 \Rightarrow v^0 = 1$ and $x^1 = \lambda$. Only the space-time with one time accounts for a finite velocity and thus a causal structure.

For a d-brane one has to assume a local coordinate frame where one component of the generalized velocity can be set to 1 ($\omega^0 = 1$). This generalized velocity component is associated with the brane "time coordinate." In fact, $\omega^0 = 1$ means that there is an integral embedding of the d-brane in the target space M, and the image of the d -brane is a sub-manifold of M. If the coordinates of M are labeled so that $x^i = z^i$, $i = 1,\ldots,D$, then x^i are internal coordinates that may be collapsed in only one coordinate — the "world line" of the d-brane.

3.2. The Quantum Mechanics of a D-brane

In this section we briefly describe the gauge-fixing approach that allows canonical quantization. This approach is mainly concerned with a choice of a coordinate time that is used as the trajectory parameter [16, 6, 13, 23]. Such choice removes the reparametrization invariance of the theory.

In a local coordinate system where $\omega^0 = 1$ and the metric is a "one-time-metric" we have:

$$L = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1\Gamma_2}\omega^{\Gamma_1}\omega^{\Gamma_2}} + \cdots + \sqrt[m]{S_m\left(\vec{\omega},\ldots,\vec{\omega}\right)}$$
$$\to A_0 + A_i\omega^i + \sqrt{1 - g_{ii}\omega^i\omega^i} + \cdots \approx A_0 + A_i\omega^i + 1 - \frac{1}{2}g_{ii}\omega^i\omega^i + \ldots .$$

Thus the Hamiltonian function is not zero anymore, so we can do canonical quantization, and the Hilbert space consists of the functions $\Psi(x) \to \Psi(z,\tilde{x})$ where $\tilde{x} = x^i$, $i = D+1,\ldots,m$. The brane coordinates z shall be treated as t in quantum mechanics in the sense that the scalar product should be an integral over the space coordinates $\tilde{x}$.

4. Relativistic Equations for Matter

Even though canonical quantization can be applied after a gauge fixing, one is not usually happy with this situation because the covariance of the theory is lost and time is a privileged coordinate. In general, there are well developed procedures for covariant quantization [16, 5, 25, 13, 24]. However, we are not going to discuss these methods. Instead, we will employ a different quantization strategy. In this section we discuss the mass-shell constraint, the Klein–Gordon equation and the Dirac equation [22] for d-branes.

4.1. The Mass-shell and Klein–Gordon Equation

Since the functional form of the canonical Lagrangian is the same for any d-brane, we use v, but it could be ω as well. We define the momentum p and generalized momentum π for our canonical Lagrangian as follow:

$$p_\Gamma = \frac{\delta L\left(\phi,\omega\right)}{\delta\omega^\Gamma} = eA_\Gamma + m\frac{g_{\Gamma\Sigma}\omega^\Sigma}{\sqrt{g\left(\vec{\omega},\vec{\omega}\right)}} + \cdots + \frac{S_{\Gamma\Sigma_1\ldots\Sigma_n}\omega^{\Sigma_1}\ldots\omega^{\Sigma_n}}{\left(S\left(\omega,\ldots,\omega\right)\right)^{1-1/n}} + \ldots,$$
$$\pi_\alpha = p_\alpha - eA_\alpha - \ldots\frac{S_{\alpha\beta_1\ldots\beta_n}v^{\beta_1}\ldots v^{\beta_n}}{\left(S\left(v,\ldots,v\right)\right)^{n/(n+1)}}\cdots = m\frac{g_{\alpha\beta}v^\beta}{\sqrt{g\left(\vec{v},\vec{v}\right)}} .$$

In the second equation we have used v instead of ω for simplicity. Notice that this generalized momentum π is consistent with the usual quantum mechanical procedure $p \to p - eA$ that is used in Yang–Mills theories, as well as with the usual GR expression $p_\alpha = mg_{\alpha\beta}v^\beta$. Now it is easy to recognize the mass-shell

constraint as a mathematical identity:

$$\frac{\vec{v}}{\sqrt{\vec{v}^2}} \cdot \frac{\vec{v}}{\sqrt{\vec{v}^2}} = 1 \Rightarrow \pi_\alpha \pi^\alpha = m^2$$

$$\Rightarrow \left(\vec{p} - e\vec{A} - \vec{S}_3(v) - \vec{S}_4(v) - \ldots\right)^2 \Psi = m^2 \Psi .$$

Notice that "gravity" as represented by the metric is gone, while the Klein–Gordon equation appears. The v dependence in the S terms reminds us about the problem related to the change of coordinates $(x, v) \rightarrow (x, p)$. So, at this stage we may proceed with the Klein–Gordon equation, if we wish.

4.2. Rund's Approach to the γ-matrices

An interesting approach to the Dirac equation has been suggested by H. Rund [20]. The idea uses the Heisenberg picture $-\mathrm{i}\hbar \dfrac{\mathrm{d}Z}{\mathrm{d}\tau} = [H, Z]$, a Hamiltonian linear in the momentum $H = \gamma^\alpha p_\alpha$, and a principle group G with generators X_i that close a Lie algebra $[X_i, X_j] = C_{ij}^k X_k$. To have the Hamiltonian H invariant under G-transformations, the γ objects should transform appropriately $[X_i, \gamma^\alpha] = (\rho(X_i))_\beta^\alpha \gamma^\beta$. The next ansatz is the important one: $X_i = (x_i)_{\alpha\beta} \gamma^\alpha \gamma^\beta$.
By using this ansatz in $[X_i, X_j] = C_{ij}^k X_k$, one writes $[X_i, (x_j)_{\alpha\beta} \gamma^\alpha \gamma^\beta] = C_{ij}^k (x_k)_{\alpha\beta} \gamma^\alpha \gamma^\beta$ and solves for $(x_j)_{\alpha\beta}$. In order for the linear system of equations to have a solution, additional algebraic conditions on the γ matrices are imposed. For the Lorentz group, this procedure gives $\Lambda^{\alpha\beta} = \frac{1}{2} \gamma^\alpha \gamma^\beta$ with $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$. Notice that in $H = \gamma^\alpha p_\alpha$ the momentum transforms according to the fundamental representation, and thus the γ-vector should transform as the conjugate one, so that H stays a scalar. The ansatz means that we are constructing the adjoint representation, which is the Lie algebra itself, by coupling two fundamental representations.

4.3. Dirac Equation from $H = 0$

Since we want γ and p to transform as vectors, it is clear that p should be a covariant derivative, but what is its structure? Consider a homogeneous Lagrangian that can be written as $L(\phi, \omega) = \omega^\Gamma p_\Gamma = \omega^\Gamma \partial L(\phi, \omega) / \partial \omega^\Gamma$ with a Hamiltonian function that is identically zero: $h = \omega^\Gamma \partial L(\phi, \omega) / \partial \omega^\Gamma - L(\phi, \omega) \equiv 0$. Notice that ω^Γ is the determinant of a matrix (the Jacobian of a transformation [7]); thus $\omega^\Gamma \rightarrow \gamma^\Gamma$ seems an interesting option for quantization. Even more, for the Dirac theory we know that γ^α are the "velocities" $(\mathrm{d}x/\mathrm{d}\tau = \partial H / \partial p)$.

If we quantize using ($h \to H$), then for the space of functions we should have: $H\Psi = 0$. By applying $\omega^\Gamma \to \gamma^\Gamma$, which means that the (generalized) velocity is considered as a vector with non-commutative components, we have $(\gamma^\Gamma p_\Gamma - L(\phi, \gamma))\,\Psi = 0$. For a 0-brane, using the canonical form of the Lagrangian (1) and the algebra of the γ matrices following Rund's approach, as described in the previous section, we have:

$$\begin{aligned} H &= \gamma^\alpha p_\alpha - L\,(\phi, \gamma) \\ &= \gamma^\alpha p_\alpha - eA_\alpha\gamma^\alpha - m\sqrt{g_{\alpha\beta}\gamma^\alpha\gamma^\beta} - \cdots - \sqrt[m]{S_m\,(\vec{\gamma}, \ldots, \vec{\gamma})}, \\ &\to \gamma^\alpha p_\alpha - eA_\alpha\gamma^\alpha - m - \cdots - \sqrt[2m]{S_{2m}g^m} - \cdots - \sqrt[2n+1]{S_{2n+1}g^{n/2}\gamma} \ldots . \end{aligned}$$

Since $g_{\alpha\beta}$ is a symmetric tensor, then $g_{\alpha\beta}\gamma^\alpha\gamma^\beta \sim g_{\alpha\beta}\{\gamma^\alpha, \gamma^\beta\} \sim g_{\alpha\beta}g^{\alpha\beta} \sim 1$. Therefore, gravity seems to leave the picture again. The symmetric structure of the extra terms S_m can be used to reintroduce g by using $\{\gamma^\alpha, \gamma^\beta\} \sim g^{\alpha\beta}$ and to reduce the powers of γ. Thus the high even terms contribute to the mass m, making it variable with $\vec{x}$ [2].

5. Conclusions and Discussions

In summary, we have discussed the structure of the matter Lagrangian L for extended objects. Imposing reparametrization invariance of the action S naturally leads to a first order homogeneous Lagrangian. In its canonical form, L contains electromagnetic and gravitational interactions, as well as interactions that are not clearly identified yet.

The non-relativistic limit for a d-brane has been defined as those coordinates where the brane is an integral sub-manifold of the target space. This gauge can be used to remove reparametrization invariance of the action S and make the Hamiltonian function suitable for canonical quantization. For the 0-brane (the relativistic particle), this also has a clear physical interpretation associated with localization and finite propagational speed.

The existence of a mass-shell constraint is universal. It is essentially due to the gravitational (quadratic in velocities) type interaction in the Lagrangian and leads to a Klein–Gordon equation. Although the Klein–Gordon equation can be defined, it is not the only way to introduce the algebra of the γ-matrices needed for the Dirac equation. The algebraic properties of the γ-matrices may be derived using the Lie group structure of the coordinate bundle; these properties are closely related to the corresponding metric tensor $g^{\alpha\beta} = \{\gamma^\alpha, \gamma^\beta\}$ and may restrict the number of terms in the Lagrangian. Once the algebraic properties of the γ-matrices are defined, one can use $v \to \gamma$ quantization in the Hamiltonian function $h = pv - L(x, v)$ to obtain the Dirac equation.

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