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# CONFORMAL MAPPINGS AND SPECIAL NETWORKS OF WEYL SPACES

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**Abstract**. In this paper, we show that a totally umbilical hypersurface of a recurrent Weyl space is conformally recurrent. Also, while a totally umbilical hypersurface of a recurrent Weyl space is conharmonically recurrent or conharmonically Ricci-recurrent, theorems concerning some special nets are proved.

## 1. Introduction

A differentiable manifold of dimension n having conformal metric tensor $g$ and symmetric connection $\nabla$ satisfying the compatibility condition

$$\nabla g = 2(TXg)$$

where $T$ is a 1-form (complementary covector field) is called a **Weyl space** which is denoted by $W_n(g, T)$. After renormalization of the metric tensor $g$

$$\breve{g} = \lambda^2 g$$

the vector field $T$ is transformed [1] into

$$\breve{T} = T + \mathrm{d} \ln \lambda$$

An object $A$ defined on $W_n(g, T)$ is called a satellite of $g$ of **weight** $\{p\}$ if it admits a transformation of the form $\breve{A} = \lambda^p A$ under the renormalization of $g$. Suppose that the metrics of $W_n$ and $W_{n+1}$ are elliptic and that they are given, respectively, by $g_{ij}\, \mathrm{d}u^i\, \mathrm{d}u^j$ and $g_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b$ which are connected by the relations

$$g_{ij} = g_{ab} x^a_i x^b_j \qquad i, j = 1, 2, \ldots, n\,, \quad a, b = 1, 2, \ldots, n+1$$

where $x_i^a$ denotes the covariant derivative of $x^a$ with respect to $u^i$. The prolonged derivative and the prolonged covariant derivative in the direction of vector $x$ of the satellite $A$ of $g$ of weight $\{p\}$ are defined by the laws, respectively,

$$\dot{\partial} A = \partial A - p(TXA), \qquad \dot{\nabla} A = \nabla A - p(TXA) \tag{1}$$

where $\partial_k$ is the partial derivative of $A$ [2–4]. By $\bar{g} = \lambda^2 g$ and second equality in (1) it follows that for every $z$, $\dot{\nabla}_z g = 0$. It is easy to see that prolonged covariant derivative preserve weights of the satellites.

The prolonged covariant derivative of $A$, relative to $W_n$ and $W_{n+1}$, are related by

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A\,. \tag{2}$$

Let $n^a$ be the contravariant components of the vector field in $W_{n+1}$ normal to $W_n$, and let it be normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\{x_a^i, n_a\}$ in $W_n$, reciprocal to the moving frame $\{x_i^a, n^a\}$ is defined by the relations [4]

$$n^a n_a = 1, \qquad n_a x_i^a = 0, \qquad n^a x_a^i = 0, \qquad x_i^a x_a^j = \delta_i^j\,. \tag{3}$$

Differentiating covariantly with respect to $u^k$ both sides of the last equality (3) and remembering that

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a \tag{4}$$

we find that $\nabla_k x_a^j$, regarded as a function of $x$'s, is a vector of $W_n$, and so it can be expressed in the form [5]

$$\dot{\nabla}_k x_a^j = \nabla_k x_a^j = \Omega_k^j n_a\,. \tag{5}$$

Let $\underset{r}{v}^i (r = 1, 2, \ldots, n)$ be the contravariant components of the $n$ independent vector fields $\underset{r}{\mathbf{v}}$ in $W_n$ which are normalized by the condition $g_{ij}\underset{r}{v}^i \underset{r}{v}^j = 1$. Following [1], we define the **covector fields** $\overset{r}{\mathbf{v}}$ satisfying the equalities

$$\underset{r}{v}^i \overset{r}{v}_j = \delta_j^i, \qquad \underset{r}{v}^i \overset{p}{v}_i = \delta_r^p \qquad r, p = 1, 2, \ldots, n\,. \tag{6}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be, respectively, the contravariant components of the vector fields $\underset{\mathbf{r}}{\mathbf{v}}$ in $W_n$ relative to $W_{n+1}$ and $W_n$. Then, we have

$$\underset{r}{v}^a = x_i^a \underset{r}{v}^i\,. \tag{7}$$

The generalised Gauss equation is obtained, in the following form [6]

$$R_{hijk} = w_{hj}w_{ik} - w_{hk}w_{ij} + \bar{R}_{bcde}x^b_h x^c_i x^d_j x^e_k \tag{8}$$

where $\bar{R}_{bcde}$ is the covariant curvature tensor of $W_{n+1}$.
A hypersurface of a Weyl space is called **totally umbilical** if the following expression holds

$$w_{ij} = \mu g_{ij} \tag{9}$$

where $\mu$ is a satellite of $g_{ij}$ with weight $\{-1\}$. From this definition, it follows that $\mu = \dfrac{M}{n}$ where $M$ is the mean curvature of the hypersurface, defined by $M = w_{ij}g^{ij}$. A hypersurface of a Weyl space is totally geodesic if

$$w_{ij} = 0\,. \tag{10}$$

We will use the following relations [7]

$$B^{ab\ldots cd}_{hi\ldots jk} = x^a_h x^b_i \ldots x^c_j x^d_k\,. \tag{11}$$

If $\underset{rp}{\bar{a}}{}^a$ and $\underset{rp}{a}{}^i$, respectively, the components of the Chebyshev vector fields of the first kind with respect to $W_{n+1}$ and $W_n$, then the following relations hold (see [5] and [8])

$$\begin{aligned}
\underset{rp}{\bar{a}}{}^a &= \underset{rp}{\kappa} n^a + \underset{rp}{a}{}^i x^a_i\,, \qquad r \neq p\\
\underset{rp}{a}{}^i &= \underset{p}{v}{}^k \dot{\nabla}_k \underset{r}{v}{}^i\,, \qquad r \neq p\\
\underset{rp}{\kappa} &= w_{ik}\underset{r}{v}{}^i \underset{p}{v}{}^k\,.
\end{aligned}$$

Let any net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ in $W_n$ be a Chebyshev net of the first kind with respect to $W_{n+1}$, in this case, the following condition holds [9]

$$\underset{rp}{\bar{a}}{}^a = 0\,. \tag{12}$$

If $\overset{r}{\bar{b}}_a$ and $\overset{r}{b}_i$ are, respectively, the components of the Chebyshev vector fields of the second kind with respect to $W_{n+1}$ and $W_n$, then the following relations hold [5, 8]

$$\overset{r}{\bar{b}}_a = (-\Omega^i_k \overset{r}{v}_i \underset{r}{v}{}^k)n_a + \overset{r}{b}_i x^i_a \overset{r}{b}_i = \underset{r\,k}{v}\, \dot{\nabla}_k \overset{r}{v}_i \Omega^i_k = w_{km}g^{mi} \tag{13}$$

(no summation over $r$).

Let any net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ in $W_n$ be a Chebyshev net of the second kind with respect to $W_{n+1}$, in this case, the following condition holds [9]

$$\overset{r}{\bar{b}}_a = 0 \,. \tag{14}$$

If $\underset{r}{\bar{c}}^a$ and $\underset{r}{c}^i$ are, respectively, the components of the geodesic vector fields of the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ with respect to $W_{n+1}$ and $W_n$, then they are connected by the relations [5, 8]

$$\underset{r}{\bar{c}}^a = \underset{rr}{\kappa} n^a + \underset{r}{c}^i x_i^a \underset{r}{c}^i = \underset{r}{v}^k \dot{\nabla}_k \underset{r}{v}^i \underset{rr}{\kappa} = w_{ik} \underset{r}{v}^i \underset{r}{v}^k \,. \tag{15}$$

Let any net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ in $W_n$ be a geodesic net with respect to $W_{n+1}$, in this case the following condition holds [9]

$$\underset{r}{\bar{c}}^a = 0 \,.$$

If $W_n$ admits of a tensor field $T_{\ldots}$ such that

$$\dot{\nabla}_k T_{\ldots} = \lambda_k T_{\ldots} \tag{16}$$

where $\lambda_k$ is non-zero vector field of $W_n$, then $W_n$ is called a **$T$-recurrent Weyl space**.

We note that since the prolonged covariant derivative preserves the weight, $\lambda_s$ is a satellite of $g_{ij}$ with weight $\{0\}$.

Let $W_n$ be a hypersurface of recurrent Weyl space $W_{n+1}$ with recurrence vector $\lambda_a$ which is not orthogonal to the hypersurface $W_n$. If we denote the tangential component of $\phi_a$ by $\phi_r$, then we have

$$\phi_k = \phi_a x_k^a \,.$$

Since $W_{n+1}$ is recurrent Weyl space, we can write

$$\dot{\nabla}_r \bar{R}_{abcd} = \phi_r \bar{R}_{abcd} \,. \tag{17}$$

According to [6], we have

$$\begin{aligned}\dot{\nabla}_r R_{hijk} &= \dot{\nabla}_r \Omega_{hijk} + \phi_e \bar{R}_{abcd} B^{abcde}_{hijkr} + \bar{R}_{abcd} B^{bcd}_{ijk} w_{hr} n^a \\ &\quad + \bar{R}_{abcd} B^{acd}_{hjk} w_{ir} n^b + \bar{R}_{abcd} B^{abd}_{hik} w_{jr} n^c + \bar{R}_{abcd} B^{abc}_{hij} w_{kr} n^d \,.\end{aligned}$$

## 2. Conformal Mappings and Special Nets of Weyl Spaces

Let $\tau$ be a conformal mapping of $W_n(g, T)$ onto $W_n^*(g^*, T^*)$. In this case, we have

$$g^* = g\,. \tag{18}$$

The covariant vector $P_k$ is defined by

$$P = T - T^* \tag{19}$$

is called the vector of the conformal mapping. Clearly, $P$ has zero weight.

Let $C$ be a smooth curve in $W_n(g, T)$ and let $C^*$ be its image under the conformal mapping $\tau$. Denote the parameters of $C$ and $C^*$ by $S$ and $S^*$, respectively. Denote the coordinates of a current point $P$ on $C$ by $x^i$ and those of the corresponding point $P^*$ by $x^*$. Then for the tangent vectors $\underset{r}{v}^i$ and $\underset{r}{v}^{*i}$ at corresponding points, we have

$$\underset{r}{v}^{*i} = \underset{r}{v}^i\,.$$

Let $\nabla$ and $\nabla^*$ be the Weyl connections of $W_n(g, T)$ and $W_n^*(g^*, T^*)$ and let the connection coefficients be denoted by $\Gamma^i_{jk}$ and $\Gamma^{*i}_{jk}$, respectively, then the tensor $T^i_{jk}$ is called the **affine deformation tensor**, where

$$T^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}\,. \tag{20}$$

Another expression for affine deformation tensor can be written in [10] as follows

$$T^i_{jk} = P_j\delta^i_k + P_k\delta^i_j - P_m g^{im} g_{jk}\,. \tag{21}$$

In this case, from the conformal transformation which is given by (1), (2), (3) and (4), the covariant curvature tensor $R_{hijk}$ transforms $R^*_{hijk}$ as in the following expression, [11]

$$R^*_{hijk} = R_{hijk} + g_{hk}P_{ij} + g_{ij}P_{hk} - g_{ik}P_{hj} - g_{hj}P_{ik} + 2g_{ih}\nabla_{[k}P_{j]} \tag{22}$$

where we have put

$$P_{ij} = \nabla_i P_j - P_i P_j + \frac{1}{2} g^{kl} g_{ij} P_k P_l$$

and

$$R^* = R + 2(n-1)P^m_m\,. \tag{23}$$

From this transformation, using (5) and (6), we can easily obtain that the conformal curvature tensor of $W_n$ Weyl space is in the following form, [12]

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} - \frac{1}{n-2}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + g_{ij} g^{hm} R_{mk} - g_{ik} g^{hm} R_{mj}) \\ &+ \frac{2}{n(n-2)}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]} + g_{ij} g^{hm} R_{[mk]} - g_{ik} g^{hm} R_{[mj]} \\ &- (n-2)\delta^h_i R_{[kj]}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) . \end{aligned} \tag{24}$$

Let us suppose that the conformal transformation (1) be a **conharmonic** one, we obtain from the above expression, [11]

$$P^h_h = g^{hk} \nabla_h P_k + \frac{1}{2}(n-2) P^h P_h = 0 . \tag{25}$$

In this case, the conharmonic curvature tensor of Weyl space is in the following form, [13]

$$\begin{aligned} K^h_{ijk} &= R^h_{ijk} - \frac{1}{n}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]} + g_{ij} g^{hm} R_{[mk]} - g_{ik} g^{hm} R_{[mj]} + 2\delta^h_i R_{[kj]}) \\ &- \frac{1}{n-2}(\delta^h_k R_{(ij)} - \delta^h_j R_{(ik)} + g_{ij} g^{hm} R_{(mk)} - g_{ik} g^{hm} R_{(mj)}) \end{aligned} \tag{26}$$

where $R_{[ij]} = \frac{1}{2}(R_{ij} - R_{ji})$ and $R_{(ij)} = \frac{1}{2}(R_{ij} + R_{ji})$. From (9), the conharmonic Ricci tensor of a Weyl space can be easily obtained in the form

$$K_{ij} = \frac{R}{2-n} g_{ij} \qquad n \neq 2 .$$

Now, we prove the following theorems about the conformally recurrent and conharmonically Ricci-recurrent Weyl spaces.

**Theorem 1.** *If $W_n$ is a totally umbilical hypersurface of a recurrent Weyl space $W_{n+1}$ then $W_n$ is also conformally recurrent.*

**Proof:** If we consider that $W_n$ is a totally umbilical hypersurface of a recurrent Weyl space $W_{n+1}$ then we have, [6]

$$\begin{aligned} \dot{\nabla}_r R_{hijk} &= \phi_r R_{hijk} + \frac{M}{n^2}[(\dot{\nabla}_j M) G_{hirk} + (\dot{\nabla}_k M) G_{hijr} + (\dot{\nabla}_i M) G_{kjrh} \\ &+ (\dot{\nabla}_h M) G_{kjir}] + \frac{2M}{n^2}(\dot{\nabla}_r M) G_{hijk} - \frac{M^2}{n^2}\phi_r G_{hijk} \end{aligned} \tag{27}$$

where $G_{hijk} = g_{hj} g_{ik} - g_{hk} g_{ij}$.

If we consider the form $\dot{\nabla}_r C_{hijk} - \phi_r C_{hijk}$, taking the prolonged covariant derivative of the conformal curvature tensor, then we obtain from (7)

$$\dot{\nabla}_r C_{hijk} = \phi_r C_{hijk} + (\dot{\nabla}_r R_{hijk} - \phi_r R_{hijk}) - \frac{M}{n^2}((\dot{\nabla}_j M)G_{hirk} + (\dot{\nabla}_k M)G_{hijr} + (\dot{\nabla}_i M)G_{kjrh} + (\dot{\nabla}_h M)G_{kjir} + (2(\dot{\nabla}_r M) - M\phi_r)G_{hijk}). \quad (28)$$

From (10) and (11), we can obtain

$$\dot{\nabla}_r C_{hijk} = \phi_r C_{hijk}$$

which is the required result. □

**Theorem 2.** *Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent $(n > 2)$. If any net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ in $W_n$ is a Chebyshev net of the first kind with respect to $W_{n+1}$, it is also a Chebyshev net of the first kind with respect to $W_n$ and the converse is also true.*

**Proof:** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci recurrent $(n > 2)$. According to [14], we say that $W_n$ is also recurrent.

If a totally umbilical hypersurface of a recurrent Weyl space is recurrent then we have, [15]

$$M = 0, \qquad \lambda_r \neq 0, n > 2. \quad (29)$$

With the help of (9), (12) and (12), we get

$$\underset{rp}{\bar{a}}{}^a = \underset{rp}{a}{}^i x_i^a, \qquad r \neq p. \quad (30)$$

From (12), (13) and (30) the proof is clear. □

**Theorem 3.** *Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent $(n > 2)$. If any net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ in $W_n$ is a Chebyshev net of the second kind with respect to $W_{n+1}$, it is also a Chebyshev net of the second kind with respect to $W_n$, and the converse is also true.*

**Proof:** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent $(n > 2)$. Then, $M = 0$. From (9) and (14), we get

$$\overset{r}{\bar{b}}_a = \overset{r}{b}_i x_a^i. \quad (31)$$

Using (14), (15) and (31) the proof is completed. □

**Theorem 4.** *Let a totally umbilical hypersurface* $W_n$ *of a recurrent Weyl space* $W_{n+1}$ *be conharmonically Ricci-recurrent* $(n > 2)$*. If any net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *in* $W_n$ *is a geodesic net with respect to* $W_{n+1}$*, it is also a geodesic net with respect to* $W_n$ *and conversely.*

**Proof:** Let a totally umbilical hypersurface $W_n$ of recurrent Weyl space $W_{n+1}$ be conharmonically Ricci-recurrent $(n > 2)$. Then, $M = 0$. In this case, using (9) and (16), we get

$$\underset{r}{\bar{c}}^a = \underset{r}{c}^i x_i^a \,. \tag{32}$$

With the help of the equations (16) and (32) and the expression $\underset{r}{\bar{c}}^a = 0$, the result is easily obtained. □

**Remark 1.** *Conharmonically recurrent Weyl space is also conharmonically Ricci-recurrent, [13].*

**Corollary 1.** *Let a totally umbilical hypersurface* $W_n$ *of recurrent Weyl space* $W_{n+1}$ *be conharmonically recurrent* $(n > 2)$*. If any net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *in* $W_n$ *is a Chebyshev net of the first kind with respect to* $W_{n+1}$*, it is also a Chebyshev net of the first kind with respect to* $W_n$ *and the converse is also true.*

**Corollary 2.** *Let a totally umbilical hypersurface* $W_n$ *of recurrent Weyl space* $W_{n+1}$ *be conharmonically recurrent* $(n > 2)$*. If any net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *in* $W_n$ *is a Chebyshev net of the second kind with respect to* $W_{n+1}$*, it is also a Chebyshev net of the second kind with respect to* $W_n$ *and conversely.*

**Corollary 3.** *Let a totally umbilical hypersurface* $W_n$ *of recurrent Weyl space* $W_{n+1}$ *be conharmonically recurrent* $(n > 2)$*. If any net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *in* $W_n$ *is a geodesic net with respect to* $W_{n+1}$*, it is also a geodesic net with respect to* $W_n$ *and conversely.*

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