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THE RELATIVISTIC CENTER OF MOMENTUM

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Abstract. Since Pryce's 1948 study of the relativistic center of momentum it is believed that the classical notion of the center of mass can be extended to special relativity theory only in "at least one Lorentz frame" (in Goldstein's words). The aim of this work is, accordingly, to study the relativistic center of mass in a way fully analogous to its Newtonian counterpart.

1. Introduction and Historical Notes

In 1948 Pryce [4] reached the conclusion that "there appears to be no wholly satisfactory definition of the (relativistic) mass-centre". He thus raised an important problem in relativistic mechanics that extends to the present day.

Gyrovector space theoretic techniques, however, allow the relativistic center of momentum to be determined uniquely in a way fully analogous to its Newtonian counterpart. Furthermore, the resulting relativistic center of momentum enjoys all the attractive features that the Newtonian one does.

In Newtonian mechanics the notion of the center of mass arises naturally in the study of motion of isolated systems of particles. Lehner and Moresch [3] stated in 1995:

> *The notion of center of mass, however, does not have a direct extension to relativistic mechanics and several studies ... on this topic have been made to define such extension through different approaches (references omitted).*
>
> *L. R. Lehner and O. M. Moresch, 1995*

Rowe therefore informed in a book that he had almost completed writing when his life was brutally terminated in the Yemen in December 1998 [5, p. 111]:

The Newtonian construction of the centre-of-mass point loses interest in special relativity because the point it identifies turns out to depend on the Galilean frame used in the construction, and because it seems evident that the particle energies should be used for the weighting of particle position instead of the masses. No other definition of a centre-of-mass point has all the attractive features that the Newtonian one does. There is a useful discussion of six alternatives for the definition of the [relativistic] centre-of-mass in a paper of M. H. L Pryce [4].

E. G. P. Rowe, 2001

The center of momentum frame is the conceptual successor to the center of mass frame of Newtonian dynamics. In 1948 Pryce [4] explored the notion of the mass-center in the special theory of relativity, reaching the conclusion that

In classical mechanics the uniform motion of the mass-centre of a free system is an expression of the conservation of momentum, and takes its simplest form when the forces are assumed to act instantaneously. When, as in relativity mechanics, this cannot be assumed, account must be taken of the momentum resident in the field through which the interactions are propagated, and this complicates the problem. But this is not the only difficulty, and even for so simple a system of two non-interacting particles, for which no field momentum need be considered, there appears to be no wholly satisfactory definition of the mass-centre.

M. H. L. Pryce, 1948

Accordingly, Goldstein [2, p. 319] believed that the classical notion of the center of mass can be extended to special relativity theory only in “at least one Lorentz frame”.

The aim of this article is to present and study the relativistic center of momentum (or, mass) in a way fully analogous to its Newtonian counterpart. The study, in turn, uncovers remarkable analogies that Newtonian and Einsteinian mechanics share.

2. Einstein Addition and the Lorentz Boost

Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{V}_c$,

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V};\ \|\mathbf{v}\| < c\} \tag{1}$$

be the set of all relativistically admissible velocities in $\mathbb{V}$, that is, all vectors $\mathbf{v} \in \mathbb{V}$ with magnitude $< c$, c being any fixed positive constant representing the vacuum speed of light. The set $\mathbb{V}_c$ of all relativistically admissible velocities in $\mathbb{V}$ is thus the open ball $\mathbb{V}_c$ of the space $\mathbb{V}$ with radius c, centered at the origin of its space. Without loss of generality the vacuum speed of light can be normalized to $c = 1$. However, we prefer to leave c as a free positive parameter, enabling classical results to be recovered in the Newtonian limit $c \to \infty$.

In physics $\mathbb{V}$ is realized by the Euclidean 3-space $\mathbb{V}_c = \mathbb{R}^3$ and, accordingly, $\mathbb{V}_c = \mathbb{R}^3_c$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible velocities.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$ be any two relativistically admissible velocities in $\mathbb{V}$. Their Einstein sum, $\mathbf{u} \oplus \mathbf{v}$, is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\} \tag{2}$$

where $\gamma_{\mathbf{u}}$ is the Lorentz factor of $\mathbf{u}$,

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{u}\|^2}{c^2}}}. \tag{3}$$

Einstein addition is noncommutative. In general

$$\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u} \tag{4}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. Moreover, Einstein addition is also nonassociative. In general

$$(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \tag{5}$$

for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_c$.

Yet, Einstein addition is a *gyrocommutative gyrogroup operation* that gives rise to a grouplike structure called a **gyrocommutative gyrogroup** [7].

Einstein addition is involved in the Lorentz boost, also known as the **pure Lorentz transformation**, that is, a Lorentz transformation without rotation. Let $(t, \mathbf{x})^t = (t, \mathbf{v}t)^t$ be a spacetime event, $\mathbf{v} \in \mathbb{V}_c$, where exponent t denotes transposition. Furthermore, let $B(\mathbf{u})$ be the Lorentz boost parametrized by $\mathbf{u} \in \mathbb{V}_c$. Then, the application of the Lorentz boost $B(\mathbf{u})$ to the spacetime event $(t, \mathbf{v}t)^t$ is given by the equation [7]

$$B(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} = \begin{pmatrix} \dfrac{\gamma_{\mathbf{u} \oplus \mathbf{v}}}{\gamma_{\mathbf{v}}} t \\ \dfrac{\gamma_{\mathbf{u} \oplus \mathbf{v}}}{\gamma_{\mathbf{v}}} (\mathbf{u} \oplus \mathbf{v}) t \end{pmatrix}. \tag{6}$$

Hence, in particular, for $t = \gamma_{\mathbf{v}}$ we have the elegant identity

$$B(\mathbf{u}) \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}} \mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u} \oplus \mathbf{v}} \\ \gamma_{\mathbf{u} \oplus \mathbf{v}} (\mathbf{u} \oplus \mathbf{v}) \end{pmatrix} \tag{7}$$

that will prove useful in the sequel.

3. The Relativistic Law of the Lever

The Lorentz boost is linear. To exploit the linearity of the Lorentz boost let us consider the linear combination of two spacetime events

$$p\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix}+q\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}=\begin{pmatrix}p\gamma_{\mathbf{a}}+q\gamma_{\mathbf{b}}\\ p\gamma_{\mathbf{a}}\mathbf{a}+q\gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}=t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\tag{8}$$

$p, q \geq 0$, $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$, where $t \geq 0$ and $\mathbf{m} \in \mathbb{V}_c$ are to be determined.
Comparing ratios between lower and upper entries in (8) we have

$$\mathbf{m}=\mathbf{m}(\mathbf{a},\mathbf{b};p,q)=\frac{p\gamma_{\mathbf{a}}\mathbf{a}+q\gamma_{\mathbf{b}}\mathbf{b}}{p\gamma_{\mathbf{a}}+q\gamma_{\mathbf{b}}}\tag{9}$$

so that, indeed, $\mathbf{m} \in \mathbb{V}_c$ as desired.

The identity (9) has an obvious interpretation in relativistic mechanics. Points of $\mathbb{V}_c$ represent relativistically admissible velocities of inertial frames all of which were coincident at time $t = 0$. Accordingly, the spacetime coordinates of the inertial frames of points of $\mathbb{V}_c$ are related to one another by the Lorentz transformation. The origin $\mathbf{0}$ of $\mathbb{V}_c$, thus, represents a rest frame $\Sigma_{\mathbf{0}}$, and a point $\mathbf{v} \in \mathbb{V}_c$ represents an inertial frame $\Sigma_{\mathbf{v}}$ with velocity $\mathbf{v}$ relative to $\Sigma_{\mathbf{0}}$. As such, the velocity of the frame $\Sigma_{\mathbf{v}}$ relative to a frame $\Sigma_{\mathbf{u}}$ is $\ominus\mathbf{u} \oplus \mathbf{v}$.

The positive numbers p and q in (9) are the rest masses of two massive objects situated at the points $\mathbf{a}$ and $\mathbf{b}$ of the velocity space $\mathbb{V}_c$ of relativistically admissible velocities. These points, accordingly, represent the velocities of the objects relative to $\Sigma_{\mathbf{0}}$. The relativistically corrected masses of the objects are, accordingly, $p\gamma_{\mathbf{a}}$ and $q\gamma_{\mathbf{b}}$. Thus, the relativistic center of mass (or, momentum) of the two objects is situated at the point $\mathbf{m}$. The point $\mathbf{m}$, in turn, represents the velocity of the relativistic center of momentum (or, mass) $\Sigma_{\mathbf{m}}$ relative to $\Sigma_{\mathbf{0}}$, as we will see from (18).

Applying the Lorentz boost $B(\mathbf{x})$, $\mathbf{x} \in \mathbb{V}_c$, to (8) in two different ways, it follows from (7) and the linearity of the Lorentz boost that

$$\begin{aligned}B(\mathbf{x})\left\{t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\}&=B(\mathbf{x})\left\{p\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix}+q\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}\right\}\\&=pB(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix}+qB(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}\\&=p\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{a}}\\ \gamma_{\mathbf{x}\oplus\mathbf{a}}(\mathbf{x}\oplus\mathbf{a})\end{pmatrix}+q\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{b}}\\ \gamma_{\mathbf{x}\oplus\mathbf{b}}(\mathbf{x}\oplus\mathbf{b})\end{pmatrix}\\&=\begin{pmatrix}p\gamma_{\mathbf{x}\oplus\mathbf{a}}+q\gamma_{\mathbf{x}\oplus\mathbf{b}}\\ p\gamma_{\mathbf{x}\oplus\mathbf{a}}(\mathbf{x}\oplus\mathbf{a})+q\gamma_{\mathbf{x}\oplus\mathbf{b}}(\mathbf{x}\oplus\mathbf{b})\end{pmatrix}\end{aligned}\tag{10}$$

and

$$B(\mathbf{x})\left\{t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\} = tB(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}$$
$$= t\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{m}}\\ \gamma_{\mathbf{x}\oplus\mathbf{m}}\mathbf{x}\oplus\mathbf{m}\end{pmatrix} \qquad (11)$$
$$= \begin{pmatrix}t\gamma_{\mathbf{x}\oplus\mathbf{m}}\\ t\gamma_{\mathbf{x}\oplus\mathbf{m}}\mathbf{x}\oplus\mathbf{m}\end{pmatrix}.$$

Comparing ratios between lower and upper entries of (10) and (11) we have

$$\mathbf{x}\oplus\mathbf{m} = \frac{p\gamma_{\mathbf{x}\oplus\mathbf{a}}(\mathbf{x}\oplus\mathbf{a}) + q\gamma_{\mathbf{x}\oplus\mathbf{b}}(\mathbf{x}\oplus\mathbf{b})}{p\gamma_{\mathbf{x}\oplus\mathbf{a}} + q\gamma_{\mathbf{x}\oplus\mathbf{b}}} \qquad (12)$$

so that by (9) and (12)

$$\mathbf{x}\oplus\mathbf{m}(\mathbf{a},\mathbf{b};p,q) = \mathbf{m}(\mathbf{x}\oplus\mathbf{a},\mathbf{x}\oplus\mathbf{b};p,q)\,. \qquad (13)$$

The identity (13) demonstrates that the structure of $\mathbf{m}$ as a function of points $\mathbf{a}$ and $\mathbf{b}$ is not distorted by left gyrotranslations. Similarly, it is not distorted by rotations in the sense that if R represents a rotation of $\mathbb{V}_c$ (that is, an automorphism of $(\mathbb{V}_c,\oplus)$ that keeps invariant the inner product that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$) then

$$R\mathbf{m}(\mathbf{a},\mathbf{b};p,q) = \mathbf{m}(R\mathbf{a},R\mathbf{b};p,q)\,. \qquad (14)$$

It follows from (13) and (14) that the point $\mathbf{m}\in\mathbb{V}_c$ possesses, as a function of the points $\mathbf{a},\mathbf{b}\in\mathbb{V}_c$, hyperbolic geometric significance. The associated relativistic mechanics interpretation of $\mathbf{m}$ as the relativistic center of momentum velocity will be uncovered in (18).

Comparing the top entries of (10) and (11) we have

$$t = \frac{p\gamma_{\mathbf{x}\oplus\mathbf{a}} + q\gamma_{\mathbf{x}\oplus\mathbf{b}}}{\gamma_{\mathbf{x}\oplus\mathbf{m}}}\,. \qquad (15)$$

But, we also have from (8)

$$t = \frac{p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}}{\gamma_{\mathbf{m}}} \qquad (16)$$

implying that the positive scalar $t = t(\mathbf{a},\mathbf{b};p,q)$ in (15) and (16) is invariant under left gyrotranslations of $\mathbf{a}$ and $\mathbf{b}$. Clearly, it is also invariant under rotations of $\mathbf{a}$ and $\mathbf{b}$ so that, being invariant under the group of motions of hyperbolic geometry, it possesses hyperbolic geometric significance. As such, we call $t = t(\mathbf{a},\mathbf{b};p,q)$ a hyperbolic geometric scalar.

Substituting $\mathbf{x} = \ominus\mathbf{m}$ in (15) we have

$$t = p\gamma_{\ominus\mathbf{m}\oplus\mathbf{a}} + q\gamma_{\ominus\mathbf{m}\oplus\mathbf{b}} \tag{17}$$

revealing that the scalar t represents the mass of an object situated at the center of momentum $\mathbf{m}$ in its rest frame $\Sigma_{\mathbf{m}}$. It is the sum of the relativistically corrected rest masses p and q in $\Sigma_{\mathbf{m}}$. The center of momentum inertial frame $\Sigma_{\mathbf{m}}$ is represented by the center of mass point $\mathbf{m}$.

Substituting $\mathbf{x} = \ominus\mathbf{m}$ in (12) we obtain the identity

$$p\gamma_{\ominus\mathbf{m}\oplus\mathbf{a}}(\ominus\mathbf{m} \oplus \mathbf{a}) + q\gamma_{\ominus\mathbf{m}\oplus\mathbf{b}}(\ominus\mathbf{m} \oplus \mathbf{b}) = 0 \tag{18}$$

revealing that $\Sigma_{\mathbf{m}}$ is the vanishing momentum inertial frame. As in classical mechanics, the frame $\Sigma_{\mathbf{m}}$ is called the relativistic center of momentum (or, energy) frame (traditionally called the center of mass frame) since the total momentum in that frame, (18), vanishes. We may note that (18) can be written, equivalently, as

$$p\gamma_{\ominus\mathbf{m}\oplus\mathbf{a}}(\ominus\mathbf{m} \oplus \mathbf{a}) = \ominus q\gamma_{\ominus\mathbf{m}\oplus\mathbf{b}}(\ominus\mathbf{m} \oplus \mathbf{b})\,. \tag{19}$$

Owing to its property (19), the relativistic center of mass $\mathbf{m}(\mathbf{a}, \mathbf{b}; p, q)$, given by (9), is called the (p, q)-midpoint of $\mathbf{a}$ and $\mathbf{b}$. It reduces to the hyperbolic midpoint $\mathbf{m}_{\mathbf{ab}}$ when the two associated masses are equal, $p = q$.

Rewriting (16) as

$$t\gamma_{\mathbf{m}} = p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}} \tag{20}$$

we obtain the two identities (19) and (20) that form the **relativistic law of the lever**. It is fully analogous to the classical law of the lever, to which it reduces in the Newtonian limit $c \to \infty$. Taking magnitudes of both sides of (19) and expressing the resulting equation in terms of rapidities, one can recover the relativistic law of the lever of Galperin [1].

The origin $\mathbf{0}$ of an Einstein gyrovector space $\mathbb{V}_c = (\mathbb{V}_c, \oplus, \otimes)$ represents the vanishing velocity of a rest frame $\Sigma_{\mathbf{0}}$.

The relativistically corrected masses are, therefore, $p\gamma_{\mathbf{a}}$ and $q\gamma_{\mathbf{b}}$ so that the total relativistic mass of the two massive objects is $p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}$. This, in turn, is equal to $t\gamma_{\mathbf{m}}$, that is, the relativistic mass of an object with rest mass t moving with velocity $\mathbf{m}$ relative to $\Sigma_{\mathbf{0}}$.

The center of momentum of the two points $\mathbf{a}$ and $\mathbf{b}$ is a single moving object with relativistic mass $p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}$ and velocity $\mathbf{m}(\mathbf{a}, \mathbf{b}; p, q)$ relative to $\Sigma_{\mathbf{0}}$, called the **(p, q)-midpoint** of the velocities $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$. The (p, q)-midpoint is **homogeneous** in the sense that it depends on the ratio p/q of the masses p and q, as we see from (9). Since it is the ratio p/q that is of interest, we call

(p/q) the **homogeneous gyrobarycentric coordinates** of $\mathbf{m}$ relative to the set $A = \{\mathbf{a}, \mathbf{b}\}$.

The rest masses p and q can be normalized by the condition $p + q = 1$, suggesting the notation

$$\mathbf{m}(\mathbf{a}, \mathbf{b}; p, 1-p) = \mathbf{m}(\mathbf{a}, \mathbf{b}; p) \tag{21}$$

$0 \leq p \leq 1$, calling it the hyperbolic p-**midpoint** of $\mathbf{a}$ and $\mathbf{b}$. Clearly, the p-midpoint possesses the symmetry

$$\mathbf{m}(\mathbf{a}, \mathbf{b}; p) = \mathbf{m}(\mathbf{b}, \mathbf{a}; 1-p)\,. \tag{22}$$

The p-midpoint $\mathbf{m}(\mathbf{a}, \mathbf{b}; p) \in \mathbb{V}_c$ gives rise to the center of momentum inertial frame $\Sigma_{\mathbf{m}}$ that moves with velocity $\mathbf{m} = \mathbf{m}(\mathbf{a}, \mathbf{b}; p)$ relative to the rest frame $\Sigma_{\mathbf{0}}$, and relative to which the center of mass of the massive objects is at rest. The center of momentum inertial frame $\Sigma_{\mathbf{m}}$ of a system of uniformly moving massive objects is, by definition, the inertial frame where the total momentum of the objects in the system vanishes.

The identity (18) demonstrates that the relativistic momentum of the moving objects, with rest masses $p, q > 0$ and with respective velocities $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$ relative to $\Sigma_{\mathbf{0}}$, vanishes in the center of momentum frame $\Sigma_{\mathbf{m}}$.

4. The Relativistic Center of Momentum

We present in this section the relativistic center of momentum (or, mass) in a way fully analogous to its Newtonian counterpart. As in the classical case, the relativistic center of momentum of an isolated system of moving massive objects is, by definition, the inertial frame relative to which the momentum of the system vanishes.

Let $(\gamma_{\mathbf{a}_k}, \gamma_{\mathbf{a}_k}\mathbf{a}_k)^t$, $\mathbf{a}_k \in \mathbb{V}_c$, $k = 1, \ldots, n$, be n spacetime points, and let

$$\sum_{k=1}^{n} m_k \begin{pmatrix} \gamma_{\mathbf{a}_k} \\ \gamma_{\mathbf{a}_k}\mathbf{a}_k \end{pmatrix} = m \begin{pmatrix} \gamma_{\mathbf{c}} \\ \gamma_{\mathbf{c}}\mathbf{c} \end{pmatrix} \tag{23}$$

$m_k \geq 0$, be a generic linear combination of these spacetime events, where $m \geq 0$ and $\mathbf{c} \in \mathbb{V}_c$ are to be determined.

Comparing ratios between lower and upper entries in (23) we have

$$\mathbf{c} = \mathbf{c}(\mathbf{a}_1, \ldots, \mathbf{a}_n; m_1, \ldots, m_n) = \frac{\sum_{k=1}^{n} m_k \gamma_{\mathbf{a}_k} \mathbf{a}_k}{\sum_{k=1}^{n} m_k \gamma_{\mathbf{a}_k}} \tag{24}$$

so that $\mathbf{c}$ lies on the convex set spanned by the points $\mathbf{a}_k$ of $\mathbb{V}_c$, $k = 1, \ldots, n$. Hence, $\mathbf{c} \in \mathbb{V}_c$ as desired.

The center of momentum frame of a system of uniformly moving massive objects is the inertial frame where the total momentum of the objects vanishes. Hence, as we will see from (32) below, the point $\mathbf{c}$ in the space $\mathbb{V}_c$ of relativistically admissible velocities represents the relativistic center of momentum reference frame $\Sigma_{\mathbf{c}}$ or, loosely, the relativistic center of mass frame. The relativistic center of mass system is fully analogous to its classical counterpart to which it is identical except that the involved masses m_k with corresponding velocities $\mathbf{a}_k$ are relativistically corrected into the relativistic masses $m_k\gamma_{\mathbf{a}_k}$, $k = 1, \ldots, n$. Indeed, Synge [6, p. 219] describes the right hand side of (24) as "the Newtonian formula for the mass-centre changed only by the substitution of relative mass $m\gamma$ for Newtonian mass".

Applying the Lorentz boost $B(\mathbf{x})$, $\mathbf{x} \in \mathbb{V}_c$, to (23) in two different ways, it follows from (7) and from the linearity of the Lorentz boost that

$$
\begin{aligned}
B(\mathbf{x})\left\{m\begin{pmatrix}\gamma_{\mathbf{c}}\\ \gamma_{\mathbf{c}}\mathbf{c}\end{pmatrix}\right\} &= \sum_{k=1}^{n} m_k B(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{a}_k}\\ \gamma_{\mathbf{a}_k}\mathbf{a}_k\end{pmatrix}\\
&= \sum_{k=1}^{n} m_k \begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{a}_k}\\ \gamma_{\mathbf{x}\oplus\mathbf{a}_k}(\mathbf{x}\oplus\mathbf{a}_k)\end{pmatrix}\\
&= \begin{pmatrix}\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{a}_k}\\ \sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{a}_k}(\mathbf{x}\oplus\mathbf{a}_k)\end{pmatrix}
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
B(\mathbf{x})\left\{m\begin{pmatrix}\gamma_{\mathbf{c}}\\ \gamma_{\mathbf{c}}\mathbf{c}\end{pmatrix}\right\} &= mB(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{c}}\\ \gamma_{\mathbf{c}}\mathbf{c}\end{pmatrix}\\
&= \begin{pmatrix}m\gamma_{\mathbf{x}\oplus\mathbf{c}}\\ m\gamma_{\mathbf{x}\oplus\mathbf{c}}(\mathbf{x}\oplus\mathbf{c})\end{pmatrix}.
\end{aligned}
\tag{26}
$$

Comparing ratios between lower and upper entries of (25) and (26) we have

$$
\mathbf{x}\oplus\mathbf{c} = \frac{\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{a}_k}(\mathbf{x}\oplus\mathbf{a}_k)}{\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{a}_k}}
\tag{27}
$$

so that, by (24) and (27)

$$
\mathbf{x}\oplus\mathbf{c}(\mathbf{a}_1,\ldots,\mathbf{a}_n;m_1,\ldots,m_n) = \mathbf{c}(\mathbf{x}\oplus\mathbf{a}_1,\ldots,\mathbf{x}\oplus\mathbf{a}_n;m_1,\ldots,m_n).
\tag{28}
$$

Identity (28) demonstrates that the structure of $\mathbf{c}$ as a function of points $\mathbf{a}_k \in \mathbb{V}_c$, $k = 1, \ldots, n$, is not distorted by a left gyrotranslation of the points by any $\mathbf{x} \in \mathbb{V}_c$.

Similarly, the structure is not distorted by rotations in the sense that if R represents a rotation of $\mathbb{V}_c$ (that is, an automorphism of $(\mathbb{V}_c, \oplus)$ that keeps invariant the inner product that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$) then

$$R\mathbf{c}(\mathbf{a}_1, \ldots, \mathbf{a}_n; m_1, \ldots, m_n) = \mathbf{c}(R\mathbf{a}_1, \ldots, R\mathbf{a}_n; m_1, \ldots, m_n)\,. \tag{29}$$

Hence the point $\mathbf{c} \in \mathbb{V}_c$ possesses, as a function of the points $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{V}_c$, hyperbolic geometric significance. Thus, in other words, $\mathbf{c}$ is a hyperbolic geometric object.

Comparing the top entries of (25) and (26) we have

$$m = \frac{\sum_{k=1}^{n} m_k \gamma_{\mathbf{x}\oplus\mathbf{a}_k}}{\gamma_{\mathbf{x}\oplus\mathbf{c}}}\,. \tag{30}$$

But, we also have from (23)

$$m = \frac{\sum_{k=1}^{n} m_k \gamma_{\mathbf{a}_k}}{\gamma_{\mathbf{c}}} \tag{31}$$

implying that the positive scalar $m = m(\mathbf{a}_1, \ldots, \mathbf{a}_n; m_1, \ldots, m_n)$ in (30) and (31) is invariant under any left gyrotranslation of the points $\mathbf{a}_k \in \mathbb{V}_c$, $k = 1, \ldots, n$, that generate its value. Clearly, it is also invariant under any rotation of its generating points. Hence, it possesses hyperbolic geometric significance. From the relativistic point of view, m is a Lorentz invariant scalar, that is, a scalar valued vector function which is invariant under the Lorentz transformation group.

From the point of view of relativistic mechanics we face here an isolated relativistic system S of n massive objects with masses m_k and respective velocities $\mathbf{a}_k$, $k = 1, \ldots, n$. The corresponding relativistically corrected masses in the frame $\Sigma_{\mathbf{0}}$ are accordingly $m_k \gamma_{\mathbf{a}_k}$, $k = 1, \ldots, n$.

To determine the center of momentum frame of the system S, that is, the frame where the total momentum of the system S vanishes, we substitute $\mathbf{x} = \ominus\mathbf{c}$ in (27) obtaining the identity

$$\sum_{k=1}^{n} m_k \gamma_{\ominus\mathbf{c}\oplus\mathbf{a}_k} (\ominus\mathbf{c} \oplus \mathbf{a}_k) = 0\,. \tag{32}$$

The resulting identity, in turn, demonstrates that the relativistic momentum vanishes in the inertial rest frame $\Sigma_{\mathbf{c}}$ of the geometric object $\mathbf{c}$. The geometric object $\mathbf{c}$ is, therefore, the center of momentum of the system S. Thus, the relativistic center of momentum $\mathbf{c}$ of masses m_k with respective velocities $\mathbf{a}_k \in \mathbb{V}_c$, $k = 1, \ldots, n$, in relativistic mechanics is just the classical center of mass of corresponding relativistically corrected masses $m_k \gamma_{\mathbf{a}_k}$, $k = 1, \ldots, n$.

Substituting $\mathbf{x} = \ominus\mathbf{c}$ in (30) we have

$$m = \sum_{k=1}^{n} m_k \gamma_{\ominus\mathbf{c}\oplus\mathbf{a}_k} \tag{33}$$

revealing the relativistic interpretation of the scalar m. It represents the mass of a fictitious object situated at the center of momentum $\mathbf{c}$ of the system S of objects with rest masses m_k, $k = 1, \dots, n$, in its rest frame $\Sigma_{\mathbf{c}}$. It is the sum of the relativistically corrected masses $m_k \gamma_{\ominus\mathbf{c}\oplus\mathbf{a}_k}$, $k = 1, \dots, n$, relative to $\Sigma_{\mathbf{c}}$, where $\Sigma_{\mathbf{c}}$ is an inertial frame relative to which the center of momentum $\mathbf{c}$ is at rest.

The center of mass $\mathbf{c}$, (24), and its mass m, (33), are clearly consistent with the classical picture.

Interestingly, the analogies shared by the center of momentum in classical and relativistic mechanics go over to corresponding analogies that Euclidean and hyperbolic centroids share as shown in [8].

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