Fourth International Conference on
Geometry, Integrability and Quantization
June 6–15, 2002, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2003, pp 330–340

PATH INTEGRAL FOR STAR EXPONENTIAL FUNCTIONS OF QUADRATIC FORMS

AKIRA YOSHIOKA and TOSHIO MATSUMOTO

Department of Mathematics, Tokyo University of Science
Kagaurazaka 1-3, Shinjyku-ku, Tokyo 162-8601, Japan

Abstract. The Moyal product is considered on the complex plane $\mathbb{C}^2$. Path integral representation of $*$-exponential function is given for a quadratic form on $\mathbb{C}^2$, $H = ax^2 + 2bxy + cy^2$ for $(x, y) \in \mathbb{C}^2$, where $a, b, c \in \mathbb{C}$.

1. Introduction

The formal deformation quantization theory was started by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [1] and the general theory is well designed studying the existence, classification and applications (cf. [2–4, 7–9]). However, if we take as a deformation parameter a number, we have no general theory of non-formal deformation quantization at present (see e. g. [5]).

In this note, we consider a deformation quantization with a deformation parameter $\hbar > 0$. The star product is given by the Moyal product formula. As is well known, any formal star product is locally isomorphic to the Moyal star product, hence we deal with local theory in this sense, but non-formal.

Let $H = ax^2 + 2bxy + cy^2$ be a quadratic form on $\mathbb{C}^2$ with $a, b, c \in \mathbb{C}$. We consider the Moyal product $*_0$ given in Definition 1 below. We will study the $*$-exponential function $\mathrm{e}_*^{tH/\mathrm{i}\hbar}$. In [6], the star exponential function is defined by solving a certain differential equation which characterizes the $*$-exponential function. The purpose of this paper is to give a path integral description of this $*$-exponential function (see Theorem 2 below).

2. $*$-Exponential Function of Quadratic Functions

We first give the Moyal product on $\mathbb{C}^2$. Let x, y be coordinate functions of $\mathbb{C}^2$ and let $\hbar$ be a positive real parameter. The canonical Poisson bracket is given by

$$\begin{aligned}\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} &= f\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g \\ &= f\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_y - \overleftarrow{\partial}_y\overrightarrow{\partial}_x\right) g\,.\end{aligned} \tag{1}$$

Using the binomial theorem formally, we set the bidifferential operators

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n = \sum_{l+m=n} \frac{n!}{l!m!}(-1)^m \left(\overleftarrow{\partial}_x\overrightarrow{\partial}_y\right)^l \left(\overleftarrow{\partial}_y\overrightarrow{\partial}_x\right)^m$$

and

$$\exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n \left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n.$$

The **Moyal product** is then given by

Definition 1.

$$f *_0 g = f \exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g\,.$$

We remark here that the product $f *_0 g$ is not necessarily convergent for arbitrary smooth functions, however it is well defined when at least one of f, g is a polynomial function.

With the Moyal product, we can define the $*$-exponential function of a quadratic form in the following way. Let us consider the quadratic form on $\mathbb{C}^2$ given by

$$H = ax^2 + 2bxy + cy^2\,, \qquad a, b, c \in \mathbb{C}\,. \tag{2}$$

The $*$-exponential function is formally given by

$$\mathrm{e}_*^{t\frac{H}{\mathrm{i}\hbar}} * f = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(\frac{H}{\mathrm{i}\hbar}\right)_*^n * f \tag{3}$$

for every polynomial function $f(x, y)$ where $\left(\frac{H}{\mathrm{i}\hbar}\right)_*^n = \underbrace{\frac{H}{\mathrm{i}\hbar} * \cdots * \frac{H}{\mathrm{i}\hbar}}_{n}\,.$

3. Normal Forms and the Invariance of $*_0$

Given a quadratic function $H = ax^2 + 2bxy + cy^2$ on $\mathbb{C}^2$, we consider the discriminant

$$D = b^2 - ac. \tag{4}$$

We will show that H with a nonvanishing discriminant can be transformed into $x^2 - y^2$ via linear transformations by $SL(2, \mathbb{C})$.

First we consider the case where H has the discriminant $D = \frac{1}{4}$. We will prove

Proposition 1. *There exists* $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{C})$ *such that*

$$H = -\frac{1}{2}w^2 + \frac{1}{2}z^2 \tag{5}$$

where

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

Such matrices $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ *are not unique and are parametrized by* $\mathbb{C}$.

We put $w = px + qy$, $z = rx + sy$ with $ps - qr = 1$ into the right hand side of (5) and consider the identity

$$ax^2 + 2bxy + cy^2 = -\frac{1}{2}(px + qy)^2 + \frac{1}{2}(rx + sy)^2.$$

Then we have

$$-p^2 + r^2 = 2a, \qquad -q^2 + s^2 = 2c, \qquad -pq + rs = 2b.$$

The identities $ps - qr = 1$ and $-pq + rs = 2b$ yield

$$(-q^2 + s^2)r = 2bs + q, \qquad (-q^2 + s^2)p = 2bq + s.$$

When $c \neq 0$, we have

$$p = \frac{1}{2c}(2bq + s), \quad r = \frac{1}{2c}(2bs + q), \quad s^2 = 2c + q^2.$$

Hence, for every $q \in \mathbb{C}$, one take p, q, r, s by the equations above. Then one can check that these p, q, r, s gives an element of $SL(2, \mathbb{C})$ and these satisfy the desired equations.

When, $c = 0$, then $D = b^2 = \dfrac{1}{4}$. If $b = \dfrac{1}{2}$, then the previous identities induce $2bq + s = q + s = 0$, hence $s = -q$. Further we see $ps - qr = -(p+r)q = 1$ which gives $p + r = -1/q$ and $-p^2 + r^2 = (-p + r)(p + r) = 2a$. Then we have

$$p = aq - \frac{1}{2q}, \qquad r = -aq - \frac{1}{2q}, \qquad s = -q\,.$$

If $b = -\dfrac{1}{2}$, the similar argument gives

$$p = -aq + \frac{1}{2q}, \qquad r = -aq - \frac{1}{2q}, \qquad s = q\,.$$

This proves Proposition 1.

Now, we consider quadratic forms with an arbitrary $D \neq 0$. We can easily reduce the problem to the previous case.

We set $\tilde{H} = H/2\sqrt{D}$. The quadratic form

$$\tilde{H} = \tilde{a}x^2 + 2\tilde{b}xy + \tilde{c}y^2$$

has discriminant $\tilde{D} = \frac{1}{4}$, where $\tilde{a} = a/2\sqrt{D}$, $\tilde{b} = b/2\sqrt{D}$ and $\tilde{c} = c/2\sqrt{D}$. Hence, we obtain

Proposition 2. *There exits* $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbf{C})$ *such that*

$$H = 2\sqrt{D}\left(-\frac{1}{2}w^2 + \frac{1}{2}z^2\right)$$

where $D = b^2 - ac$ $(\neq 0)$ *is the discriminant and*

$$w = px + qy, \qquad z = rx + sy\,.$$

Remark 1. *By a similar manner one can also transform* H *via* $SL(2, \mathbb{C})$ *with* $D \neq 0$ *into the normal form* $2\sqrt{D}xy$.

Now, we consider the invariance of the Moyal product under the linear transformation of $SL(2, \mathbb{C})$. Let us consider $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{C})$ and put coordinate functions by

$$w = px + qy, \qquad z = rx + sy\,. \tag{7}$$

We then easily see that the commutator satisfies

$$[w, z] = w *_0 z - z *_0 w = i\hbar\,.$$

Furthermore, by a direct calculation we have the following invariance property of the Moyal product:

Proposition 3. *The Moyal product can be expressed in terms of the coordinate functions w, z as follows:*

$$f *_0 g = f \exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial_w} \wedge \overrightarrow{\partial_z}\right) g\,.$$

4. Path-Integral Representation

In this section we give a path-integral representation of the $*$-exponential function of quadratic forms. Fist we consider $H = -\frac{1}{2}x^2 + \frac{1}{2}y^2$ and then by the propositions in the previous section, we obtain the $*$-exponential functions for a general H with $D \neq 0$.

In the sequel, we write $* = *_0$ for simplicity.

4.1. $*$-exponential of $-\frac{1}{2}x^2 + \frac{1}{2}y^2$

First, we give the $*$-exponential function of $-\frac{1}{2}x^2 + \frac{1}{2}y^2$.

The basic tool is the following Mehler's formula:

Lemma 1. *Let $H_n(t)$ be the Hermite polynomial of degree n. Then it holds*

$$\mathrm{e}^{-x^2-y^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(x)H_n(y) = \frac{1}{\sqrt{1-z^2}}\exp\left(-\frac{1}{1-z^2}(x^2+y^2-2zxy)\right).$$

Step 1. For the first step of path integral, we consider the product $\exp\left(t\frac{H}{\mathrm{i}\hbar}\right) * \exp\left(s\frac{H}{\mathrm{i}\hbar}\right)$ for $H = (-x^2+y^2)/2$. In what follows, we will show the formula:

Proposition 4.

$$\exp\left(t\frac{H}{\mathrm{i}\hbar}\right) * \exp\left(s\frac{H}{\mathrm{i}\hbar}\right) = \frac{1}{1+ts/4}\exp\left(\frac{t+s}{1+ts/4}\frac{H}{\mathrm{i}\hbar}\right).$$

First, we remark the basic relation for the Hermite polynomials $H_n(x)$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \mathrm{e}^{-x^2} = (-1)^n\,\mathrm{e}^{-x^2}H_n(x), \qquad n = 0, 1, 2, \ldots \tag{8}$$

and hence we see

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp\left(\frac{t}{\mathrm{i}\hbar}\frac{x^2}{2}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n \mathrm{e}^{-y^2}\Big|_{y=\sqrt{\frac{\mathrm{i}t}{2\hbar}}x} \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\right)^n$$
$$= (-1)^n\,\mathrm{e}^{-y^2} H_n(y) \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\right)^n\Bigg|_{y=\sqrt{\frac{\mathrm{i}t}{2\hbar}}x} . \tag{9}$$

Then the relation (8) yields

Lemma 2.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp\left(\frac{t}{\mathrm{i}\hbar}\frac{x^2}{2}\right) = (-1)^n \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\right)^n \exp\left(\frac{t}{\mathrm{i}\hbar}\frac{x^2}{2}\right) H_n\left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}x\right) .$$

By definition we have

$$\exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial_x}\wedge\overrightarrow{\partial_y}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n\left(\overleftarrow{\partial_x}\wedge\overrightarrow{\partial_y}\right)^n .$$

Using formally the binomial theorem

$$\left(\overleftarrow{\partial_x}\wedge\overrightarrow{\partial_y}\right)^n = \sum_{l+m=n}\frac{n!}{l!m!}(-1)^m\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y}\right)^l\left(\overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)^m$$

we have

$$\exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial_x}\wedge\overrightarrow{\partial_y}\right) = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\mathrm{i}\hbar}{2}\right)^l\overleftarrow{\partial_x}^l\overrightarrow{\partial_y}^l\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\mathrm{i}\hbar}{2}\right)^k\overleftarrow{\partial_y}^k\overrightarrow{\partial_x}^k . \tag{10}$$

Using (10) we obtain

$$\exp\left(t\frac{H}{\mathrm{i}\hbar}\right)*\exp\left(s\frac{H}{\mathrm{i}\hbar}\right) = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\mathrm{i}\hbar}{2}\right)^l\partial_x^l\,\mathrm{e}^{-\frac{t}{\mathrm{i}\hbar}\frac{x^2}{2}}\partial_y^l\,\mathrm{e}^{\frac{s}{\mathrm{i}\hbar}\frac{y^2}{2}}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\mathrm{i}\hbar}{2}\right)^k\partial_y^k\,\mathrm{e}^{\frac{t}{\mathrm{i}\hbar}\frac{y^2}{2}}\partial_x^k\,\mathrm{e}^{-\frac{s}{\mathrm{i}\hbar}\frac{x^2}{2}}$$

Lemma 2 gives

$$\exp\left(t\frac{H}{\mathrm{i}\hbar}\right)*\exp\left(s\frac{H}{\mathrm{i}\hbar}\right)$$
$$= \exp\left(t\tfrac{H}{\mathrm{i}\hbar}\right)\sum_{l=0}^{\infty}\tfrac{1}{l!}\left(\tfrac{\mathrm{i}\hbar}{2}\right)^l\left(\tfrac{t}{2\mathrm{i}\hbar}\right)^{l/2}H_l\left(\sqrt{\tfrac{t}{2\mathrm{i}\hbar}}x\right)\left(\tfrac{\mathrm{i}s}{2\hbar}\right)^{l/2}H_l\left(\sqrt{\tfrac{\mathrm{i}s}{2\hbar}}y\right)$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\mathrm{i}\hbar}{2}\right)^k \left(\frac{\mathrm{i}t}{2\hbar}\right)^{k/2} H_k \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\, y\right) \left(\frac{s}{2\mathrm{i}\hbar}\right)^{k/2} H_k \left(\sqrt{\frac{s}{2\mathrm{i}\hbar}}\, x\right) \exp\left(s\frac{H}{\mathrm{i}\hbar}\right)$$

and then it holds

$$\begin{aligned} &\exp\left(t\frac{H}{\mathrm{i}\hbar}\right) * \exp\left(s\frac{H}{\mathrm{i}\hbar}\right) \\ &\quad = \exp\left(\frac{t+s}{\mathrm{i}\hbar}H\right) \sum_{l=0}^{\infty} \frac{1}{2^l l!} \left(\frac{\mathrm{i}\sqrt{ts}}{2}\right)^l H_l \left(\sqrt{\frac{t}{2\mathrm{i}\hbar}}\, x\right) H_l \left(\sqrt{\frac{\mathrm{i}s}{2\hbar}}\, y\right) \\ &\quad \times \sum_{k=0}^{\infty} \frac{1}{2^k k!} \left(\frac{-\mathrm{i}\sqrt{ts}}{2}\right)^k H_k \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\, y\right) H_k \left(\sqrt{\frac{s}{2\mathrm{i}\hbar}}\, x\right) \end{aligned} \tag{11}$$

Applying the Mehler's formula, we have

$$\begin{aligned} &\sum_{l=0}^{\infty} \frac{1}{2^l l!} \left(\frac{\mathrm{i}\sqrt{ts}}{2}\right)^l H_l \left(\sqrt{\frac{t}{2\mathrm{i}\hbar}}\, x\right) H_l \left(\sqrt{\frac{\mathrm{i}s}{2\hbar}}\, y\right) \\ &\quad = \frac{1}{\sqrt{1+ts/4}} \exp \frac{1}{\mathrm{i}\hbar} \left(t\frac{x^2}{2} - s\frac{y^2}{2}\right) \exp \frac{1}{1+ts/4} \left(-\frac{t}{\mathrm{i}\hbar}\frac{x^2}{2} + \frac{s}{\mathrm{i}\hbar}\frac{y^2}{2} - \mathrm{i}\frac{ts}{2\hbar}xy\right) \end{aligned}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{2^k k!} \left(\frac{-\mathrm{i}\sqrt{ts}}{2}\right)^k H_k \left(\sqrt{\frac{\mathrm{i}t}{2\hbar}}\, y\right) H_k \left(\sqrt{\frac{s}{2\mathrm{i}\hbar}}\, x\right) \\ &\quad = \frac{1}{\sqrt{1+ts/4}} \exp \frac{1}{\mathrm{i}\hbar} \left(-t\frac{y^2}{2} + s\frac{x^2}{2}\right) \exp \frac{1}{1+ts/4} \left(-\frac{s}{\mathrm{i}\hbar}\frac{x^2}{2} + \frac{t}{\mathrm{i}\hbar}\frac{y^2}{2} + \mathrm{i}\frac{ts}{2\hbar}xy\right). \end{aligned}$$

Substituting these identities into (11), we obtain Proposition 4.

Step 2. For the next step, we consider the iterated product of exponential functions $e^{t_1 \tilde{H}} * \cdots * e^{t_n \tilde{H}}$, where we put $\tilde{H} = \frac{H}{\mathrm{i}\hbar} = (-x^2 + y^2)/2\mathrm{i}\hbar$.

In what follows, we will prove the formula:

Proposition 5.

$$e^{t_1 \tilde{H}} * \cdots * e^{t_n \tilde{H}} = \frac{1}{c_n(t)} \exp\left(\frac{s_n(t)}{c_n(t)}\tilde{H}\right) \tag{a}$$

where

$$c_n(t) = 1 + \sum_{k;2\le 2k\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} (t_{i_1}/2)\dots(t_{i_{2k}}/2) \tag{b}$$

and

$$s_n(t) = 2 \sum_{k;1\le 2k+1\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n} (t_{i_1}/2)\dots(t_{i_{2k+1}}/2). \tag{c}$$

We will prove the formula by induction with respect to the length of the product n.

For $n = 2$, the functions $c_2(t)$ and $s_2(t)$ in the formula are

$$c_2(t) = 1 + \sum_{k;2\le 2k\le 2} \sum_{1\le i_1<i_2<\cdots<i_{2k}\le 2} (t_{i_1}/2)\dots(t_{i_{2k}}/2) = 1 + \frac{1}{4}t_1t_2$$

$$s_2(t) = 2 \sum_{k;1\le 2k+1\le 2} \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le 2} (t_{i_1}/2)\dots(t_{i_{2k+1}}/2) = t_1 + t_2.$$

Then the formula (4) shows that the formula (5) is true when $n = 2$.

We assume that the formula is valid for n and we will show that it is true in the case $n+1$. We put

$$\mathrm{e}^{t_1\tilde{H}} * \cdots * \mathrm{e}^{t_n\tilde{H}} * \mathrm{e}^{t_{n+1}\tilde{H}} = \frac{1}{c_n(t)} \exp\left(\frac{s_n(t)}{c_n(t)}\tilde{H}\right) * \mathrm{e}^{t_{n+1}\tilde{H}}.$$

Using the formula (4) we see that the right hand side is

$$\frac{1}{c_n}\frac{1}{1+\frac{1}{4}(s_n/c_n)t_{n+1}} \exp\left(\frac{s_n/c_n + t_{n+1}}{1+\frac{1}{4}(s_n/c_n)t_{n+1}}\tilde{H}\right)$$
$$= \frac{1}{c_n + \frac{1}{4}s_nt_{n+1}} \exp\left(\frac{s_n + c_nt_{n+1}}{c_n+\frac{1}{4}s_nt_{n+1}}\tilde{H}\right).$$

We calculate by using the assumption

$$c_n + \frac{1}{4}s_nt_{n+1} = 1 + \sum_{k;2\le 2k\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} (t_{i_1}/2)\dots(t_{i_{2k}}/2)$$
$$+ \left(\sum_{k;1\le 2k+1\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n} (t_{i_1}/2)\dots(t_{i_{2k+1}}/2)\right)(t_{n+1}/2).$$

The third term of the right hand side is written as

$$\sum_{k;2\le 2k\le n+1} \sum_{1\le i_1<i_2<\cdots<i_{2k-1}\le n, i_{2k}=n+1} (t_{i_1}/2)\dots(t_{i_{2k-1}}/2)(t_{i_{2k}}/2)$$

which gives $c_n + \dfrac{1}{4}s_nt_{n+1} = c_{n+1}$.

As to s_n we find

$$s_n + c_nt_{n+1} = 2 \sum_{k;1\le 2k+1\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n} (t_{i_1}/2)\dots(t_{i_{2k+1}}/2)$$

$$
\begin{aligned}
&+\left(1+\sum_{k;2\le 2k\le n}\ \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n}(t_{i_1}/2)\dots(t_{i_{2k}}/2)\right)t_{n+1}\\
&=\sum_{i=1}^{n+1}t_i+2\sum_{k;2\le 2k\le n-1}\ \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n}(t_{i_1}/2)\dots(t_{i_{2k+1}}/2)\\
&+2\sum_{k;2\le 2k\le n-1}\ \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n,i_{2k+1}=n+1}(t_{i_1}/2)\dots(t_{i_{2k}}/2)(t_{i_{2k+1}}/2)\\
&+2\sum_{k;n\le 2k;2k+1\le n+1}\ \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n,i_{2k+1}=n+1}(t_{i_1}/2)\dots(t_{i_{2k}}/2)(t_{i_{2k+1}}/2)\\
&=s_{n+1}\,.
\end{aligned}
$$

Thus, we obtain that the formula holds for $n+1$ and thus we get the proof.

Step 3. As a third step, we take the limit of the iterated products in the previous step.

For $t>0$, we divide the interval $[0,t]$ into N equal segments for every positive integer N. We will then take the limit $N\to\infty$ and we will check the convergence of the iterated product. We will find also the limit explicitly. Then we will obtain the star exponential function of the quadratic form $-\frac{x^2}{2}+\frac{y^2}{2}$.

Let us put $\Delta t=t/N$. First, we consider $c_N(\Delta t)$, $s_N(\Delta t)$ and their limit $N\to\infty$. Form the formulae (12), (12) we have

$$
\begin{aligned}
c_N(\Delta t)&=1+\sum_{k;2\le 2k\le n}(\Delta t/2)^{2k}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n}1\\
s_N(\Delta t)&=2\sum_{k;1\le 2k+1\le n}(\Delta t/2)^{2k+1}\sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n}1\,.
\end{aligned}
$$

Notice that

$$
\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n}1=\binom{N}{2k}=\frac{N!}{(N-2k)!(2k)!}
$$

and

$$
\sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n}1=\binom{N}{2k+1}=\frac{N!}{(N-2k-1)!(2k+1)!}\,.
$$

Hence we have

$$
c_N(\Delta t)=\sum_{k;0\le 2k\le N}a_k(t/2)^{2k},\qquad s_N(\Delta t)=2\sum_{k;1\le 2k+1\le N}b_k(t/2)^{2k+1}
$$

where the coefficients are given by

$$a_k = \frac{1}{(2k)!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{2k-1}{N}\right)$$

and

$$b_k = \frac{1}{(2k+1)!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{2k}{N}\right).$$

It is easy to see that the coefficients of $c_N(\Delta t)$ and $s_N(\Delta t)$ converge as $N \to \infty$ and the limits are

$$\lim_{N\to\infty} c_N(\Delta t) = \cosh\frac{t}{2}, \qquad \lim_{N\to\infty} s_N(\Delta t) = 2\sinh\frac{t}{2}.$$

Thus, we have

Theorem 1. *The N-iterated product converges*

$$\lim_{N\to\infty} \mathrm{e}^{(t/N)\tilde{H}} * \cdots * \mathrm{e}^{(t/N)\tilde{H}} = \frac{1}{\cosh\frac{t}{2}}\,\mathrm{e}^{2\tilde{H}\tanh\frac{t}{2}}$$

where $\tilde{H} = H/(\mathrm{i}\hbar)$ *and* $H = -\dfrac{x^2}{2} + \dfrac{y^2}{2}$.

4.2. $*$-Exponential in the General Case

We consider the $*$-exponential function for $H = ax^2 + 2bxy + cy^2$ with $a, b, c \in \mathbb{C}$ and $D = b^2 - ac \neq 0$.

Using Proposition 2, we have $H = 2\sqrt{D}\left(-\frac{1}{2}w^2 + \frac{1}{2}z^2\right)$, where $w = px + qy$, $z = rx + sy$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{C})$.

Notice that the commutator of w, z satisfies $[w, z] = \mathrm{i}\hbar$. Then the transformation formula in Proposition 3 yields

Theorem 2. *The $*$-exponential function of H is*

$$\mathrm{e}_*^{t\frac{H}{\mathrm{i}\hbar}} = \frac{1}{\cosh\sqrt{D}t}\exp\left(\frac{H}{\sqrt{D}}\tanh\sqrt{D}t\right).$$

Acknowledgements

This work is partially supported by Grant-in-Aid for Scientific Research (#13640088), Ministry of Education, Science and Culture, Japan. The authors would like to express also their thanks to Professor Kazuyuki Fujii for discussion and helpful comments.

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