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BASIC ASPECTS OF SOLITON THEORY

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Abstract. This is a review of the main ideas of the inverse scattering method (ISM) for solving nonlinear evolution equations (NLEE), known as soliton equations. As a basic tool we use the fundamental analytic solutions $\chi^{\pm}(x,\lambda)$ of the Lax operator $L(\lambda)$. Then the inverse scattering problem for $L(\lambda)$ reduces to a Riemann–Hilbert problem. Such construction has been applied to wide class of Lax operators, related to the simple Lie algebras. We construct the kernel of the resolvent of $L(\lambda)$ in terms of $\chi^{\pm}(x,\lambda)$ and derive the spectral decompositions of $L(\lambda)$. Thus we can solve the relevant classes of NLEE which include the NLS equation and its multi-component generalizations, the N-wave equations, etc. Applying the dressing method of Zakharov and Shabat we derive the N-soliton solutions of these equations.

Next we explain that the ISM is a natural generalization of the Fourier transform method. As appropriate generalizations of the usual exponential function we use the so-called "squared solutions" which are constructed again in terms of $\chi^{\pm}(x,\lambda)$ and the Cartan–Weyl basis of the relevant Lie algebra. One can prove the completeness relations for the "squared solutions" which in fact provide the spectral decompositions of the recursion operator Λ.

These decompositions can be used to derive all fundamental properties of the corresponding NLEE in terms of Λ: i) the explicit form of the class of integrable NLEE; ii) the generating functionals of integrals of motion; iii) the hierarchies of Hamiltonian structures. We outline the importance of the classical R-matrices for extracting he involutive integrals of motion.

1. Introduction

The modern development of the soliton theory in the last three decades of the 20th century has lead to a number of important applications and developments in several areas of contemporary physics and mathematics, see [41, 12, 6, 3, 2]. In this review I will outline the basic ideas of the **inverse scattering method** (ISM) on the