Geometry,
Integrability
and
Quantization VII

GEOMETRY AND TOPOLOGY OF FINITE-GAP VORTEX FILAMENTS

THOMAS A. IVEY

College of Charleston, Charleston, South Carolina 29424, USA

Abstract. In this talk, I will discuss the vortex filament flow, and its connection with the nonlinear Schrödinger equation (NLS), the geometry of solutions to the filament flow that correspond to finite-gap solutions of NLS, and in particular, the relationship between geometric properties of the filament to features of the Floquet spectrum of a periodic NLS potential. These will be illustrated by the example of elastic rod centerlines. I will also discuss the geometric implications of symmetric Floquet spectra, and the topological implications of certain isoperiodic deformations of the spectrum that open up real double points.

1. The Vortex Filament Flow

The **vortex filament flow** is an evolution equation for space curves, given by

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2} = \kappa \mathbf{B}, \qquad \gamma \in \mathbb{R}^3 \tag{1}$$

where x is arclength, κ is Frenet curvature, and $\mathbf{B}$ is the binormal. This was proposed in 1904 by da Rios (a student of Levi-Civita) as a model for self-induced motion of vortex lines in an incompressible fluid. An intuitive explanation of this model is that when a vortex line bends, the compression of streamlines inside the bend pushes the vortex line in a direction perpendicular to the osculating plane, with velocity proportional to the curvature. For example, circles translate with constant velocity (like a smoke ring). However, most planar initial curves immediately become non-planar under this flow.

1.1. The Hasimoto Map

The vortex filament flow is regarded as completely integrable because of Hasimoto's discovery [6] that one can map solutions of (1) to solutions of the nonlinear