THE BIHARMONIC STRESS-ENERGY TENSOR AND THE GAUSS MAP

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Abstract. We consider the energy and bienergy functionals as variational problems on the set of Riemannian metrics and present a study of the biharmonic stress-energy tensor. This approach is then applied to characterize weak conformality of the Gauss map of a submanifold. Finally, working at the level of functionals, we recover a result of Weiner linking Willmore surfaces and pseudo-umbilicity.

1. Introduction

The guiding principle of variational theory is that geometric objects can be selected according to whether or not they minimize certain functionals and, since Morse theory, critical points can prove sufficiency. Once this criterion is chosen, the adequate Euler–Lagrange equation will characterise maps particularly well adapted to our geometric framework. However, roles can be reversed and metrics can be viewed as variables and required to fit with a map and complete the picture. Other than the duality of these approaches, the theory of general relativity has put metrics firmly in centre of the stage and the characterisation of Einstein metrics as (constrained) critical points of the total curvature has created a new viewpoint on the usual functionals, in particular the various energies defined for maps between manifolds.

Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds of dimension m, respectively n. Assuming that M is compact we can define the

energy of ϕ to be

$$E(\phi) = \int_M e(\phi)\, v_g$$

where $e(\phi) = \frac{1}{2}|\mathrm{d}\phi|^2$ is (half) the Hilbert–Schmidt norm.

Call a map **harmonic** if it is a critical point of E, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} E(\phi_t) = 0$, for any smooth deformation $\{\phi_t\}$ of ϕ. The corresponding Euler–Lagrange equation characterizes harmonicity

$$\tau(\phi) = g^{ij}\left(\frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\sigma}\phi^\beta_i \phi^\sigma_j\right)\frac{\partial}{\partial y^\alpha} = 0$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\alpha_{\beta\sigma}$ are the Christoffel symbols of g and h.

On non-compact manifolds, this equation serves as definition.

If M is compact, the set $\mathcal{G}$ of Riemannian metrics on M is an infinite dimensional manifold and its tangent space at g is identified with symmetric $(0,2)$-tensors

$$T_g\mathcal{G} = C(\odot^2 T^* M).$$

For a deformation $\{g_t\}$ of g we denote $\omega = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} g_t \in T_g\mathcal{G}$.

Now, fix $\phi : M \to (N, h)$ and define the functional $\mathcal{F} : \mathcal{G} \to \mathbb{R}$ by

$$\mathcal{F}(g) = E(\phi)$$

where $E(\phi)$ is computed with respect to the metrics g and h.

Sanini obtained the Euler–Lagrange equation for $\mathcal{F}$.

Theorem 1 ([11]). *Let $\phi : M \to (N, h)$ and assume that M is compact, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{F}(g_t) = \frac{1}{2}\int_M \langle \omega, e(\phi)g - \phi^* h\rangle\, v_g$$

*so g is a critical point of $\mathcal{F}$ if and only if the **stress-energy tensor** $S = e(\phi)g - \phi^* h$ vanishes.*

This naturally extends into a definition on non-compact domains and Baird and Eells have proved

Theorem 2 ([1]). *Let $\phi : (M, g) \to (N, h)$ be a map between Riemannian manifolds, then*

$$\mathrm{Div}\, S(X) = -\langle \tau(\phi), \mathrm{d}\phi(X)\rangle \quad \text{for all} \quad X \in C(TM).$$

Therefore, if ϕ is harmonic then $\mathrm{Div}\, S = 0$.

The vanishing of S is a strong condition which can be spelled out as

Theorem 3 ([1, 11]). *Let $\phi : (M, g) \to (N, h)$. Then $S = 0$ if and only if either $m = 2$ and ϕ is conformal, or $m > 2$ and ϕ is constant.*

Note that a homothetic transformation of the domain can render $\mathcal{F}$ arbitrarily large or small, since $\mathcal{F}(tg) = t^{\frac{m-2}{2}}\mathcal{F}(g)$, for a positive constant t. To avoid this, impose $\mathrm{vol}(M, g_t) = \mathrm{vol}(M, g)$, i.e., $\{g_t\}$ is an isovolumetric deformation, in this case ω is orthogonal to g as vectors in $T_g\mathcal{G}$, i.e.,

$$(\omega, g) = \int_M \langle \omega, g \rangle \, v_g = 0$$

and g is a critical point of $\mathcal{F}$ with respect to isovolumetric deformations of g if and only if $S = \lambda g$, where λ is a real constant.

Theorem 4 ([11]). *Let $\phi : (M, g) \to (N, h)$. Then $S = \lambda g$ if and only if either $m = 2$ and ϕ is conformal, or $m > 2$ and ϕ is a homothety.*

2. The Biharmonic Case

Let $\phi : (M, g) \to (N, h)$ be a smooth map between the Riemannian manifolds M and N. Assuming that M is compact one can define the **bienergy** of ϕ by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g.$$

The map ϕ is called **biharmonic** if it is a critical point of E_2 and Jiang derived its Euler–Lagrange equation.

Theorem 5 ([4]). *Let $\phi : (M, g) \to (N, h)$ and assume M compact. Then ϕ is biharmonic if and only if*

$$\tau_2(\phi) = -\Delta\tau(\phi) - \mathrm{trace}\, R^N(\mathrm{d}\phi, \tau(\phi)) \, \mathrm{d}\phi = 0.$$

In this paper we use the sign conventions $\Delta\sigma = -\mathrm{trace}\,\nabla\mathrm{d}\sigma$, $\sigma \in C(\phi^{-1}TN)$, and $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

Obviously, any harmonic map is biharmonic, therefore we are interested in non-harmonic biharmonic maps, which we call **proper biharmonic**.

Two examples of proper biharmonic maps are:

1. The inclusion $\mathbf{i} : \mathbb{S}^n(\frac{1}{\sqrt{2}}) \to \mathbb{S}^{n+1}$ is proper biharmonic.

2. Let $\psi : M \to \mathbb{S}^n(\frac{1}{\sqrt{2}})$ be a harmonic map with $e(\psi)$ constant. Then the composition map $\phi = \mathbf{i} \circ \psi$ is proper biharmonic.

For an account of biharmonic maps see [8] and *The bibliography of biharmonic maps* [6].

To a map $\phi : (M, g) \to (N, h)$, Jiang associates in [5] the symmetric $(0, 2)$ tensor

$$S_2(X, Y) = \left(\tfrac{1}{2}|\tau(\phi)|^2 + \langle \mathrm{d}\phi, \nabla\tau(\phi)\rangle\right)\langle X, Y\rangle$$
$$- \langle \mathrm{d}\phi(X), \nabla_Y\tau(\phi)\rangle - \langle \mathrm{d}\phi(Y), \nabla_X\tau(\phi)\rangle$$

and proved
$$\operatorname{Div} S_2(X) = -\langle \tau_2(\phi), \mathrm{d}\phi(X) \rangle. \tag{2.1}$$
Therefore, if $\tau_2(\phi) = 0$ then $\operatorname{Div} S_2 = 0$.

As for harmonic maps, the expression of S_2 can be deduced from a variational problem.

Theorem 6 ([7]). *Fix $\phi : M \to (N, h)$, assume M compact and define $\mathcal{F}_2 : \mathcal{G} \to \mathbb{R}$ to be*
$$\mathcal{F}_2(g) = E_2(\phi)$$
then
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_2(g_t) = -\frac{1}{2} \int_M \langle \omega, S_2 \rangle \, v_g.$$
So g is a critical point of $\mathcal{F}_2$ if and only if $S_2 = 0$.

From (2.1) we obtain

Proposition 1. *If $\phi : (M, g) \to (N, h)$ is*:

    a) *a Riemannian immersion, then $\operatorname{Div} S_2 = 0$ if and only if $\tau_2(\phi)$ is normal.*

    b) *a submersion (not necessarily Riemannian), then $\operatorname{Div} S_2(\phi) = 0$ if and only if $\tau_2(\phi) = 0$.*

This allows us to obtain new examples of proper biharmonic maps.

Proposition 2 ([7]). *Let $\phi : (M, g) \to (N, h)$ be a submersion with basic tension field, i.e., $\tau(\phi) = \xi \circ \phi$, $\xi \in C(TN)$, and ξ is a Killing vector field. If M is compact then ϕ is harmonic, while if M is non-compact then ϕ is a proper biharmonic if and only if the norm of ξ is constant (non-zero).*

Example 1. Let (M^m, g) and (N^n, h) be Riemannian manifolds and $f \in C^\infty(M)$ a positive function. Consider the warped product manifold $M \times_{f^2} N$, then the projection π onto the first term is a Riemannian submersion and $\tau(\pi) = n \operatorname{grad}(\ln f) \circ \pi$. If $\ln f$ is an affine function on M then $\operatorname{grad}(\ln f)$ is a Killing vector field of constant norm and π is biharmonic.

Example 2. For any vector field ξ, the tangent bundle TM can be endowed with a Sasaki-type metric such that the canonical projection is a Riemannian submersion and $\tau(\pi) = -(m + 1)\xi \circ \pi$ ([9]). If ξ is Killing of constant norm then π is biharmonic.

If $\tau(\phi) = 0$ then $S_2 = 0$ but the converse, i.e., $S_2 = 0$ (a critical point of $\mathcal{F}_2$) implies $\tau(\phi) = 0$ (an absolute minimum of $\mathcal{F}_2$) is less straight-forward. Note that, in general, $S_2 = 0$ does not imply harmonicity; for example, the non-geodesic curve $\gamma(t) = t^3 a$, $a \in \mathbb{R}^n$, has $S_2 = 0$. Remember also that for harmonicity, when $m > 2$, $S = 0$ implies ϕ constant.

The vanishing of S_2 implies harmonicity in some situations (cf. [7]):

1) curves parametrized by the arc-length

2) $\phi : (M^2, g) \to (N, h)$

3) $\phi : (M^m, g) \to (N, h)$, $m > 2$, and $\operatorname{rank} \phi \leq m - 1$

4) $\phi : (M^m, g) \to (N, h)$, $m > 2$, and ϕ submersion

5) $\phi : (M^m, g) \to (N, h)$, $m \neq 4$, M compact ([4])

6) $\phi : (M^m, g) \to (N, h)$ Riemannian immersion, $m \neq 4$.

Dimension four plays a special role for the domain manifold, as we can see from the followings

Theorem 7 ([5]). *Let* $\phi : (M^4, g) \to (N, h)$ *be a non-minimal Riemannian immersion, then* $S_2 = 0$ *if and only if* ϕ *is pseudo-umbilical.*

To generalize this result, we have to consider conformal immersions:

Proposition 3 ([7]). *Let* $\phi : (M^4, g = e^{2\rho} \phi^* h) \to (N, h)$ *be a conformal immersion,* M *compact. Then* $S_2 = 0$ *if and only if* ρ *is constant and* $\overline{\phi} : (M^4, \phi^* h) \to (N, h)$ *is pseudo-umbilical.*

Proposition 4 ([7]). *Let* $\phi : (M^4, g) \to (N^4, h)$ *be a local diffeomorphism, i.e.,* $\operatorname{rank} \phi = 4$, M *compact. Then* $S_2 = 0$ *if and only if* $\tau(\phi) = 0$.

Proposition 5 ([7]). *Let* $\phi : (M^4, g) \to (N, h)$ *be a map such that* $\operatorname{rank} \phi \leq 3$. *Then* $S_2 = 0$ *if and only if* $\tau(\phi) = 0$.

Then we consider the deformations which preserve the domain metric:

Theorem 8 ([7]). *Let* $\phi : (M^m, g) \to (N, h)$ *be a Riemannian immersion. Then* $S_2 = \lambda g$ *if and only if either* $m = 4$ *and* ϕ *is pseudo-umbilical, or* $m \neq 4$ *and* ϕ *is pseudo-umbilical with* $|\tau(\phi)|$ *constant.*

We end this section with the study of the behaviour of S_2 under conformal changes of the domain metric.

Proposition 6. *Consider* $\phi : (M^m, g) \to (N^n, h)$, $\tilde{\phi} : (M, \tilde{g} = tg) \to (N, h)$, $\phi = \tilde{\phi} \circ 1$, *where* $1 : (M, g) \to (M, \tilde{g})$ *is the identity map and* t *is a positive constant. Then* $\tilde{S}_2 = \frac{1}{t} S_2$, *therefore* $\tilde{S}_2 = 0$ *if and only if* $S_2 = 0$.

For surfaces we get

Proposition 7. *Let* $\phi : (M^2, g) \to (N^n, h)$ *and* $\tilde{\phi} : (M, \tilde{g} = e^{2\rho} g) \to (N, h)$, $\phi = \tilde{\phi} \circ 1$, $\rho \in C^\infty(M)$:

 a) $\tilde{S}_2 = 0$ *if and only if* $S_2 = 0$ *and, in this case, the maps are harmonic.*

 b) *if* $\langle \tau(\phi), d\phi(X) \rangle = 0$ *for all* $X \in C(TM)$, *then* $\tilde{S}_2 = e^{-2\rho} S_2$.

For domains of higher dimension we obtain two "rigidity" results:

Proposition 8. *Let M^m be compact, $m > 2$, $m \neq 4$. Consider $\phi : (M^m, g) \to (N^n, h)$ such that $\langle \tau(\phi), \mathrm{d}\phi(X) \rangle = 0$ for all $X \in C(TM)$ and $\tilde{\phi} : (M, \tilde{g} = \mathrm{e}^{2\rho} g) \to (N, h)$. Then $\tilde{S}_2 = 0$ if and only if $\mathrm{d}\phi(\mathrm{grad}\,\rho) = 0$ and $S_2 = 0$, and both maps must then be harmonic. When ϕ is a Riemannian immersion, $\tilde{S}_2 = 0$ if and only if ρ is constant and $S_2 = 0$.*

Proposition 9. [7] *Let $\phi : (M^4, g) \to (N^n, h)$ be a non-minimal Riemannian immersion and assume that M is compact. Let $\tilde{\phi} : (M, \tilde{g} = \mathrm{e}^{2\rho} g) \to (N, h)$, then $\tilde{S}_2 = 0$ if and only if ρ is constant and $S_2 = 0$. In this case ϕ is pseudo-umbilical.*

3. The Tensor S_2 and the Gauss Map

Let M^m be an oriented submanifold of $\mathbb{R}^n$, $p \in M$ an arbitrary point and $\{X_i\}_{i=1}^m$ a positive oriented geodesic basis centered around p. On a neighbourhood U of p, the Gauss map associated to M can be written

$$G : M \to G(n, m)$$
$$G(q) = X_1(q) \wedge \cdots \wedge X_m(q) \quad \text{for all} \quad q \in U.$$

Since

$$\mathrm{d}G_q(X_i) = \sum_{j=1}^m X_1(q) \wedge \cdots \wedge X_{j-1}(q) \wedge \left(\nabla^0_{X_i} X_j \right)(q) \wedge X_{j+1}(q) \wedge \cdots \wedge X_m(q)$$

where ∇^0 is the canonical connection on $\mathbb{R}^n$, at p we have

$$\mathrm{d}G_p(X_i) = \sum_{j=1}^m X_1(p) \wedge \cdots \wedge X_{j-1}(p) \wedge B_p(X_i, X_j) \wedge X_{j+1}(p) \wedge \cdots \wedge X_m(p)$$

where B denotes the second fundamental form of M.

Complete $\{X_i(p)\}_{i=1}^m$ to an orthonormal basis $\{X_\alpha(p)\}_{\alpha=1}^n$ of $\mathbb{R}^n$. Let $\alpha \in \{1, \ldots, n\}$ and $a \in \{m+1, \ldots, n\}$, then

$$B_p(X_i, X_j) = \sum_a b_{ij}^a(p) X_a(p)$$

and

$$\mathrm{d}G_p(X_i) = \sum_a \sum_j b_{ij}^a(p) X_1(p) \wedge \cdots \wedge X_{j-1}(p) \wedge X_a(p) \wedge X_{j+1}(p) \wedge \cdots \wedge X_m(p).$$

Now, the m-subspace $X_1(p) \wedge \cdots \wedge X_{j-1}(p) \wedge X_a(p) \wedge X_{j+1}(p) \wedge \cdots \wedge X_m(p)$, can be identified with $X_j^*(p) \otimes X_a(p)$ ([3]), so that

$$\mathrm{d}G_p(X_i) = \sum_a \sum_j b_{ij}^a(p) X_j^*(p) \otimes X_a(p).$$

The canonical metric g_{can} on $G(n, m)$ is defined by requiring that

$$\{X_j^*(p) \otimes X_a(p) : j = 1, \ldots, m, \ a = m + 1, \ldots, n\}$$

is an orthonormal basis of $T_{G(p)}G(n, m)$. By direct computation, we obtain

$$g_{\text{can}}(\mathrm{d}G_p(X_i), \mathrm{d}G_p(X_k)) = \sum_j \langle B_p(X_i, X_j), B_p(X_k, X_j) \rangle$$

where $\langle \, , \rangle$ is the canonical metric on $\mathbb{R}^n$. By the Gauss Lemma

$$g_{\text{can}}(\mathrm{d}G_p(X_i), \mathrm{d}G_p(X_k)) = -\operatorname{Ricci}_p(X_i, X_k) + m\langle H(p), B_p(X_i, X_k)\rangle$$

where H is the mean curvature vector field. Therefore

$$(G^* g_{\text{can}})(p) = m\langle H(p), B_p\rangle - \operatorname{Ricci}_p \,.$$

Now

$$S^G = e(G)g - G^* g_{\text{can}} = (\operatorname{Ricci} - \frac{r}{2}g) + \frac{m^2}{2}|H|^2 g - m\langle H, B\rangle$$

$$= (\operatorname{Ricci} - \frac{r}{2}g) + \frac{1}{2}|\tau(\mathbf{i})|^2 g - \langle \tau(\mathbf{i}), \nabla \mathrm{d}\mathbf{i}\rangle$$

$$= (\operatorname{Ricci} - \frac{r}{2}g) - \frac{1}{2}S_2^{\mathbf{i}} + \frac{1}{4}|\tau(\mathbf{i})|^2 g$$

where $g = \langle \, , \rangle$, $\mathbf{i}$ is the canonical inclusion of M in $\mathbb{R}^n$ and $r = \operatorname{trace} \operatorname{Ricci}$ is the scalar curvature.

Proposition 10. *Assume M^2 is an orientable surface in $\mathbb{R}^n$, then the following conditions are equivalent:*

   a) $S^G = 0$

   b) *G is weakly conformal*

   c) *M^2 is pseudo-umbilical*

   d) $S_2^{\mathbf{i}} = \frac{1}{2}|\tau(\mathbf{i})|^2 g.$

Proposition 11. *Assume that $m > 2$, then any two of the following statements implies the third:*

   a) $S_2^{\mathbf{i}} = fg$, *where $f \in C^\infty(M)$*

   b) *M is Einstein*

   c) *G is weakly conformal.*

Remark 1. *We have also*

   a) *if $S_2^{\mathbf{i}} = fg$ and G is weakly conformal then $S_2^{\mathbf{i}} = \frac{4-m}{2m}|\tau(\mathbf{i})|^2 g$, $G^* g_{\text{can}} = \frac{2}{m}e(G)g$ and*

$$\operatorname{Ricci} = \frac{|\tau(\mathbf{i})|^2 - 2e(G)}{m}g$$

i.e., M is Einstein. Moreover, in this case, $r = |\tau(\mathbf{i})|^2 - 2e(G)$ must be constant.

b) *if $S_2^{\mathbf{i}} = fg$ and $\mathrm{Ricci} = cg$, c constant, then G is weakly conformal and*

$$e(G) = \frac{|\tau(\mathbf{i})|^2 - mc}{2}.$$

Moreover, in this case, $|\tau(\mathbf{i})|^2 - mc \geq 0$, and, if M has constant mean curvature then G is homothetic. We conclude that if M^m, $m > 2$, is an Einstein pseudo-umbilical submanifold of $\mathbb{R}^n$, with constant mean curvature when $m \neq 4$, then its Gauss map is homothetic.

Since $\mathrm{Div}\left(\mathrm{Ricci} - \frac{r}{2}g\right) = 0$ we re-obtain Jiang's result:

Theorem 9 ([5])**.** *Let M^m be an oriented submanifold of $\mathbb{R}^n$. Then the tensors S^G and $S_2^{\mathbf{i}}$ are related by*

$$\mathrm{Div}\, S^G + \frac{1}{2}\,\mathrm{Div}\, S_2^{\mathbf{i}} - \frac{1}{4}\mathrm{d}(|\tau(\mathbf{i})|^2) = 0.$$

Since Ruh and Vilms [10] proved that G is harmonic if and only if the mean curvature vector field is parallel, we conclude:

Corollary 1. *Let M^m be an oriented submanifold of $\mathbb{R}^n$, then:*

a) *if the manifold M has constant mean curvature, then $\mathrm{Div}\, S_2^{\mathbf{i}} = 0$ if and only if $\mathrm{Div}\, S^G = 0$*

b) *if G is harmonic then $\mathrm{Div}\, S_2^{\mathbf{i}} = 0$.*

4. On a Result of Weiner

Inspired by the above technique on the Gauss map, we conclude with a result on Willmore surfaces in $\mathbb{R}^n$ due to Weiner [12].

Let $\phi : (M, g) \to \mathbb{R}^n$ be a Riemannian immersion, i.e., $g = \phi^*\langle\,,\,\rangle$, assume M oriented. We have

$$G^* g_{\mathrm{can}} = m\langle H, B\rangle - \mathrm{Ricci} = \langle \tau(\phi), \nabla\mathrm{d}\phi\rangle - \mathrm{Ricci}$$

and

$$e(G) = \frac{1}{2}m^2|H|^2 - \frac{1}{2}r.$$

Assume $m = 2$, therefore

$$e(G) = 2|H|^2 - K$$

where K is the Gaussian curvature of (M, g), and integrating,

$$\int_M e(G)\, v_g = 2\int_M |H|^2\, v_g - 2\pi\chi(M).$$

Consider a one-parameter family of immersions $\{\phi_t\}$, $\phi_0 = \phi$ such that $\phi_t :$ $(M, g_t) \to \mathbb{R}^n$ is a Riemannian immersion, i.e., $g_t = \phi_t^* \langle , \rangle$. All previous formulas hold for ϕ_t, so, for any t

$$\int_M e(G_t)\, v_{g_t} = 2 \int_M |H_t|^2\, v_{g_t} - 2\pi\chi(M).$$

The right-hand side consists of the Willmore functional plus the Euler–Poincaré characteristic, a topological invariant. Compute

$$W = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M e(G_t)\, v_{g_t} = 2 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M |H_t|^2\, v_{g_t}.$$

Put $h = g_{\mathrm{can}}$, then

$$
\begin{aligned}
2W =\ & \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M g^{ij}(x,t) G_i^\alpha(x,t) h_{\alpha\beta}(G(x,t)) G_j^\beta(x,t)\, v_{g_t(x)} \\
=\ & \int_M \frac{\partial g^{ij}}{\partial t}(x,0) G_i^\alpha(x) h_{\alpha\beta}(G(x)) G_j^\beta(x)\, v_g \\
& + \int_M g^{ij}(x) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\{ G_i^\alpha(x,t) h_{\alpha\beta}(G(x,t)) G_j^\beta(x,t) \right\}\, v_g \\
& + \int_M g^{ij}(x) G_i^\alpha(x) h_{\alpha\beta}(G(x)) G_j^\beta(x) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v_{g_t(x)}.
\end{aligned}
$$

Let

$$W_1 = \int_M g^{ij}(x) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\{ G_i^\alpha(x,t) h_{\alpha\beta}(G(x,t)) G_j^\beta(x,t) \right\}\, v_g.$$

so

$$
\begin{aligned}
2W =\ & \int_M \frac{\partial g^{ij}}{\partial t}(x,0) G_i^\alpha(x) h_{\alpha\beta}(G(x)) G_j^\beta(x)\, v_g \\
& + \int_M \left(4|H|^2 - r \right) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v_{g_t(x)} + W_1.
\end{aligned}
$$

Recall that

$$\frac{\partial g^{ij}}{\partial t}(x,0) = -g^{ik} g^{jl} \omega_{kl} \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v_{g_t(x)} = \langle \tfrac{1}{2} g, \omega \rangle v_g.$$

Replacing we obtain

$$
\begin{aligned}
2W =\ & -\int_M \langle \omega, G^* h \rangle\, v_g + \int_M \left(4|H|^2 - r \right) \langle \tfrac{1}{2} g, \omega \rangle\, v_g + W_1 \\
=\ & \int_M \langle 2|H|^2 g - 2\langle H, B \rangle, \omega \rangle\, v_g + W_1.
\end{aligned}
$$

Clearly if G is harmonic (so $W_1 = 0$) and M^2 pseudo-umbilical in $\mathbb{R}^n$ (i.e., $|H|^2 g - \langle H, B \rangle = 0$) then it is Willmore.

To obtain a (partial) converse of the above statement, we first establish the link between ω and $V = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi_t$, which we assume to be a normal vector field. Since $\omega = \frac{\partial g_{ij}}{\partial t}(x,0)\,\mathrm{d}x^i\mathrm{d}x^j$ and $g_{ij}(x,t) = \sum_{\alpha=1}^{n}\Phi_i^{\alpha}(x,t)\Phi_j^{\alpha}(x,t)$, $i,j = 1,2$, then

$$\frac{\partial g_{ij}}{\partial t}(x,0) = 2\sum_{\alpha}\frac{\partial^2\Phi^{\alpha}}{\partial x^i\partial t}(x,0)\phi_j^{\alpha}(x) = 2\sum_{\alpha}\frac{\partial V^{\alpha}}{\partial x^i}(x)\phi_j^{\alpha}(x)$$

$$= 2\langle\nabla_{\partial_i}V,\mathrm{d}\phi(\partial_j)\rangle = -2\langle V,\nabla_{\partial_i}\mathrm{d}\phi(\partial_j)\rangle$$

$$= -2\langle V,\nabla\mathrm{d}\phi(\partial_i,\partial_j) - \mathrm{d}\phi(\nabla_{\partial_i}\partial_j)\rangle = -2\langle V,B(\partial_i,\partial_j)\rangle$$

and hence $\omega = -2\langle V,B\rangle$, where $\langle V,B\rangle(X,Y) = \langle V,B(X,Y)\rangle$. Therefore

$$\langle |H|^2 g - \langle H,B\rangle,\omega\rangle = -2\langle |H|^2 g - \langle H,B\rangle,\langle V,B\rangle\rangle$$

$$= -2|H|^2\langle g,\langle V,B\rangle\rangle + 2\langle\langle H,B\rangle,\langle V,B\rangle\rangle$$

but

$$\langle g,\langle V,B\rangle\rangle = \sum_i\langle V,B(X_i,X_i)\rangle = m\langle V,H\rangle = 2\langle V,H\rangle$$

and

$$\langle\langle H,B\rangle,\langle V,B\rangle\rangle = \sum_{i,j}\langle H,B(X_i,X_j)\rangle\langle V,B(X_i,X_j)\rangle$$

$$= \sum_{i,j}\left(\sum_a H^a B^a(X_i,X_j)\right)\left(\sum_b V^b B^b(X_i,X_j)\right)$$

$$= \sum_b\left(\sum_{i,j,a}H^a B^a(X_i,X_j)B^b(X_i,X_j)\right)V^b$$

where $a,b = 3,\ldots,n$. On the other hand, the contraction $\langle\langle H,B\rangle,B\rangle$ is the normal vector field defined by

$$\sum_{i,j}\langle H,B(X_i,X_j)\rangle B(X_i,X_j) = \sum_b\sum_{i,j}\left(\sum_a H^a B^a(X_i,X_j)\right)B^b(X_i,X_j)\eta^b$$

where $\{\eta^b\}$ is a normal frame, therefore

$$\langle\langle\langle H,B\rangle,B\rangle,V\rangle = \sum_b\left(\sum_{i,j}\left(\sum_a H^a B^a(X_i,X_j)\right)B^b(X_i,X_j)\right)V^b.$$

Hence $\langle\langle H,B\rangle,\langle V,B\rangle\rangle = \langle\langle\langle H,B\rangle,B\rangle,V\rangle$ and

$$\langle |H|^2 g - \langle H,B\rangle,\omega\rangle = -4|H|^2\langle H,V\rangle + 2\langle\langle\langle H,B\rangle,B\rangle,V\rangle$$

$$= \langle -4|H|^2 H + 2\langle\langle H,B\rangle,B\rangle,V\rangle.$$

This shows that if we assume that G is harmonic and M^2 is Willmore then

$$\int_M \langle -4|H|^2 H + 2\langle\langle H, B\rangle, B\rangle, V\rangle \, v_g = 0$$

for all normal variations V as required by the Willmore problem, and, therefore, we have $-4|H|^2 H + 2\langle\langle H, B\rangle, B\rangle = 0$. To conclude we need to show that $-2|H|^2 H + \langle\langle H, B\rangle, B\rangle = 0$, or, since $H = \frac{1}{2}\langle g, B\rangle$, $\langle -|H|^2 g + \langle H, B\rangle, B\rangle = 0$, implies $-|H|^2 g + \langle H, B\rangle = 0$, i.e., M^2 is pseudo-umbilical.

Decompose B into its trace and traceless parts: $B = H \otimes g + S$, with trace $S = 0$, then M^2 is pseudo-umbilical if and only if $\langle S, H\rangle = 0$ (umbilical being $S = 0$). Then

$$
\begin{aligned}
0 &= \langle -|H|^2 g + \langle H, B\rangle, B\rangle \\
&= \langle -|H|^2 g + \langle H, H \otimes g + S\rangle, H \otimes g + S\rangle \\
&= \langle -|H|^2 g + |H|^2 g + \langle H, S\rangle, H \otimes g + S\rangle \\
&= \langle H, \text{trace}\, S\rangle H + \sum_{i,j} \langle H, S(X_i, X_j)\rangle S(X_i, X_j)
\end{aligned}
$$

therefore $\sum_{i,j} \langle H, S(X_i, X_j)\rangle S(X_i, X_j) = 0$ and taking its inner-product with H, yields $\langle S, H\rangle = 0$.

Therefore we recover (part of) Weiner's result:

Theorem 10 ([12]). *Let $\phi : M^2 \to \mathbb{R}^n$ be a Riemannian immersion of a compact oriented surface into $\mathbb{R}^n$, such that its Gauss map is harmonic. Then M^2 is a Willmore surface if and only if it is pseudo-umbilical.*

Remark 2. *Recall Chen and Yano's result [2]: A submanifold of $\mathbb{R}^n$ is pseudo-umbilical with parallel mean curvature vector field if and only if it is minimal in a hypersphere of $\mathbb{R}^n$. So a minimal surface in $\mathbb{S}^{n+1}$ is a Willmore surface of $\mathbb{R}^n$.*

Remark 3. *The only compact oriented Riemannian immersed Willmore surface in $\mathbb{R}^3$ of constant mean curvature is the sphere.*

Acknowledgements

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