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LIE TRANSFORMATION GROUPS AND GEOMETRY

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Abstract. We present geometrical aspects of Lie groups and reductive homogeneous spaces, and some resent results on homogeneous geodesics and homogeneous Einstein metrics. The article is based on the four lectures given in Varna, June 2007.

Contents

1. Introduction

The present article summarizes most of the four lectures that I have presented during the Varna Conference on Geometry, Integrability and Quantization, June 2007. They are based on my book *An Introduction to Lie groups and the Geometry of Homogeneous Spaces* [4], with additional recent results on homogeneous geodesics and homogeneous Einstein metrics.

The theory of Lie groups (i.e., a manifold with a group structure) is one of the classical well established areas of mathematics. It made its appearance at the end of the nineteenth century in the works of S. Lie, whose aim was to apply algebraic methods to differential equations and geometry. During the past one hundred years the concepts and methods of the theory of Lie groups entered into many areas of mathematics and theoretical physics.

The basic method of the theory of Lie groups, which makes it possible to obtain deep results with striking simplicity, consists in reducing questions concerning Lie groups to certain problems of linear algebra. This is done by assigning to every Lie group G its "tangent algebra" $\mathfrak{g}$. This is the tangent space of G at the identity element e, equipped with a natural Lie algebra structure. To a large extend, the Lie algebra $\mathfrak{g}$ determines the group G, and for every homomorphism $f : G \to H$ of Lie groups, a homomorphism $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ of their Lie algebras determines f to a large extend.

The question "what is geometry?" is a question that was emerged through the various attempts to prove Euclid's fifth postulate. After C. F. Gauss' *Theorema Egregium* (curvature is an intrinsic property of a surface) there were two main directions in the development of geometry. The first, was the theory of Riemannian manifolds, developed by B. Riemann, and is a generalization of Gauss' theory of surfaces.

The other direction was developed by F. Klein in his *Erlangen program*, according to which the object of geometry is a G-space M, that is a set M with a given group G of transformations. If the group acts transitively, that is for all $p, q \in M$ there exists an element in G which transforms p into q, then the G-space is called **homogeneous**. As a result, if we pick any point $o \in M$, we can identify M with the set G/H of left cosets, where H is the subgroup of G consisting of those elements which map o to itself. Therefore, the homogeneous geometry of such a space $M = G/H$ is the study of those geometrical properties and of those subsets of M, which are invariant under G. By varying the group G, we obtain different geometries (e.g., Euclidean, affine, projective, etc). As a result, if we know the value of a geometrical object (e.g., curvature) at a point of M, then we can calculate it at any other point.

Using the identification of a homogeneous space M with the quotient G/H, several geometrical problems can be reformulated in terms of the group G and the subgroup H. In particular, if G and H are Lie groups the problems can be further reformulated in terms of their infinitesimal objects, i.e., the Lie algebra $\mathfrak{g}$ of G and its Lie subalgebra $\mathfrak{h}$ associated to H. The major benefit of such an infinitesimal approach is that difficult nonlinear problems (from geometry, analysis or differential equations) can be reduced to linear algebra. This is essentially done by use of the canonical isomorphism $T_o(G/H)$, of the tangent space of G/H at the identity coset $o = eH$, with the quotient $T_e(G)/T_e(H) = \mathfrak{g}/\mathfrak{h}$.

After Cartan's classification of semisimple Lie groups, two important classes of homogeneous spaces were classified, namely symmetric spaces and flag manifolds. Flag manifolds are adjoint orbits of a compact semisimple Lie group, and equivalently homogeneous spaces of the form $G/C(T)$, where T is a torus in G. They have many applications in real and complex analysis, topology, geometry, dynamical systems, and physics.

There is actually a third direction in the development of geometry, which is Cartan's theory of connections on a fiber bundle (*espàces généralizés*), which was used as an appropriate mathematical framework in recent physical theories (Yang-Mills theory, quantum gravity). I will not enter in this topic.

The object of these lectures was to present some aspects of Lie groups and homogeneous spaces, as well as their geometrical objects defined on them, such as invariant metrics and curvature. As an application of the theory, I included a section in homogeneous Einstein metrics with some old and new results. I had in mind an audience of graduate students with a background on linear algebra and an introductory course on differential manifolds.

2. Lie Groups

A Lie group is a an abstract group with a smooth structure.

Definition 1. *A set G is a Lie group if and only if*

1) *G is a group*
2) *G is a smooth manifold*
3) *The operation $G \times G \to G, \quad (x, y) \mapsto xy^{-1}$ is smooth.*

Examples 1. 1) The sets $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ are abelian Lie groups under addition.

2) The sets $\mathbb{R}^*$, $\mathbb{C}^*$, $\mathbb{H}^*$ are Lie groups under multiplication. The first two are abelian, the third is not.

3) The set $\mathrm{M}_n\mathbb{R}$ of all $n \times n$ real matrices (respectively $\mathrm{M}_n\mathbb{C}$, $\mathrm{M}_n\mathbb{H}$) which is identified with the set $\mathrm{End}(\mathbb{R}^n)$ (respectively $\mathrm{End}(\mathbb{C}^n)$, $\mathrm{End}(\mathbb{H}^n)$) of all endomorphisms (i.e., linear maps) of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$, $\mathbb{H}^n$).

4) The set $\mathrm{GL}_n\mathbb{R}$ of all invertible real matrices, which is identified with the set $\mathrm{Aut}(\mathbb{R}^n)$ of all automorphisms of $\mathbb{R}^n$. Similarly we can define the Lie groups $\mathrm{GL}_n\mathbb{C}$ and $\mathrm{GL}_n\mathbb{H}$.

5) The circle $\mathbb{S}^1 \subset \mathbb{C}^*$ and the three-sphere $\mathbb{S}^3 \subset \mathbb{H}^*$.

6) The torus $\mathbb{S}^1 \times \mathbb{S}^1$.

In general, if G and H are Lie groups then the product $G \times H$ is also a Lie group.

To obtain more examples we need the following notion.

Definition 2. a) *A **Lie subgroup** H of a Lie group G is an abstract subgroup of G which is also an immersed submanifold of G.*

b) *A **closed subgroup** of a Lie group G is an abstract subgroup and a closed subset of G.*

Proposition 1 (Cartan)**.** *If H is a closed subgroup of a Lie group G, then H is a submanifold, so a Lie subgroup of G. In particular, it has the induced topology.*

It is possible to have a Lie subgroup which is not a closed subset. The standard example is the line of irrational slope $\phi : \mathbb{R} \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1$, $t \mapsto (\mathrm{e}^{2\pi \mathrm{i} t}, \mathrm{e}^{2\pi \mathrm{i} \alpha t})$, α irrational. The map ϕ is an one to one homomorphism, and an immersion. It is known that its image is a dense subset of the torus, so it is not an embedding (e.g., [12]).

By use of the above proposition we can obtain more examples of Lie groups.

7) The orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}_n\mathbb{R};\, AA^t = I\}$. By using the implicit function theorem we obtain that the dimension of $\mathrm{O}(n)$ is $\frac{1}{2}n(n-1)$.
8) The unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}_n\mathbb{C};\, AA^* = I\}$ and the symplectic group $\mathrm{Sp}(n) = \{A \in \mathrm{GL}_n\mathbb{H};\, AA^* = I\}$. Their dimensions are n^2 and $2n^2 + n$ respectively.
9) The special orthogonal group $\mathrm{SO}(n)$, and the special unitary groups $\mathrm{SU}(n)$ consisting of matrices in $\mathrm{O}(n)$ and $\mathrm{U}(n)$ of determinant 1.

Subgroups of $\mathrm{GL}_n\mathbb{K}$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are known as the *classical groups*.

We have the following simple isomorphisms: $\mathrm{SO}(1) \cong \mathrm{SU}(1) \cong \{I\}$, $\mathrm{O}(1) \cong \mathbb{S}^0 = \mathbb{Z}_2$, $\mathrm{U}(1) \cong \mathrm{SO}(2) \cong \mathbb{S}^1$, $\mathrm{SU}(2) \cong \mathbb{S}^3 \cong \mathrm{Sp}(1)$.

A result of Hopf states that $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ are the only spheres that admit a Lie group structure.

2.1. The Tangent Space of a Lie Group – Lie Algebras

There are two important maps in a Lie group G, called *translations*.

For $a \in G$, we define the **left translation** $L_a : G \to G$ by $g \mapsto ag$ and the **right translation** $R_a : G \to G$ by $g \mapsto ga$. These maps are diffeomorphisms, and can be used to get around in a Lie group. In fact, any $a \in G$ can be moved to the identity element e by $L_{a^{-1}}$, and $(\mathrm{d}L_{a^{-1}})_a : T_aG \to T_eG$ is a vector space isomorphism.

Proposition 2. *Any Lie group is G parallelizable, i.e., its tangent bundle is trivial.*

Proof: The map $X_g \mapsto (g, \mathrm{d}L_{g^{-1}}X_g)$ gives the desired isomorphism $TG \cong G \times T_eG$. □

Definition 3. *A vector field X on a Lie group G is called* ***left-invariant*** *if $X \circ L_a = \mathrm{d}L_a(X)$ for all $a \in G$.*

As a consequence, if X is a left-invariant vector field then $X_a = (\mathrm{d}L_a)_e(X_e)$ for all $a \in G$, that is its value is determined by X_e.

The set $\mathfrak{g}$ of all left-invariant vector fields on G is a real vector space, and this vector space can be identified with the tangent space of G at the identity, as the next proposition shows.

Proposition 3. $\mathfrak{g} \cong T_eG$.

Proof: We define the map $\mathfrak{g} \to T_eG$ by $X \mapsto X_e$. Its inverse is $T_eG \ni v \mapsto X^v$, where $X^v_g = (\mathrm{d}L_g)_e(v)$ is a left-invariant vector field. $\square$

It is easy to see that the set $\mathfrak{g}$ is closed under the bracket operation of vector fields, that is if $X, Y \in \mathfrak{g}$ then $[X, Y] \in \mathfrak{g}$. This bracket provides $\mathfrak{g}$ with a real *Lie algebra* structure. This means that $[\ ,\ ]$ is bilinear, antisymmetric, and satisfies the Jacobi identity: $\mathrm{Cyclic}([X, [Y, Z]]) = 0$ for all $X, Y, Z \in \mathfrak{g}$. Using the above isomorphism this Lie algebra structure can be translated to T_eG by $[u, v] = [X^u, X^v]_e$ ($u, v \in T_eG$).

Definition 4. *The Lie algebra of a Lie group G is the vector space T_eG equipped with the Lie bracket defined above.*

Examples 2. 1) The cross product operation $[x, y] = x \times y$ in $\mathbb{R}^3$ defines a Lie algebra structure.

2) The Lie algebra of $G = (\mathbb{R}^n, +)$ is $\mathfrak{g} = \mathbb{R}^n$ with bracket $[x, y] = 0$.

3) The operation $[A, B] = AB - BA$ defines a Lie algebra structure in $\mathrm{M}_n\mathbb{R} \cong \mathbb{R}^{n^2}$.

4) The Lie algebra of $\mathrm{GL}_n\mathbb{R}$ (i.e., the tangent space at the identity I) is $\mathrm{M}_n\mathbb{R} = \mathfrak{g}$ (in fact it is an open submanifold of a Euclidean space). What is the Lie algebra bracket? To each $X \in \mathfrak{g}$ we associate the $n \times n$ matrix $A = (a_{ij})$ of components of X_e, so that $X_e = \sum_{i,j} \left(\frac{\partial}{\partial x^{ij}}\Big|_e \right)$, and write $A = \mu(X)$. By explicit inspection of components one can show that $\mu([X, Y]) = \mu(X)\mu(Y) - \mu(Y)\mu(X)$, giving the Lie algebra structure on $\mathfrak{g} = \mathrm{M}_n\mathbb{R}$.

In order to be able to find explicitly the Lie algebra of various Lie groups, we need to give an alternative description of a Lie group. In fact, this description is close to Lie's original concept of a Lie group.

2.2. Infinitesimal Description of a Lie Group

Definition 5. *An one-parameter subgroup of G is a smooth homomorphism $\varphi : (\mathbb{R}, +) \to G$.*

Examples 3. 1) The map $\varphi(t) = \mathrm{e}^t$ is a one-parameter subgroup in $G = \mathbb{R}$.

2) Given a vector $v \in \mathbb{R}^n$, the map $\varphi(t) = tv$ is a one-parameter subgroup in $\mathbb{R}^n$.

3) Similarly $\varphi(t) = \mathrm{e}^{\mathrm{i}t}$ is an one-parameter subgroup in $G = \mathbb{S}^1 = \mathrm{U}(1)$.

4) The map $\varphi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is an one-parameter subgroup in $G = \mathrm{U}(2)$.

It can be shown that a path $\varphi(t)$ in a Lie group G is a one-parameter subgroup if and only if the velocity of $\varphi(t)$ is constant and $\varphi(0) = e$.
The main result is the following one.

Theorem 1. *The map $\varphi \mapsto \mathrm{d}\varphi_0(1)$ defines a one to one correspondence between one-parameter subgroups of G and T_eG.*

Proof: Let $v \in T_eG$ and $X^v_g = (\mathrm{d}L_g)_e(v)$ be the value of the corresponding left-invariant vector field. Let $\varphi : (-\epsilon, \epsilon) \to G$ be the unique integral curve of X^v such that $\varphi(0) = e$ and $\mathrm{d}\varphi_t = X^v_{\varphi(t)}$. Then φ is a homomorphism, and extend it to all $\mathbb{R}$ by $\varphi_v(t) = \varphi(\frac{t}{n})^n$ for large n. Then the map $v \mapsto \varphi_v$ is the inverse of $\varphi \mapsto \mathrm{d}\varphi_0(1)$. $\square$

Corollary 1. *For each $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup $\varphi_X : \mathbb{R} \to G$ such that $\phi'_X(0) = X$.*

Definition 6. *The exponential map of G is the map $\exp : \mathfrak{g} \to G$ given by $\exp(X) = \varphi_X(1)$.*

It follows that $\exp(tX) = \varphi_X(t)$, therefore

Corollary 2. *The curve $\gamma(t) = \exp(tX)$ $(X \in \mathfrak{g})$ is the unique homomorphism in G with $\gamma'(0) = X$.*

The following proposition summarizes some properties of the exponential map.

Proposition 4. 1) *The exponential map is smooth, and $\mathrm{d}\exp_0 : \mathfrak{g} \to \mathfrak{g}$ is the identity map.*
2) $\exp(tX + sX) = \exp(tX) \cdot \exp(sX)$.
3) $\exp(tX)\exp(tY) = \exp(t(X + Y) + \frac{t^2}{2}[X, Y] + o(t^2))$ (*Campbell-Baker-Hausdorff formula*).
4) *If G is compact and connected, then $\exp$ is onto.*
5) *If $\theta : G \to H$ is a homomorphism of Lie groups, then $\mathrm{d}\theta_e : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, and $\theta \circ \exp = \exp \circ\, \mathrm{d}\theta_e$.*

Examples 4. 1) If $G = \mathbb{R}^*$, then $\mathfrak{g} = \mathbb{R}$ and $\exp(t) = \mathrm{e}^t$.
2) If $G = \mathrm{GL}_n\mathbb{R}$, then $\mathfrak{g} = \mathrm{M}_n\mathbb{R}$ and $\exp(A) = \mathrm{e}^A$ (usual matrix exponentiation).
3) We will show that the Lie algebra of $\mathrm{O}(n) = \{A \in \mathrm{GL}_n\mathbb{R};\ A^t = A^{-1}\}$ is $\mathfrak{o}(n) = \{A \in \mathrm{M}_n\mathbb{R};\ A^t = -A\}$, the set of all skew-symmetric matrices. Hence, the dimension of $\mathrm{O}(n)$ is $\frac{1}{2}n(n-1)$.

Let $\gamma(s)$ be a curve in $\mathrm{M}_n\mathbb{R}$ with $\gamma(0) = I$ that lies in $\mathrm{O}(n)$, i.e., $\gamma(s)^t\gamma(s) = I$. Differentiating at $s = 0$ we obtain that $\gamma'(0)^t = -\gamma'(0)$, thus $T_I\mathrm{O}(n) \subset \mathfrak{o}(n)$. To show the opposite inclusion, we need to use the fact (exercise) that for any matrix X, $(\mathrm{e}^X)^t = (\mathrm{e}^X)^{-1}$ if and only if $X^t = -X$. Then, if $A \in \mathfrak{o}(n)$, then $\gamma(s) = \mathrm{e}^{sA}$ is a curve in $\mathrm{M}_n\mathbb{R}$ with $\gamma(0) = I$ and $\gamma(\mathbb{R}) \subset \mathrm{O}(n)$. Differentiating at $s = 0$ it follows that $\gamma'(0) = A \in T_I\mathrm{O}(n)$, so $\mathfrak{o}(n) \subset T_I\mathrm{O}(n)$.

4) The Lie algebra of $\mathrm{U}(n)$ is $\mathfrak{u}(n) = \{A \in \mathrm{M}_n\mathbb{C};\ \bar{A} = -A^t\}$, the set of all skew-Hermitian matrices.
5) The Lie algebra of $\mathrm{SL}_n\mathbb{R}$, of the set of all real matrices with determinant one, is $\mathfrak{sl}(n) = \{A \in \mathrm{M}_n\mathbb{R};\ \mathrm{tr}\, A = 0\}$.

Jumping a bit ahead, we mention that if a Lie group G is given a Riemannian metric which is invariant under L_g and R_g, then $\exp : \mathfrak{g} \to G$ is the usual exponential map for G at e. In this case the one-parameter subgroups of G are the geodesics through e.

2.3. Lie's Fundamental Theorems

The precise relationship between a Lie group and its Lie algebra is described by the following statements, which are due, in a direct or indirect manner, to S. Lie.

1) Given a Lie algebra $\mathfrak{g}$ there is a Lie group G whose Lie algebra is $\mathfrak{g}$.
2) There exists an one to one correspondence between connected immersed subgroups H of a Lie group G and subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ (the Lie algebra of G). This correspondence is given by $H \mapsto \mathfrak{h} = T_eH$. Normal subgroups of G correspond to ideals in $\mathfrak{g}$.
3) If G_1, G_2 are Lie groups with Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, and if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as Lie algebras, then G_1 and G_2 are locally isomorphic (in fact they have the same covering space). For example, $\mathbb{S}^3 \cong \mathrm{Sp}(1)$ and $\mathrm{SO}(3) \cong \mathbb{RP}^3$ are locally isomorphic, but not isomorphic.
4) The category of Lie algebras and homomorphisms is isomorphic to the category of connected, simply connected Lie groups and homomorphisms.

2.4. The Adjoint Representation

We need a measure of the non-commutativity of a Lie group, and this can be provided by an important representation, called the adjoint representation. Furthermore, this can be used to define important invariants of a Lie group, other from its dimension and the center.

For $g \in G$, let $\sigma(g) : G \to G$ be the **inner automorphism** $\sigma(g)(h) = ghg^{-1}$.

Definition 7. 1) *The adjoint representation of a Lie group G is the $(smooth)$ homomorphism* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ *given by* $\mathrm{Ad}(g) = (\mathrm{d}\sigma(g))_e : T_eG \to T_eG$.

2) *The adjoint representation of a Lie algebra $\mathfrak{g}$ is the homomorphism* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *given by* $\mathrm{ad}(X) = (\mathrm{d}\,\mathrm{Ad})_e(X)$.

It follows that $\ker \mathrm{Ad} = \mathrm{Z}(G)$ the center of G, and $\ker \mathrm{ad} = \mathrm{Z}(\mathfrak{g})$. If G is connected the Lie algebra of $\mathrm{Z}(G)$ is $\mathrm{Z}(\mathfrak{g})$.

Proposition 5. *If G is a matrix group (i.e., $G \subset \mathrm{GL}_n\mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) then*

1) $\mathrm{Ad}(g)X = gXg^{-1}$ *for all $g \in G$, $X \in \mathfrak{g}$.*
2) $\mathrm{ad}(X)(Y) = [X, Y]$ *for all $X, Y \in \mathfrak{g}$. In fact this is true for any Lie group.*
3) *For any $g \in G$ and $X \in \mathfrak{g}$,* $\exp \circ \mathrm{ad}(X) = \mathrm{Ad} \circ \exp(X)$.

Examples 5. 1) If G is abelian, then both Ad and ad are trivial (i.e., $\mathrm{Ad}(g) = \{\mathrm{Id}\}$). This is the case for $\mathrm{SO}(1)$, $\mathrm{SO}(2) \cong \mathrm{U}(1)$, $\mathrm{O}(1)$, $\mathrm{O}(2)$.

2) Trying to compute $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{Aut}(\mathfrak{su}(2))$, consider the basis

$$\left\{ X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\}$$

of $\mathfrak{su}(2)$, and let

$$A = \begin{pmatrix} x + \mathrm{i}y & u + \mathrm{i}v \\ -u + \mathrm{i}v & x - \mathrm{i}y \end{pmatrix} \in \mathrm{SU}(2).$$

We know that $\mathrm{Ad}(A)B = ABA^{-1}$, so by finding the matrices $\mathrm{Ad}(A)X_1$, $\mathrm{Ad}(A)X_2$, $\mathrm{Ad}(A)X_3$ we can obtain the matrix representation of $\mathrm{Ad}(A)$ (this is a 3×3 matrix).

In fact one can do more: Using the following Proposition 6 it follows that $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{O}(3)$, and since $\mathrm{SU}(2) \cong \mathbb{S}^3$, then $\det(\mathrm{Ad}\, g) = 1$, therefore Ad is a homomorphism from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$. It can be shown that this homomorphism is onto.

Using language of the more advanced representation theory, it can be shown that the complexified adjoint representation of $\mathrm{SU}(n)$ is given by $\mathrm{Ad}^{\mathrm{SU}(n)} \otimes \mathbb{C} = \mu_n \otimes \bar{\mu}_n - 1$, where $\mu_n : \mathrm{SU}(n) \to \mathrm{SU}(n)$ is the standard representation of $\mathrm{SU}(n)$ and 1 is the trivial representation.

3) If $\lambda_n : \mathrm{SO}(n) \to \mathrm{SO}(n)$ is the standard representation of $\mathrm{SO}(n)$, then $\mathrm{Ad}^{\mathrm{SO}(n)} = \Lambda^2\lambda_n$, the second exterior power of λ_n.

Towards studying the geometry of a Lie group the following notion is very important.

Definition 8. 1) *Let $\mathfrak{g}$ be a Lie algebra. The Killing form of $\mathfrak{g}$ is the symmetric bilinear form* $\mathrm{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *given by* $\mathrm{B}(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. *The Killing form of a Lie group is the Killing form of its Lie algebra.*

2) *A Lie algebra $\mathfrak{g}$ is called semisimple if B is non-degenerate. This is equivalent to* $\mathrm{Z}(\mathfrak{g}) = 0$.
3) *A Lie group G is semisimple if its Lie algebra is semisimple. This is equivalent to the fact that $\mathrm{Z}(G)$ is discrete.*

The groups $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{U}(n)$ are not semisimple. Table 1 gives the Killing form of the classical Lie groups.

Table 1. The Killing form of the classical Lie groups

G	B
$\mathrm{U}(n)$	$2n \operatorname{tr} XY - 2 \operatorname{tr} X \operatorname{tr} Y$
$\mathrm{SU}(n)$	$2n \operatorname{tr} XY$
$\mathrm{SO}(n)$	$(n-2) \operatorname{tr} XY$
$\mathrm{Sp}(n)$	$2(n+1) \operatorname{tr} XY$

The Killing form can be used to define an inner product on a compact semisimple Lie group:

Theorem 2. *Let G be a compact semisimple Lie group. Then the Killing form* B *is negative definite. The converse is true if G is connected.*

This theorem is a consequence of the fact that a compact Lie group G admits an inner product $\langle\ ,\ \rangle$ on its Lie algebra $\mathfrak{g}$, which is Ad-*invariant*, i.e.,

$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y \rangle = \langle X, Y \rangle \quad \text{for all} \quad g \in G,\ X, Y \in g.$$

Proposition 6. 1) *The Killing form* B *of a Lie group G is* Ad*-invariant. As a consequence, for all $g \in \mathfrak{g}$ the operator* $\mathrm{Ad}(g)$ *is* B*-orthogonal, that is* $\mathrm{Ad}(G) \subset \mathrm{O}(\mathfrak{g})$.

2) *For any $Z \in \mathfrak{g}$ the operator* $\mathrm{ad}(Z)$ *is skew-symmetric with respect to* B, *that is* $\mathrm{B}(\mathrm{ad}(Z)X, Y) + \mathrm{B}(X, \mathrm{ad}(Z)Y) = 0$. *Equivalently,* $\mathrm{B}([X, Z], Y) = \mathrm{B}(X, [Z, Y])$.

Definition 9. *A semisimple Lie algebra is called simple if it is non-abelian and it has no non-trivial ideals.*

Simple Lie algebras are the "building blocks" of semisimple Lie algebras, since a semisimple Lie algebra is a direct product of simple ideals.

2.5. Maximal Tori and the Classification Theorem

Definition 10. 1) *A **torus** $\mathbb{T}$ in a Lie group G is a Lie subgroup isomorphic to a product* $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

2) *A **maximal torus** in G is a torus $\mathbb{T}$ if whenever $\mathbb{T} \subset \mathbb{S} \subset G$, with $\mathbb{S}$ a torus, then $\mathbb{T} = \mathbb{S}$.*

Examples 6. 1) The set of matrices $\mathbb{T}_n = \{\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_n})\}$ is a maximal torus in $\mathrm{U}(n)$. By adding the condition $\theta_1 + \cdots + \theta_n = 0$, this set is a maximal torus in $\mathrm{SU}(n)$.

2) If $\mathrm{rot}\,\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$, then the set of the block diagonal matrices of type $\{\mathrm{diag}\,\{\mathrm{rot}\,\theta_1, \dots, \mathrm{rot}\,\theta_{2n}\}\}$ is a maximal torus in $\mathrm{SO}(2n)$, and $\{\mathrm{diag}(\mathrm{rot}\,\theta_1, \dots, \mathrm{rot}\,\theta_{2n}, 1)\}$ is a maximal torus in $\mathrm{SO}(2n+1)$.

The following proposition gives a characterization of tori.

Proposition 7. *A Lie group H is a torus if and only if H is compact, connected, and abelian.*

The next theorem essentially summarizes the central theory of maximal tori in a Lie group.

Theorem 3. *Let G be a compact, and connected Lie group. Then*

1) *Any element in G is contained in some maximal torus.*
2) *Any two maximal tori $\mathbb{T}_1$, $\mathbb{T}_2$ are conjugate in G, that is $g\mathbb{T}_1g^{-1} = \mathbb{T}_2$ for some $g \in G$.*
3) *If $\mathbb{T}$ is a maximal torus in G, then $G = \bigcup_{g\in G} g\mathbb{T}g^{-1}$.*

Several well known theorems of linear algebra can be interpreted by the above theorem. For example, if $G = \mathbb{U}(n)$ then any unitary matrix can be diagonalized.

Due to part above the following concept is well defined.

Definition 11. *The rank of a compact and connected Lie group G is the dimension of a maximal torus.*

Denoting by $\mathrm{rk}(G)$ the rank of the Lie group G, then $\mathrm{rk}(\mathrm{U}(n)) = \mathrm{rk}(\mathrm{SO}(2n)) = \mathrm{rk}(\mathrm{SO}(2n+1)) = n$, and $\mathrm{rk}(\mathrm{SU}(n)) = n - 1$.

There is an analogous concept for Lie algebras. A **Cartan subalgebra** of a Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$. In turns out that the Lie algebra $\mathfrak{t}$ of a maximal torus $\mathbb{T}$ of G, is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G.

For example, if $G = \mathrm{U}(n)$, then $\mathfrak{g}$ consists of all $n \times n$ skew-symmetric complex matrices, and the Lie algebra of $\mathbb{T}_n$ is the set $\{\mathrm{diag}(\mathrm{i}c_1, \dots, \mathrm{i}c_n)\,;\, c_i \in \mathbb{R}\}$. This is a Cartan subalgebra of $\mathfrak{u}(n)$.

The maximal tori are used for the classification of compact and connected Lie groups, which it is summarized in the following theorem.

Theorem 4. 1) *Let G be a compact, and connected Lie group. Then there exists a Lie group $\widetilde{G}$ which is a finite covering of G and so that $\widetilde{G} \cong \mathbb{S} \times H$, where $\mathbb{S}$ is a torus, and H a compact, connected, and simply connected Lie group.*

2) *Every compact, connected, and simply connected Lie group is isomorphic to a product of simple, compact, connected, and simply connected Lie groups.*

3) *The simple, compact, connected, and simply connected Lie groups are the following*: $\mathrm{SU}(n)$ $(n \geq 2)$, $\widetilde{\mathrm{SO}}(2n+1)$ (≥ 3), $\widetilde{\mathrm{SO}}(2n)$ $(n \geq 4)$, $\mathrm{Sp}(n)$ $(n \geq 2)$, G_2, F_4, E_6, E_7, E_8.

The first four simple groups are the classical groups, and their corresponding Lie algebras are denoted by A_{n-1}, B_n, C_n, and D_n, respectively. The remaining five are called the *exceptional Lie groups*, and their definition is more complicated. The subscript denotes their rank, and their dimensions are 14, 52, 78, 133 and 278, respectively. The group $\widetilde{\mathrm{SO}}(n)$ is also denoted by $\mathrm{Spin}(n)$, and is known as the **spinor group**.

The analysis and proof of the classification theorem is a laborious process, and reduces to the classification of complex semisimple Lie algebras, that was achieved by E. Cartan. This is a slightly easier process, since it uses elementary, but non trivial linear algebra. The bottom line, is that the complex semisimple Lie algebras can be classified by certain combinatorial graphs, called *Dynkin diagrams*.

3. Homogeneous Spaces

3.1. Group Actions and Examples

Definition 12. *Let G be a Lie group, and H a closed subgroup. Then the space G/H of left cosets is called a homogeneous space.*

Proposition 8. *The space G/H has a natural manifold structure. The projection $G \to G/H$ is a submersion, and it is a principal fiber bundle with group H.*

The group G acts on G/H by $a \cdot gH = agH$. This action is transitive. In fact, every transitive action is represented in this way:

Proposition 9. *Let $G \times M \to M$ be a transitive action of a Lie group G on a manifold M, and let $H = G_m = \{g \in G;\, g \cdot m = m\}$ be the isotropy subgroup of $m \in M$. Then*

1) *H is a closed subgroup of G.*
2) *The manifold G/H is diffeomorphic to M.*
3) *The orbit $G \cdot m$ is diffeomorphic to G/G_m.*

In this case we say traditionally, that the Lie group G is "represented" as a group of diffeomorphisms or "transformations" of M.

Definition 13. *A Riemannian homogeneous space is a Riemannian manifold (M, g) on which the isometry group $I(M)$ acts transitively.*

It is a result of Myers and Steenrod that $I(M)$ is a Lie group.

Examples 7. 1) Spheres. The group $\mathrm{O}(n+1)$ acts transitively on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and the isotropy subgroup at $(1, 0, \ldots, 0)$ can be identified with $\mathrm{O}(n)$, therefore $\mathbb{S}^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$. By restriction of the action to $\mathrm{SO}(n+1)$ we also obtain that $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$.
Similarly, $\mathbb{S}^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n)$, and $\mathbb{S}^{4n+3} \cong \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$.

2) Grassmann manifolds. The group $\mathrm{SO}(n)$ acts transitively on the set $\mathrm{Gr}_k\mathbb{R}^n = \{E \subset \mathbb{R}^n\,;\, E \text{ a subspace of } \mathbb{R}^n, \dim E = k\}$. This is called a **real Grassmann manifold**. It follows that $\mathrm{Gr}_k\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(n-k))$.
A special case is the real projective space $\mathbb{R}\mathrm{P}^n = \mathrm{Gr}_1\mathbb{R}^n$.

3) Flag manifolds. The group $\mathrm{O}(n)$ acts on the set of flags $\mathbb{F}_{k_1,\ldots,k_l} = \{\mathbf{x} = (E_{k_1}, \ldots, E_{k_l})\,;\, E_{k_i} \in \mathrm{Gr}_{k_i}\mathbb{R}^n, \dim E_{k_i} = k_i \text{ and } E_{k_1} \subset \cdots \subset E_{k_l} \subset \mathbb{R}^n\}$, by $A \cdot \mathbf{x} = (AE_{k_1}, \ldots, AE_{k_l})$.
The isotropy subgroup at the point $E_{k_i} = \mathrm{span}\{e_1, \ldots, e_{k_i}\}$ can be identified with the block diagonal matrices $A = \mathrm{diag}(A_1, A_2, \ldots) \in \mathrm{O}(n)$ with A_i an orthogonal matrix. It follows that $\mathbb{F}_{k_1,\ldots,k_l} \cong \mathrm{O}(n)/\mathrm{O}(k_1) \times \mathrm{O}(k_2 - k_1) \times \cdots \times \mathrm{O}(n - k_l)$. This is called a **real flag manifold**. More generally, a flag manifold is a homogeneous space of the form $G/C(T)$, where G is a semisimple compact Lie group, and $C(T)$ the centralizer of a torus T in G. Flag manifolds for a simple Lie group G can be classified in terms of "painted" Dynkin diagrams. For more details on flag manifolds we refer to [5] and references therein.

4) Stiefel manifolds. A k-frame in $\mathbb{R}^n$ is a set of k linear independent vectors in $\mathbb{R}^n$. A **real Stiefel manifold** is the set $V_k\mathbb{R}^n$ of all k frames in $\mathbb{R}^n$. The groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ act on $V_k\mathbb{R}^n$ and the isotropy subgroups at $(e_1, \ldots, e_k)$ are identified with $\mathrm{O}(n-k)$ and $\mathrm{SO}(n-k)$, respectively, therefore $V_k\mathbb{R}^n \cong \mathrm{O}(n)/\mathrm{O}(n-k) \cong \mathrm{SO}(n)/\mathrm{SO}(n-k)$. Notice the special cases $V_1\mathbb{R}^n \cong \mathbb{S}^{n-1}$, $V_n\mathbb{R}^n \cong \mathrm{O}(n)$, and $V_2\mathbb{R}^n \cong T_1\mathbb{S}^{n-1}$ (the unit tangent bundle).

5) Symmetric spaces. This an important class of homogeneous spaces, but we will not enter into any further analysis here. Briefly, a symmetric space is a Riemannian manifold (M, g) which is defined by the geometrical condition that its curvature tensor is locally parallel, i.e., $\nabla R = 0$.

In concluding this section, we remark that various non-Euclidean geometries are realized as examples of coset spaces. Also, if is possible that a manifold M is represented as a homogeneous space in more than one ways, i.e., $M = G/H = G'/H'$. Finding all such possible presentations is a difficult problem in general.

3.2. Reductive Homogeneous Spaces

Let G/H be a homogeneous space and $\pi : G \to G/H$ the canonical projection. Consider the derivative $\mathrm{d}\pi_e : G \to T_o(G/H)$, where $o = eH$. Then an easy computation shows that $\ker \mathrm{d}\pi_e = \mathfrak{h}$, the Lie algebra of H, hence $\mathfrak{g}/\mathfrak{h} \cong T_o(G/H)$. This motivates the following

Definition 14. *A homogeneous space is called* ***reductive*** *if there exists a subspace* $\mathfrak{m}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *and* $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ *for all* $h \in H$*, that is* $\mathfrak{m}$ *is* $\mathrm{Ad}(H)$*-invariant.*

The last condition implies that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and the converse is true if H is connected. So, if G/H is reductive, then $\mathfrak{m} \cong T_o(G/H)$. For example, if G is a compact and semisimple Lie group, we can take $\mathfrak{m} = \mathfrak{h}^{\perp}$ with respect to an Ad-invariant inner product of $\mathfrak{g}$.

Examples 8. 1) Let $G/H = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$.
Then $\mathfrak{h} = \{\mathrm{diag}\{\mathrm{i}a, \mathrm{i}b, \mathrm{i}c\};\ a + b + c = 0\}$, and with respect to the Killing form $\mathrm{B}(X, Y) = 6 \operatorname{tr} XY$ of $\mathrm{SU}(3)$, we obtain that

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a_1 + \mathrm{i}b_1 & a_2 + \mathrm{i}b_2 \\ -a_1 + \mathrm{i}b_1 & 0 & a_3 + \mathrm{i}b_3 \\ -a_2 + \mathrm{i}b_2 & -a_3 + \mathrm{i}b_3 & 0 \end{pmatrix} \right\}.$$

2) Let $M = G/H = \mathrm{SO}(5)/\mathrm{U}(2)$.
Then $\mathfrak{h} = \left\{ \begin{pmatrix} a\mathrm{i} & b + \mathrm{i}c \\ -b + \mathrm{i}c & d\mathrm{i} \end{pmatrix} ;\ a, b, c, d \in \mathbb{R} \right\}$, and we embed it to $\mathfrak{so}(5)$, first by using the identification $\mathfrak{u}(2) \cong \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{u}(2) \end{pmatrix}$, and then by use of the embedding $X + \mathrm{i}Y \hookrightarrow \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$.
Then with respect to the Killing form $\mathrm{B}(X, Y) = 3 \operatorname{tr} XY$ of $\mathrm{SO}(5)$, it follows that

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & b_1 & 0 & b_3 \\ -a_2 & -b_1 & 0 & -b_3 & 0 \\ -a_3 & 0 & b_3 & 0 & -b_1 \\ -a_4 & -b_3 & 0 & b_1 & 0 \end{pmatrix} ;\ a_i, b_j \in \mathbb{R},\ i = 1, \ldots, 4,\ j = 1, 3 \right\}.$$

3.3. The Isotropy Representation

For $a \in G$, let $\tau_a : G/H \to G/H$ be the *left translation* defined by $\tau_a(gH) = agH$. This is a diffeomorphism.

Definition 15. *The* ***isotropy representation*** *of G/H is the homomorphism $\chi : H \to \mathrm{GL}(T_o(G/H))$ given by $\chi(h) = (\mathrm{d}\tau_h)_o$.*

It is possible to have a more concrete description of χ when G/H is a reductive homogeneous space. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and consider the restricted adjoint representation of G, $\mathrm{Ad}^G\big|_H : H \to \mathrm{Aut}(\mathfrak{g})$. Then $\mathfrak{h}$ and $\mathfrak{m}$ are $\mathrm{Ad}^G\big|_H$-invariant subspaces of $\mathfrak{g}$, therefore we obtain that $\mathrm{Ad}^G\big|_H = \mathrm{Ad}^H \oplus \mathrm{Ad}^{G/H}$, where the first summand is the adjoint representation of H, and the second summand is the isotropy representation of G/H, $\chi \equiv \mathrm{Ad}^{G/H} : H \to \mathrm{Aut}(\mathfrak{m})$. Its precise relationship to the adjoint representation of G is given by $\mathrm{Ad}^{G/H}(h)X = \mathrm{Ad}^G(h)X$ for all $h \in H$, $X \in \mathfrak{g}$.

Definition 16. *A homogeneous space is called isotropy irreducible if its isotropy representation is irreducible.*

Examples 9. 1) Let $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Recall that $\mathrm{Ad}^{\mathrm{SO}(n)} = \Lambda^2\lambda_n$, where λ_n is the standard representation of $\mathrm{SO}(n)$. Then

$$\mathrm{Ad}^{\mathrm{SO}(n+1)}\Big|_{\mathrm{SO}(n)} = \Lambda^2\lambda_{n+1}\Big|_{\mathrm{SO}(n)} = \Lambda^2(\lambda_n \oplus 1) = \Lambda^2\lambda_n \oplus \Lambda^2 1 \oplus (\lambda_n \otimes 1).$$

The first summand is the adjoint representation of $\mathrm{SO}(n)$, the second is zero, and the third is identified with λ_n, which is the isotropy representation of $\mathbb{S}^n$. This is irreducible.

2) Let $M = \mathrm{SO}(5)/\mathrm{U}(2)$. It is easier to consider the complexified isotropy representation of $\mathrm{SO}(5)$. Then

$$\begin{aligned}\mathrm{Ad}^{\mathrm{SO}(5)} \otimes \mathbb{C}\Big|_{\mathrm{U}(2)} &= \Lambda^2(\lambda_5 \otimes \mathbb{C})\Big|_{\mathrm{U}(2)} = \Lambda^2(\mu_2 \oplus \bar{\mu}_2 \oplus 1) \\ &= \Lambda^2\mu_2 \oplus \Lambda^2\bar{\mu}_2 \oplus (\mu_2 \otimes \bar{\mu}_2) \oplus (\mu_2 \otimes 1) \oplus (\bar{\mu}_2 \otimes 1).\end{aligned}$$

The third summand is the complexified adjoint representation of $\mathrm{U}(2)$. The rest of the summands contribute to the isotropy representation of $\mathrm{SO}(5)/\mathrm{U}(2)$, which consists exactly of two irreducible summands, namely $[\Lambda^2\mu_2 \oplus \Lambda^2\bar{\mu}_2] \oplus [(\mu_2 \otimes 1) \oplus (\bar{\mu}_2 \otimes 1]$. Their dimensions are two and four, respectively. This decomposition induces an $\mathrm{Ad}(\mathrm{U}(2))$-invariant decomposition of the tangent space $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, into two non-equivalent irreducible submodules.

4. Geometry of Compact Lie Groups and Homogeneous Spaces

4.1. Invariant Metrics

We will develop the geometry of a compact Lie group and a homogeneous space in a parallel manner. Of course a Lie group G is identified with the homogeneous space $G/\{e\}$, so its geometry is a special case. However, various formulas for a compact Lie group have their own value, so we will treat them separately.

Definition 17. *A Riemannian metric g on a Lie group G is called left-invariant if $g(u,v)_x = g((\mathrm{d}L_a)_x u, (\mathrm{d}L_a)_x v)_{L_a(x)}$ for all $a, x \in G$ and $u, v \in T_x G$.*

This means that the diffeomorphism L_a is an isometry. Similarly, a Riemannian metric is called *right-invariant* if the right translation R_a is an isometry. A metric which is both left-invariant and right-invariant is called *bi-invariant*.

Proposition 10. *There exists an one to one correspondence between left-invariant metrics on G and scalar products on its Lie algebra.*

Proof: Let g be a left-invariant metric on G. Then for any $X, Y \in \mathfrak{g}$ the function $g(X,Y) : G \to \mathbb{R}$ is constant due to the left-invariance, therefore it defines a scalar product on $\mathfrak{g}$. Conversely, if $\langle\ ,\ \rangle_e$ is a scalar product on $\mathfrak{g}$, then the metric given by $g(x,y)_a = \langle (\mathrm{d}L_{a^{-1}})_a x, (\mathrm{d}L_{a^{-1}})_a y\rangle_e$ ($a \in G$, $x, y \in T_a G$) is a left-invariant metric on G. □

A compact Lie group G possesses a bi-invariant metric. In fact, it can be shown that G admits a G-invariant integral $\int_G f(g)\,\mathrm{d}g$. Then, by fixing a scalar product $\langle\ ,\ \rangle_e$ on $\mathfrak{g}$, define a bi-invariant metric on G by $\langle u, v\rangle = \int_G \langle \mathrm{Ad}(g)u, \mathrm{Ad}(g)v\rangle_e\,\mathrm{d}g$.

Proposition 11. *There exists an one to one correspondence between bi-invariant metrics on G and Ad-invariant scalar products on $\mathfrak{g}$.*

We now turn to homogeneous spaces $M = G/H$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition.

Definition 18. *A metric g on M is called G-invariant if for all $a \in G$ the diffeomorphism $\tau_a : G/H \to G/H$ is an isometry, i.e., $g(X,Y) = g(\mathrm{d}\tau_a(X), \mathrm{d}\tau_a(Y))$ for all $X, Y \in T_o(G/H)$.*

Proposition 12. *There exists an one to one correspondence between G-invariant metrics g on $M = G/H$ and $\mathrm{Ad}^{G/H}$-invariant scalar products $\langle\ ,\ \rangle$ on $\mathfrak{m}$, i.e., $\langle X, Y\rangle = \langle \mathrm{Ad}^{G/H}(h)X, \mathrm{Ad}^{G/H}(h)Y\rangle$ for all $X, Y \in \mathfrak{m}$, $h \in H$.*

This proposition is a special case of a general phenomenon, where G-invariant objects on a homogeneous space G/H (e.g., (p,q)-tensors), correspond to $\mathrm{Ad}^{G/H}$-invariant objects on $T_o(G/H) \cong \mathfrak{m}$.

4.2. Connections and Curvature

Let G be a compact Lie group with a left-invariant metric g. Then the Riemannian connection is given by

$$\nabla_X Y = \frac{1}{2}([X,Y] - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X)$$

where $(\mathrm{ad}_X)^*$ is the formal adjoint operator of ad_X. Equivalently,

$$g(\nabla_X Y, Z) = \frac{1}{2}\{g(Z,[X,Y]) + g(Y,[Z,X]) + g(X,[Z,Y])\}.$$

Using a left-invariant metric it is quite complicated to handle other geometrical objects such as curvature or geodesics. However, if we restrict ourselves to bi-invariant metrics, formulas simplify.

Proposition 13. *Let G be a Lie group with a bi-invariant metric. Then for any $X, Y, Z \in \mathfrak{g}$*

1) $\nabla_X Y = \frac{1}{2}[X, Y]$.
2) *Geodesics starting at* e *are the one-parameter subgroups* $\exp tX$.
3) *The curvature tensor is given by* $R(X,Y)Z = \frac{1}{2}[[X,Y],Z]$.
4) *The sectional curvature is given by* $K(X,Y) = \frac{1}{4}\frac{\langle [X,Y],[X,Y]\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$.
5) *The Ricci curvature is given by* $\mathrm{Ric}(X,Y) = \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle$, *where* $\{E_i\}$ *is an othonormal basis of* $\mathfrak{g}$.
6) *If G is compact and the bi-invariant metric is the Killing form, then the scalar curvature is* $S = \frac{1}{4}\dim G$.

If G is semisimple and compact, then with respect to a bi-invariant metric, the Ricci curvature is given by $\mathrm{Ric}(X,Y) = -\frac{1}{4}B(X,Y)$, that is G is an **Einstein manifold**.

We now turn to homogeneous spaces. Let g be a G-invariant metric on a homogeneous space $M = G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$, and $o = eH$. For any $X \in \mathfrak{g}$ we define the vector field $X^*_o = \frac{\mathrm{d}}{\mathrm{d}t}(\exp tX)\cdot o\Big|_{t=0}$. This is a Killing vector field (i.e., its flows are isometries), and satisfies $[X^*, Y^*] = -[X,Y]^*$. Recalling the canonical projection $\pi : G \to G/H$, we have that $\mathrm{d}\pi(X) = X^*_o$ and $\mathrm{d}\pi(X_\mathfrak{m}) = X^*_o$. Here $X_\mathfrak{m}$ denotes the component of X in $\mathfrak{m}$.

Proposition 14. *Let $X, Y \in \mathfrak{m}$. Then the Riemannian connection of g is given by*

$$\mathfrak{m} \cong T_o(G/H) \ni \nabla_{X^*}Y^*\big|_o = -\frac{1}{2}[X,Y]_\mathfrak{m} + U(X,Y)$$

where $U : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ is determined by the identity

$$2\langle U(X,Y), Z\rangle = \langle [Z,X]_\mathfrak{m}, Y\rangle + \langle X, [Z,Y]_\mathfrak{m}\rangle$$

for all $Z \in \mathfrak{m}$.

There is a particularly simple class of reductive homogeneous spaces. For a semisimple and compact Lie group G, we know that every bi-invariant metric determines an Ad-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. The restriction $\langle\ ,\ \rangle|_\mathfrak{m}$ induces a

G-invariant Riemannian metric, called *normal*. If $\langle\ ,\ \rangle = -B$, this metric is called *standard*.

Formulas for the various curvatures for a general reductive homogeneous spaces are quite complicated (see, e.g., [4, 8, 27]). We only mention the following

Proposition 15. *The curvature tensor of a reductive homogeneous space G/H is determined by the following equation*

$$\begin{aligned}\langle R(X,Y)X,Y\rangle = &-\frac{3}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle - \frac{1}{2}\langle [X,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, Y\rangle \\ &-\frac{1}{2}\langle [Y,[Y,X]_{\mathfrak{m}}]_{\mathfrak{m}}, X\rangle + \langle U(X,Y), U(X,Y)\rangle \\ &-\langle U(X,X), U(Y,Y)\rangle + \langle Y, [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}\rangle\end{aligned}$$

for all $X, Y \in \mathfrak{m}$.

If $U \equiv 0$ then G/H is called **naturally reductive**. From geometrical viewpoint, this condition is equivalent to the fact that all geodesics are the one-parameter subgroups $\exp tX \cdot o$ ($X \in \mathfrak{m}$). Homogeneous spaces that have this property are, for example, the symmetric spaces. Such geodesics are called **homogeneous geodesics**, and have been studied in various occasions by various people (e.g., Kajher, Arnold, Kostant).

The following proposition is due to O. Kowalski, L. Vanhecke and E. Vinberg.

Proposition 16. *Let $M = G/H$ be a homogeneous space. Then the orbit $\gamma(t) = \exp(tX)\cdot o$ is a geodesic in M if and only if $\langle [X,Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0$ for all $Y \in \mathfrak{m}$.*

Spaces with the property that all geodesics are homogeneous are known as **g.o. spaces**, and have been studied by Kowalski and his collaborators. It is an active area of research. For example D. Alekseevky and the author classified all flag manifolds, which are g.o. spaces (see [1]). The main theorem is the following:

Theorem 5. *The only flag manifolds $M = G/K$ of a simple Lie group G which admit an invariant metric with homogeneous geodesics, not homothetic to the standard metric, are the manifolds $\mathrm{Com}(\mathbb{R}^{2\ell+2}) = \mathrm{SO}(2\ell+1)/\mathrm{U}(\ell))$ of complex structures in $\mathbb{R}^{2\ell+2}$ and the complex projective space $\mathbb{CP}^{2\ell-2} = \mathrm{Sp}(\ell)/U(1)\cdot \mathrm{Sp}(\ell-1)$. These manifolds admit a one-parameter family g_λ, $\lambda > 0$ of invariant metrics (up to a scaling). All these metrics have homogeneous geodesics and are weakly symmetric. The metric g_1 is the standard metric. It has the full connected isometry group $\mathrm{SO}(2\ell+2)$ (respectively $\mathrm{SU}(2\ell-1)$) and is the standard metric of the symmetric space $\mathrm{Com}(\mathbb{R}^{2\ell+2}) = \mathrm{SO}(2\ell+2)/\mathrm{U}(\ell+1)$ (respectively $\mathbb{CP}^{2\ell-2} = \mathrm{SU}(2\ell-1)/\mathrm{U}(2\ell-2)$). All the other metrics g_λ, $\lambda \neq 1$ have the full connected isometry group $\mathrm{SO}(2\ell+1)$ (respectively $\mathrm{Sp}(\ell)$). In particular, the corresponding spaces are not naturally reductive as Riemannian manifolds.*

Note that for $\ell = 2$ we obtain $\mathrm{Sp}(2)/\mathrm{U}(1)\cdot\mathrm{Sp}(1) \cong \mathrm{SO}(5)/\mathrm{U}(2)$, which is the 6-dimensional non-naturally reductive g.o. space in the work of Kowalski and Vanhecke [19], in which they classified homogeneous g.o. spaces of dimension ≤ 6.

5. Homogeneous Einstein Metrics

5.1. Brief Introduction

Which are the "best" metrics on a Riemannian manifold (M, g)? Motivated from the two-dimensional case, where the good metrics are the ones with constant Gauss curvature, in higher dimensions we need to look at the various curvatures of a manifold. Constancy of the sectional curvature is a very strong condition, and constancy of the scalar curvature is a very weak one. Therefore, we are lead to impose constancy of the Ricci curvature, which is equivalent to the equation $\mathrm{Ric}(g) = cg$, where c is some constant (called *Einstein constant*). Manifolds that satisfy this equation are called *Einstein manifolds*. The terminology, as expected, is related to general relativity. In the four-dimensional case the equation $\mathrm{Ric}(g) = cg$ is equivalent to the Einstein's field equations with cosmological constant.

If (M, g) is compact, then g is an Einstein metric if and only if g is a critical point of the scalar curvature functional $T : \mathcal{M}_1 \to \mathbb{R}$ given by $T(g) = \int_M S_g\,\mathrm{dvol}_g$, on the set Riemannian metrics of unit volume. This is an old result of D. Hilbert. For references on Einstein manifolds we refer to the book of Besse [8] and the survey of Wang [28].

If $M = G/H$ is a homogeneous space, where G, H are compact, then the G-invariant Einstein metrics on M are precisely the critical points of T restricted to $\mathcal{M}_1^G$, the set of G-invariant metrics of unit volume. This is a direct consequence of R. Palais' principle of "symmetric criticality."

General problem: Find (if possible all) G-invariant metrics on a homogeneous space G/H.

The problem is difficult even for a compact semisimple Lie group. It is still an open problem to find all left-invariant metrics in this case. Is this set finite or infinite?

In 1979 D'Atri and Ziller [13] obtained many Einstein metrics on G, which are naturally reductive. In 1973 Jensen [16] obtained examples of Einstein metrics by a fiber bundle construction.

If the Einstein constant is *positive*, then G/H is compact. Examples of such manifolds are $\mathbb{S}^n$ and $\mathbb{CP}^n$ with the standard metrics, symmetric spaces of compact type, and isotropy irreducible spaces. These admit a unique (up to scalar) Einstein metric, and were classified by J. Wolf in 1968. In 1985 M. Wang and W. Ziller classified all normal homogeneous Einstein manifolds [31].

Einstein metrics on flag manifolds are not unique. Explicit solutions were obtained for various examples of flag manifolds by D. Alekseevsky, the author, M. Kimura, Y. Sakane, and E. Rodionov. A complete description remains open.

There exist compact homogeneous spaces with *no* G-invariant Einstein metrics, as it was shown in [29].

If the Einstein constant is *zero*, then D. Alekseevsky and Kimel'fel'd have shown that a homogeneous Ricci flat manifold is flat.

Finally, if the Einstein constant is *negative*, then G/H is not compact. Examples of such manifolds are $\mathbb{RH}^n$ with the standard metric, and symmetric spaces of non-compact type. A result of Dotti and Miatello in 1982 says that if G is a unimodular solvable Lie group, then any Einstein left-invariant metric on G is flat. There is a lot of active research in the non-compact case.

General existence results is difficult to obtain. We mention the results of Jensen [15], Wang and Ziller [29], and more recently a new existence approach by Böhm, Wang and Ziller [10]. This was used by Böhm and Kerr [9] to show the following

Proposition 17. *Every compact simply connected homogeneous space of dimension* ≤ 11 *admits at least one invariant Einstein mertic.*

In dimension 12 there are examples of non-existence.

The structure of the set of invariant Einstein metrics on a given homogeneous manifold is not well understood in general. The situation is only clear for isotropy irreducible spaces, partly for flag manifolds, and for some special types of homogeneous spaces studied by Nikonorov, Lomshakov and Firsov [21]. A finiteness conjecture, due to Ziller, says that if the isotropy representation χ of a homogeneous space consists of pairwise inequivalent irreducible components, then the set of Einstein metrics is finite.

5.2. The Variational Approach for Einstein Metrics

There are two direct methods for finding Einstein metrics on a homogeneous space. The first is the direct computation of the Ricci curvature. This has been successful in various cases such as flag manifolds (Arvanitoyeorgos, Rodionov). The second is the variational approach, where Einstein metrics are the critical points of the scalar curvature functional, as explained in the previous section. For both cases the Einstein equation reduces to an algebraic system of equations, which in some cases can be solved explicitly.

We will give applications of the variational method for two cases, namely Einstein metrics on flag manifolds, and Stiefel manifolds.

Let $M = G/H$ be a homogeneous space of a compact semisimple Lie group, with reducive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with respect to $-\mathrm{B}$. Recall that G-invariant

Einstein metrics g on G/H correspond to $\mathrm{Ad}^{G/H}$-invariant scalar products $\langle\ ,\ \rangle$ on $\mathfrak{m}$.

Let $\{e_a\}$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. Then according to [29] the scalar curvature of g is given by

$$S(\langle\ ,\ \rangle) = -\frac{1}{2}\sum_a \mathrm{B}(e_a, e_b) - \frac{1}{4}\sum_{a,b}\langle [e_a, e_b]_{\mathfrak{m}}, [e_a, e_b]_{\mathfrak{m}}\rangle. \qquad (1)$$

We assume that the isotropy representation of G/H decomposes into a direct sum $\chi = \chi_1 \oplus \cdots \oplus \chi_s$ of irreducible subrepresentations, which are pairwise inequivalent. If they are not, then the description of G-invariant metrics is more complicated. Then the tangent space of $\mathfrak{m}$ decomposes into a direct sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ of irreducible pairwise inequivalent $\mathrm{Ad}(H)$-submodules.

Then any $\mathrm{Ad}^{G/H}$-invariant scalar product on $\mathfrak{m}$ has the form

$$\langle\ ,\ \rangle = x_1 \left.(-B)\right|_{\mathfrak{m}_1} + \cdots + x_s \left.(-B)\right|_{\mathfrak{m}_s}, \qquad x_i > 0.$$

Let $d_\alpha = \dim \mathfrak{m}_\alpha$ and $\{e^j_\alpha\}$ be a $-$B-orthonormal basis of $\mathfrak{m}_\alpha$ ($1 \leq j \leq d_\alpha$). Define the numbers $[\alpha\beta\gamma] = \sum_{i,j,k} \mathrm{B}([e^i_\alpha, e^j_\beta], e^k_\gamma)^2$, where i, j, k vary from 1 to d_α, d_β and d_γ, respectively. These numbers are symmetric, and independent of the basis, but depend on the decomposition of $\mathfrak{m}$. Then the scalar curvature (1) takes the form

$$S = \frac{1}{2}\sum_{i=1}^{s}\frac{d_i}{x_i} - \frac{1}{4}\sum_{\alpha,\beta,\gamma}[\alpha\beta\gamma]\frac{x_\gamma}{x_\alpha x_\beta}\,.$$

The volume condition takes the form $V = \prod_{i=1}^{s} x_i^{d_i} - 1 = 0$. Therefore, the solutions of the Einstein equation are the solutions of the Lagrange system $\nabla S = \lambda(\nabla V)$.

Flag Manifolds.

Consider the flag manifold $G/H = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3))$ ($n = n_1 + n_2 + n_3$). The isotropy representation decomposes into three non-equivalent irreducible components, inducing the decomposition $\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$, hence $\mathrm{SU}(n)$-invariant metrics depend on three positive parameters x_{12}, x_{13}, x_{23}.

The Lagrange system reduces to the following algebraic system of three equations

$$n_i + n_j + \frac{1}{2}\sum_{k\neq i,j}\frac{n_k}{x_{ik}x_{jk}}(x_{ij}^2 - (x_{ik} - x_{jk})^2) = x_{ij}$$

which has four solutions listed in Table 2.

It turns out that the first three Einstein metrics are also Kähler. If $n_1 = n_2 = n_3$ then the fourth metric is the standard metric.

Table 2

x_{12}	x_{13}	x_{23}
$n_1 + n_2$	$n_1 + 2n_2 + n_3$	$n_2 + n_3$
$n_1 + n_2 + 2n_3$	$n_1 + n_3$	$n_2 + n_3$
$n_1 + n_2$	$n_1 + n_3$	$2n_1 + n_2 + n_3$
$n_1 + n_2$	$n_1 + n_3$	$n_2 + n_3$

Invariant Einstein metrics on the full flag manifold $\mathrm{SU}(n)/\mathbb{T}^n$, where the torus $\mathbb{T}^n = \mathrm{S}(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1))$ a maximal torus, are not completely classified (except for $n = 3, 4$). The only known Einstein metrics are the finite number of $\frac{n!}{2}$ Kähler-Einstein metrics, the standard metric, a class of n Einstein metrics found by the author [3], and a class of n Einstein metrics found by Sakane [26].

For Einstein metrics on other flag manifolds we refer to [3, 20, 26].

Stiefel Manifolds.

Let $G/H = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ be a real Stiefel manifold. The simplest case $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is an irreducible symmetric space, therefore it admits up to scale a unique invariant Einstein metric. It was Kobayashi [18] who proved first the existence of an invariant Einstein metric on $T_1S^n = \mathrm{SO}(n)/\mathrm{SO}(n-2)$. Later on, Sagle [25] proved that the Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-k)$ admit at least one homogeneous invariant Einstein metric. For $k \geq 3$ Jensen [16] found a second metric. In the same work he also proved that the quaternionic Stiefel manifold $\mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ admits at least two homogeneous invariant Einstein metrics. Einstein metrics on $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ are completely classified. If $n = 3$ the group $\mathrm{SO}(3)$ has a unique Einstein metric. If $n \geq 5$ it was shown by Back and Hsiang [7] that $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ admits exactly one homogeneous invariant Einstein metric. The same result was obtained by Kerr [17]. The Stiefel manifold $\mathrm{SO}(4)/\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics which follows from the classification of five-dimensional homogeneous Einstein manifolds due to Alekseevsky, Dotti, and Ferraris [2]. We also refer to [10, pp 727-728] for further discussion. For $k \geq 3$ there is no obstruction for existence of more than two homogeneous invariant Einstein metrics on Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-k)$.

In a recent joint work with Dzhepko and Nikonorov [6] we developed a method of finding invariant Einstein metrics on certain homogeneous spaces of classical Lie groups, and as a consequence we obtained new Einstein metrics on real and quaternionic Stiefel manifolds. For specifically, let G be a compact Lie group and H a closed subgroup so that G acts almost effectively on G/H. We investigate G-invariant metrics on G/H with additional symmetries, and the hope is to find among them, Einstein metrics coming from with simpler systems of algebraic equations.

Let K be a closed subgroup of G with $H \subset K \subset G$, and suppose that $K = L' \times H'$, where $\{e_{L'}\} \times H' = H$. It is clear that $K \subset N_G(H)$, the normalizer of H in G. If we denote $L = L' \times \{e_{H'}\}$, then the group $\widetilde{G} = G \times L$ acts on G/H by $(a,b)\cdot gH = agb^{-1}H$, and it turns out that the isotropy subgroup at eH is $\widetilde{H} = \{(a,b);\, ab^{-1} \in H\}$.

The set $\mathcal{M}^G$ of G-invariant metrics on G/H is finite dimensional. We consider the subset $\mathcal{M}^{G,K}$ of $\mathcal{M}^G$ corresponding to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$ (and not only $\mathrm{Ad}(H)$-invariant).

Let $\rho \in \mathcal{M}^{G,K}$. The action $\widetilde{G}$ on $(G/H, \rho)$ is isometric, so any metric form $\mathcal{M}^{G,K}$ can be identified with a metric in $\mathcal{M}^{\widetilde{G}}$ and vice-versa. Therefore, we may think of $\mathcal{M}^{\widetilde{G}}$ as $\mathcal{M}^{G,K}$, which is a subset of $\mathcal{M}^G$. Since metrics in $\mathcal{M}^{G,K}$ correspond to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$, we call these metrics *$\mathrm{Ad}(K)$-invariant metrics on G/H.*

We apply the above construction for $G = \mathrm{SO}(n)$ and $\mathrm{Sp}(n)$, and prove existence of Einstein metrics in the set $\mathcal{M}^{G,K}$ for various choices of the subgroup $K = L' \times H'$. Let $n \in \mathbb{N}$ and $k_1, k_2, \dots, k_s, k_{s+1}, \dots, k_{s+t}$ be natural numbers such that $k_1 + \cdots + k_s = l$, $k_{s+1} + \cdots + k_{s+t} = m$, $l + m = n$. Let $G = \mathrm{SO}(n)$ and $K = L' \times H'$, where $L' = \mathrm{SO}(k_1) \times \cdots \times \mathrm{SO}(k_s)$ and $H' = \mathrm{SO}(k_{s+1}) \times \cdots \times \mathrm{SO}(k_{t+s})$. The embedding of K in G is the standard one. Analogously, we consider $G = \mathrm{Sp}(n)$ and $K = L' \times H'$, where $L' = \mathrm{Sp}(k_1) \times \cdots \times \mathrm{Sp}(k_s)$ and $H' = \mathrm{Sp}(k_{s+1}) \times \cdots \times \mathrm{Sp}(k_{t+s})$.

We consider the simple case $\mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$ ($s = 2$, $t = 1$) and investigate $\mathrm{SO}(k_1+k_2+k_3) \times \mathrm{SO}(k_1) \times \mathrm{SO}(k_2)$-invariant Einstein metrics. Here $L' = \mathrm{SO}(k_1) \times \mathrm{SO}(k_2)$, and these metrics depend on five parameters x_1, x_2, x_{12}, x_{13}, x_{23}. The scalar curvature of such a metric is given by

$$
\begin{aligned}
S = {} & \frac{k_1(k_1-1)(k_1-2)}{8(n-2)} \cdot \frac{1}{x_1} + \frac{k_2(k_2-1)(k_2-2)}{8(n-2)} \cdot \frac{1}{x_2} \\
& + \frac{1}{2}\left(\frac{k_1k_2}{x_{(1,2)}} + \frac{k_1k_3}{x_{(1,3)}} + \frac{k_2k_3}{x_{(2,3)}}\right) - \frac{1}{8(n-2)}\Bigg(k_1k_2(k_1-1)\frac{x_1}{x_{(1,2)}^2} \\
& + k_1k_3(k_1-1)\frac{x_1}{x_{(1,3)}^2} + k_2k_3(k_2-1)\frac{x_2}{x_{(2,3)}^2} + k_1k_2(k_2-1)\frac{x_2}{x_{(1,2)}^2}\Bigg) \\
& - \frac{1}{4(n-2)}k_1k_2k_3\left(\frac{x_{(1,2)}}{x_{(1,3)}x_{(2,3)}} + \frac{x_{(1,3)}}{x_{(1,2)}x_{(2,3)}} + \frac{x_{(2,3)}}{x_{(1,2)}x_{(1,3)}}\right).
\end{aligned}
$$

The volume condition is

$$
x_1^{d_1} x_2^{d_2} x_{(1,2)}^{d_{(1,2)}} x_{(1,3)}^{d_{(1,3)}} x_{(2,3)}^{d_{(2,3)}} = \mathrm{const}
$$

where $d_i = \frac{1}{2}k_i(k_i-1)$ and $d_{(i,j)} = k_ik_j$.

By the Lagrange method the problem for finding $\mathrm{Ad}(K)$-invariant Einstein metrics reduces to the algebraic system

$$x_2x_{23}^2\Big((k_1-2)x_{12}^2x_{13}^2+k_2x_1^2x_{13}^2+k_3x_1^2x_{12}^2\Big)$$
$$=x_1x_{13}^2\Big((k_2-2)x_{12}^2x_{23}^2+k_3x_2^2x_{12}^2+k_1x_2^2x_{23}^2\Big)$$

$$x_{13}\Big((k_2-2)x_{12}^2x_{23}^2+k_3x_2^2x_{12}^2+k_1x_2^2x_{23}^2\Big)$$
$$=x_2x_{23}\Big(2(k_1+k_2+k_3-2)x_{12}x_{13}x_{23}-(k_1-1)x_1x_{13}x_{23}$$
$$-(k_2-1)x_2x_{13}x_{23}+k_3x_{12}^3-k_3x_{12}x_{13}^2-k_3x_{12}x_{23}^2\Big)$$

$$x_{13}\Big(2(k_1+k_2+k_3-2)x_{12}x_{13}x_{23}-(k_1-1)x_1x_{13}x_{23}-(k_2-1)x_2x_{13}x_{23}$$
$$+k_3x_{12}^3-k_3x_{12}x_{13}^2-k_3x_{12}x_{23}^2\Big)=x_{12}\Big(2(k_1+k_2+k_3-2)x_{12}x_{13}x_{23}$$
$$-(k_1-1)x_1x_{12}x_{23}+k_2x_{13}^3-k_2x_{12}^2x_{13}-k_2x_{13}x_{23}^2\Big)$$

$$x_{23}\Big(2(k_1+k_2+k_3-2)x_{12}x_{13}x_{23}-(k_1-1)x_1x_{12}x_{23}$$
$$+k_2x_{13}^3-k_2x_{12}^2x_{13}-k_2x_{13}x_{23}^2\Big)=x_{13}\Big(2(k_1+k_2+k_3-2)x_{12}x_{13}x_{23}$$
$$-(k_2-1)x_2x_{12}x_{13}+k_1x_{23}^3-k_1x_{12}^2x_{23}-k_1x_{13}^2x_{23}\Big).$$

It turns out ([6]) that a solution to the above system exists only when $k_1=k_2=k$ ($k\geq 3$), and let $k_3=l$. One of the results we obtain is the following

Proposition 18. *If $l>k\geq 3$ then the Stiefel manifold $\mathrm{SO}(2k+l)/\mathrm{SO}(l)$ admits at least four $\mathrm{SO}(2k+l)\times\mathrm{SO}(k)\times\mathrm{SO}(k)$-invariant Einstein metrics. Two of these metrics are Jensen's metrics found in*[16]*, and other two metrics are new.*

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References

[1] Alekseevsky D. and Arvanitoyeorgos A., *Riemannian Flag Manifolds with Homogeneous Geodesics*, Trans. Amer. Math. Soc. **359** (2007) 293–305.

[2] Alekseevsky D., Dotti I. and Ferraris C., *Homogeneous Ricci Positive 5-Manifolds*, Pacific J. Math. **175** (1996) 1–12.

[3] Arvanitoyeorgos A., *New Invariant Einstein Metrics on Generalized Flag Manifolds*, Trans. Amer. Math. Soc. **337** (1993) 981–995.

[4] Arvanitoyeorgos A., *An Introduction to Lie Groups and the Geometry of Homogeneous Spaces*, Amer. Math. Soc. STML 22, 2003.

[5] Arvanitoyeorgos A., *Geometry of Flag Manifolds*, Int. J. Geom. Meth. Mod. Phys. **3** (2006) 1–18.

[6] Arvanitoyeorgos A., Dzhepko V. and Nikonorov Yu., *Invariant Einstein Metrics on Some Homogeneous Spaces of Classical Lie Groups*, Canad. J. Math. (to appear).

[7] Back A. and Hsiang W., *Equivariant Geometry and Kervaire Spheres*, Trans. Amer. Math. Soc. **304** (1987) 207–227.

[8] Besse A., *Einstein Manifolds*, Springer, Berlin Heidelberg New York, 1987.

[9] Böhm C. and Kerr M., *Low Dimensional Homogenous Einstein Manifolds*, Trans. Amer. Math. Soc. **358** (2006) 1455–1468.

[10] Böhm C., Wang M. and Ziller W., *A Variational Approach for Compact Homogeneous Einstein Manifolds*, Geom. Func. Anal. **14** (2004) 681–733.

[11] Böhm C., *Homogeneous Einstein Metrics and Simplicial Complexes*, J. Diff. Geom. **67** (2004) 79–165.

[12] Bourbaki N., *Topologie générale*, Herman, Paris, 1947.

[13] D'Atri J. and Ziller W., *Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups*, Memoirs Amer. Math. Soc. **18** (1979).

[14] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Amer. Math. Soc., 2001.

[15] Jensen G., *The Scalar Curvature of Left Invariant Riemannian Metrics*, Indiana J. Math. **20** (1971) 1125–1144.

[16] Jensen G., *Einstein Metrics on Principal Fiber Bundles*, J. Diff. Geom. **8** (1973) 599–614.

[17] Kerr M., *New Examples of Homogeneous Einstein Metrics*, Michigan J. Math. **45** (1998) 115–134.

[18] Kobayashi S., *Topology of Positive Pinched Kähler Manifolds*, Tôhoku Math. J. **15** (1963) 121–139.

[19] Kowalski O. and Vanhecke L., *Riemannian Manifolds with Homogeneous Geodesics*, Boll. Un. Math. Ital. B **7** (1991) 189–246.

[20] Kimura M., *Homogeneous Einstein Metrics on Certain Kähler C-Spaces*, Adv. Stud. Pure Math. **18** (1990) 303–320.

[21] Lomshakov A., Nikonorov Yu. and Firsov E., *Invariant Einstein Metrics on Three-Locally-Symmetric Spaces* (in Russian), Mat. Tr. **6** (2003) 80–101; Engl. Transl. in Siberian Adv. Math. **14** (2004) 43–62.

[22] Nikonorov Yu., *On a Class of Homogeneous Compact Einstein Manifolds* (in Russian), Sibirsk. Mat. Zh. **41** (2000) 200–205; Engl. Transl. in Siberian Math. J. **41** (2000) 168–172.

[23] Onishchik A. (Ed), *Lie Groups and Lie Algebras I*, Encyclopaedia of Mathematical Sciences **20**, Springer, Berlin, 1993.

[24] O'Neil B., *Semi-Riemannian Geometry with Applications in Relativity*, Academic Press, New York, 1983.

[25] Sagle A., *Some Homogeneous Einstein Manifolds*, Nagoya Math. J. **39** (1970) 81–106.

[26] Sakane Y., *Homogeneous Einstein Metrics on Flag Manifolds*, Lobachevskii J. Math. **4** (1999) 71–87.

[27] Wallach N., *Compact Homogeneous Riemannian Manifolds with Strictly Positive Curvature*, Ann. Math. **96** (1972) 277–295.

[28] Wang M., *Einstein Metrics from Symmetry and Bundle Constructions*, In: Surveys in Differential Geometry: Essays on Einstein Manifolds, 287–325, Surv. Differ. Geom. vol. VI, Int. Press, Boston, 1999.

[29] Wang M. and Ziller W., *Existence and Non-Existence of Homogeneous Einstein Metrics*, Invent. Math. **84** (1986) 177–194.

[30] Wang M. and Ziller W., *Einstein Metrics with Positive Scalar Curvature*, In: Curvature and Topology of Riemannian Manifolds, Springer Lecture Notes in Mathematics vol. 1201, pp 319–336.

[31] Wang M. and Ziller W., *On Normal Homogeneous Einstein Metrics*, Ann. Sci. Ec. Norm. Sup. **18** (1985) 563–633.

[32] Warner F., *Foundations of Differential Manifolds and Lie Groups*, Springer, Berlin, 1983.