

ONE REMARK ON VARIATIONAL PROPERTIES OF GEODESICS IN PSEUDORIEMANNIAN AND GENERALIZED FINSLER SPACES

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Abstract. A new variational property of geodesics in (pseudo-)Riemannian and Finsler spaces has been found.

1. Introduction

Let us assume an n -dimensional **Finsler space** F_n with local coordinates $x \equiv (x^1, \dots, x^n)$ on the underlying manifold M_n , and a (positive definite) metric form with local expression

$$ds^2 = g_{ij}(x, \dot{x}) dx^i dx^j. \quad (1)$$

Here $g_{ij}(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n)$ are components of the metric tensor, and (x, \dot{x}) denote adapted local coordinates on the tangent bundle TM , i.e., $(\dot{x}^1, \dots, \dot{x}^n)$ are coordinates of the “tangent vector” \dot{x} at x . Metric depends on “positions” and “velocities” in general.

In the Finsler space F_n there exists a (fundamental) function $F(x, \dot{x})$ which is homogeneous of the second degree in \dot{x}^i and satisfies

$$g_{ij}(x, \dot{x}) = \frac{\partial^2 F(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}.$$

Particularly, the equality

$$F(x, \dot{x}) = g_{ij}(x, \dot{x}) dx^i dx^j$$

holds [3]. As it is well known, in the particular case when components of the metric tensor depend only on position coordinates (i.e., are independent of “velocity coordinates” \dot{x}) the Finsler space F_n turns out to be a **Riemannian space** V_n .

2. Pseudo-Riemannian and (Generalized) Finslerian Spaces

In what follows, the signature of the (non-degenerate) metric form is supposed to be arbitrary (we no more restrict ourselves onto positive definite metrics only) so that we can write

$$ds^2 = eg_{ij}(x, \dot{x}) dx^i dx^j, \quad e = \pm 1 \quad (2)$$

and the sign is determined in such a way that $ds^2 \geq 0$.

In short, we will call such metrics and spaces **Finslerian metrics** and **Finsler spaces** again, or *Riemannian*, respectively (more usually, they are called pseudo-Riemannian, or semi-Riemannian).

The arc length of a curve γ , given by parametrization $x^i = x^i(t)$, is given in a Finsler or Riemannian space (in our sense) by the integral

$$s = \int_{t_0}^{t_1} \sqrt{eg_{ij}(x(t), \dot{x}(t)) \dot{x}^i(t) \dot{x}^j(t)} dt, \quad \dot{x}^i(t) = \frac{dx^i(t)}{dt}. \quad (3)$$

It is well known [3], that this integral is stationary in a Finsler space if and only if its extremals are **geodesic curves** determined by the equations

$$\ddot{x}^h + 2G^h(x, \dot{x}) = \varrho(t) \dot{x}^h \quad (4)$$

where $\varrho(t)$ is a function, g^{ij} are components of the matrix inverse to (g_{ij}) , and

$$G^h = \frac{1}{2} g^{ij} \left(\frac{\partial^2 F(x, \dot{x})}{\partial \dot{x}^j \partial x^k} \dot{x}^k - \frac{\partial F(x, \dot{x})}{\partial \dot{x}^j} \right)$$

are components of the Berwald connection. Let us emphasize that extremals of the integral of length are independent of reparametrization of geodesics. In Riemannian spaces, [2, 3], the components read

$$G^h = \frac{1}{2} \Gamma_{ij}^h(x) \dot{x}^i \dot{x}^j$$

where Γ_{ij}^h are the Christoffels of second type.

Many authors define a **geodesic** in V_n as an extremal curve of the integral

$$I = \int_{t_0}^{t_1} g_{ij}(x) \dot{x}^i \dot{x}^j dt. \quad (5)$$

Extremals of this variational problem are those geodesics which satisfy the equations (4) with $\varrho(t) \equiv 0$.

Analogous situation is in Finsler spaces (in our generalized sense). Extremal curves of the integral (5) are determined together with their parameter, which is used to be called *canonical*. Note that particularly, arc length in V_n or F_n , respectively, is always canonical.

3. Generalized Variational Problem of Geodesics

In a Riemannian or in a Finsler space (in a more general sense explained above) consider the following more general **variational problem**

$$I = \int_{t_0}^{t_1} f(e g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j) d\tau \quad (6)$$

where e takes the values ± 1 , and $f(\tau)$ is a differentiable real-valued function (at least of class two) defined on some open domain $D \subset \mathbb{R}$ which is regular on D in the sense that $f'(\tau) \neq 0$ for all $\tau \in D$.

As an immediate consequence of the Euler-Lagrange equations for the Lagrange function $\mathcal{L} = f(e g_{ij} \dot{x}^i \dot{x}^j)$, it can be checked that the extremals satisfy the equations

$$\ddot{x}^h + 2G^h(x, \dot{x}) = -\frac{d}{dt} (\ln |f'(e g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta)|) \dot{x}^h. \quad (7)$$

We can prove the following theorem.

Theorem 1. *In (generalized) Finsler or Riemannian spaces, respectively, geodesic lines parameterized by a canonical parameter, which satisfy the condition*

$$e g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = k \in D$$

are extremals of the integral (6).

Theorem 2. *Consider (all) extremals of the integral (6) in a Finsler space (or in a Riemannian space, respectively). All curves arising under all possible regular reparameterizations of extremal curves belong to extremals, too, if and only if the function f takes the form $f(x) \equiv \alpha \sqrt{x}$ where α is some non-zero constant.*

Theorem 3. *All possible extremals of the integral (6) are just those geodesics which figure in Theorem 1 and Theorem 2. More precisely, in the particular case $f(x) \equiv \alpha \sqrt{x}$, $0 \neq \alpha = \text{const}$, they are represented by all unparameterized geodesics (i.e., geodesics under all possible regular reparameterizations), while for all other functions f , satisfying the above assumptions of the problem (6), extremals are represented just by canonically parameterized geodesics only.*

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References

- [1] Eisenhart L., *Riemannian Geometry*, Princeton Univ. Press, Princeton, 1926.
- [2] Radulovich Zh., Mikesh J. and Gavril'chenko M., *Geodesic Mappings and Deformations of Riemannian Spaces*, Podgorica, Odessa, 1997.
- [3] Rund H., *The Differential Geometry of Finsler Spaces*, Springer, Berlin, 1959.
- [4] Sinyukov N., *Geodesic Mappings of Riemannian Spaces*, Moscow, Nauka, 1979.