ON THE KAUP-KUPERSHMIDT EQUATION. COMPLETENESS RELATIONS FOR THE SQUARED SOLUTIONS

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Abstract. We regard a cubic spectral problem associated with the Kaup-Kupershmidt equation. For this spectral problem we prove a completeness of its "squared" solutions and derive the completeness relations which they satisfy. The spectral problem under consideration can be naturally viewed as a $\mathbb{Z}_3$-reduced Zakharov-Shabat problem related to the algebra $\mathfrak{sl}(3, \mathbb{C})$. This observation is crucial for our considerations.

1. Introduction

The **Kaup-Kupershmidt equation** (KKE) is a $1+1$ nonlinear evolution equation given by

$$\partial_t f = \partial_{x^5}^5 f + 10 f \partial_{x^3}^3 f + 25 \partial_x f \partial_{x^2}^2 f + 20 f^2 \partial_x f \tag{1}$$

where $f \in C^\infty(\mathbb{R}^2)$ and ∂_x stands for the partial derivative with respect to the variable x. It is S-integrable, i.e., it possesses a scalar Lax representation $\partial_t \mathcal{L} = [\mathcal{L}, \mathcal{A}]$ with Lax operators of the form

$$\mathcal{L} = \partial_{x^3}^3 + 2f\partial_x + \partial_x f \tag{2}$$

$$\mathcal{A} = 9\partial_{x^5}^5 + 30f\partial_{x^3}^3 + 45\partial_x f\partial_{x^2}^2 + (20f^2 + 35\partial_{x^2}^2 f)\partial_x + 10\partial_{x^3}^3 f + 20f\partial_x f. \tag{3}$$

It proves to be convenient to work not with scalar but with one-order matrix Lax operators. That is why we factorize the scattering operator $\mathcal{L}$ (see [4])

$$\mathcal{L} = (\partial_x - u)\partial_x(\partial_x + u) \tag{4}$$

where the new function $u(x, t)$ is interrelated with $f(x, t)$ via a Miura transformation as follows

$$f = \partial_x u - \frac{1}{2}u^2. \tag{5}$$

Taking into account that $\mathcal{L}$ defines the spectral problem $\mathcal{L}\psi = \lambda^3\psi$ one obtains that the matrix scattering operator reads

$$\mathcal{L} \to L = \partial_x + \mathcal{Q} - \lambda\mathcal{J} \tag{6}$$

where

$$\mathcal{Q} = \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -u \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

As we shall see further this represents a $\mathbb{Z}_3$-reduced generalized Zakharov-Shabat problem related to the algebra $\mathfrak{sl}(3,\mathbb{C})$. This is a crucial point in our considerations. That is why we are going to sketch in the next section some basic facts on the Lie algebras theory and on the inverse scattering transform.

The purpose of this work is to apply the general methods developed in [6] for proving completeness of the squared solutions of the corresponding **generalized Zakharov-Shabat system** (GZS). This shall be done in Section 3 (see Theorem 1).

2. Preliminaries

In this section we are going to remind the reader briefly all necessary facts on the theory of simple complex Lie algebras and inverse scattering transform and introduce the notation we aim to use later on. For a more detailed information about the theory of Lie algebras we refer to the book [7] while those who want to find a profound exposition of the inverse scattering method are referred to [11, 12].

2.1. Lie Algebras

Let $\mathfrak{g}$ be a simple Lie algebra, i.e., it does not have proper ideals. Its Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is the maximal commutative subalgebra. The dimension of $\mathfrak{h}$ is called rank of $\mathfrak{g}$. For $\mathfrak{sl}(r + 1, \mathbb{C})$ the Cartan subalgebra is r-dimensional and consists of all traceless diagonal matrices of $\mathfrak{h}$. The basis of $\mathfrak{h}$ bears the name a **Cartan basis**. The Cartan basis $\{H_k\}_{k=1}^r$ for $\mathfrak{sl}(r + 1, \mathbb{C})$ reads

$$H_k = E_{kk} - \frac{1}{r+1}\sum_{j}^{r+1} E_{jj}, \qquad k = 1, \ldots, r \tag{7}$$

where $(E_{ij})_{mn} = \delta_{im}\delta_{jn}$ is the Weyl basis of $\mathfrak{sl}(r + 1, \mathbb{C})$.

Any root $\alpha \in \mathfrak{h}^*$ satisfies by definition the equality

$$[H_k, E_\alpha] = \alpha(H_k)E_\alpha \tag{8}$$

where $E_\alpha \in \mathfrak{g}$ is called a root vector. The set of all roots Δ is known as a root system of the Lie algebra $\mathfrak{g}$. For $\mathfrak{sl}(r + 1)$ the roots can be presented by

$$e_i - e_j, \qquad i \neq j, \qquad i, j = 1, \ldots, r + 1 \tag{9}$$

where $\{e_k\}_{k=1}^r$ forms an orthonormal basis in the Euclidean space $\mathbb{R}^r$. The corresponding root vectors are

$$E_{e_i - e_j} = E_{ij}. \tag{10}$$

A root is called positive (negative) if its first nonzero component is positive (negative). Thus one introduces ordering in Δ and it splits into a subset of all positive roots $\Delta^+ \equiv \{\alpha \in \Delta; \alpha > 0\}$ and subset of all negative roots $\Delta^- \equiv \{\alpha \in \Delta; \alpha < 0\}$.

Positive roots $\{\alpha_j\} \in \Delta$ are said to be simple if all of them are linearly independent and $\alpha_i - \alpha_j \notin \Delta$. The simple roots form a "basis" in the set of all root Δ, i.e., each root is a linear combination of them. The set of all simple roots for $\mathfrak{sl}(r+1)$ is presented by

$$\alpha_j = e_j - e_{j+1}. \tag{11}$$

A positive root $\alpha_{\max}$ is called maximal if $\alpha_{\max} + \alpha \notin \Delta$ for any $\alpha \in \Delta^+$. By analogy one can introduce the notion of minimal root $\alpha_{\min}$ – it is a negative root to satisfy $\alpha_{\min} - \alpha \notin \Delta$. In the case of $\mathfrak{sl}(r+1)$ the maximal root is given by

$$\alpha_{\max} = e_1 - e_{r+1}, \qquad \alpha_{\min} = -\alpha_{\max}. \tag{12}$$

The set $\mathscr{A}$ of all simple roots and the minimal one bears the name system of admissible roots.

A reflection $S_\alpha : \mathbb{R}^r \to \mathbb{R}^r$ with respect to the hyperplane orthogonal to a root α leaves the set of roots Δ invariant. The symmetries of Δ form a finite group called Weyl group. A Coxeter automorphism $\mathcal{C}$ is a transformation induced by the reflections with respect to the simple roots, namely

$$\mathcal{C} = S_{\alpha_1} \circ S_{\alpha_2} \circ \cdots \circ S_{\alpha_r}.$$

One can prove that $\mathcal{C}$ is a finite order automorphism, i.e., there is an integer h such that $\mathcal{C}^h = \mathrm{Id}$. The number h is called Coxeter number. The Coxeter number for $\mathfrak{sl}(r+1, \mathbb{C})$ is simply $r+1$.

Due to Cartan's theorem every simple Lie algebra possesses a nondegenerate scalar product – the Killing form defined by

$$\langle X, Y \rangle \equiv \mathrm{tr}(\mathrm{ad}_X \, \mathrm{ad}_Y). \tag{13}$$

For $\mathfrak{sl}(r+1)$ the Killing metric has the form

$$\langle X, Y \rangle = \frac{1}{2} \mathrm{tr}(XY). \tag{14}$$

Another notion we are going to use is the so-called Casimir element (operator). The **Casimir operator** of some simple Lie algebra $\mathfrak{g}$ belongs to its universal enveloping

algebra and can be presented as quadratic polynomial of some elements of $\mathfrak{g}$. For example the second Casimir operator P of $\mathfrak{sl}(r+1)$ looks as follows

$$P = \sum_{j=1}^{r} H_j \otimes H_j + \sum_{\alpha \in \Delta} E_\alpha \otimes E_{-\alpha}. \tag{15}$$

It has the important property

$$P(A \otimes B) = (B \otimes A)P \quad \text{for all} \quad A, B \in \mathrm{SL}(r+1). \tag{16}$$

2.2. Spectral Problem for a Generic L Operator

Consider a generic (nonreduced) GZS

$$L\psi = (\mathrm{i}\partial_x + Q(x) - \lambda J)\psi = 0 \tag{17}$$

where J is a real nondegenerate Cartan element, i.e., $J \in \mathfrak{h} \subset \mathfrak{g}$ and $\alpha(J) \neq 0$ for all roots α, while $Q(x)$ is a linear combination of the Weyl generators E_α of $\mathfrak{g}$

$$Q(x) = \sum_{\alpha \in \Delta} Q_\alpha(x) E_\alpha. \tag{18}$$

The scattering operator under consideration differs from that one in the first section by a multiplication by an imaginary unit (compare with (6)). This is not a principal issue and it is just a matter of technical convenience.

The fundamental solutions $\psi(x, \lambda)$ take values in the Lie group G corresponding to $\mathfrak{g}$. In the simplest case of zero boundary conditions, i.e., $\lim_{x \to \pm\infty} Q(x) = 0$ the continuous part of the spectrum of L fills up the real axis of the complex λ-plane.

A basic notion in the theory of inverse scattering transform is Jost solution. The **Jost solutions** behave at infinity as plane waves, namely

$$\lim_{x \to \pm\infty} \psi_\pm(x, \lambda) \mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}. \tag{19}$$

The transition matrix between the Jost solutions $T(\lambda) = \hat{\psi}_+(x, \lambda)\psi_-(x, \lambda), \lambda \in \mathbb{R}$ is called a scattering matrix. The Jost solutions and the scattering matrix are defined only on the imaginary axis. One can prove [5, 10] there exist fundamental solutions $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ which possess analytical properties in the upper ($\Im\lambda > 0$) and lower ($\Im\lambda < 0$) half plane $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. The fundamental solutions $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ are interrelated via

$$\chi^-(x, \lambda) = \chi^+(x, \lambda)G(\lambda), \qquad \lambda \in \mathbb{R} \tag{20}$$

$$G(\lambda) = \hat{S}^-(\lambda)S^+(\lambda) = \hat{D}^-(\lambda)\hat{T}^+(\lambda)T^-(\lambda)D(\lambda).$$

The matrices $S^\pm(\lambda), T^\pm(\lambda)$ have a triangular form while $D^\pm(\lambda)$ are diagonal and they represent factors in the Gauss decomposition of the scattering matrix $T(\lambda)$

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda).$$

2.3. Algebraic Reductions

Let us consider the action of a discrete group G_R called a **reduction group** by Mikhailov [9] on the set fundamental solutions $\{\psi(x, \lambda)\}$ as follows

$$\mathrm{Ad}_C\, \psi(x, \kappa(\lambda)) = \tilde{\psi}(x, \lambda)$$

where Ad stands for the adjoint action of G in the Lie algebra $\mathfrak{g}$ induced by G_R. We require that the linear problem

$$(\mathrm{i}\partial_x + \mathcal{Q} - \lambda\mathcal{J})\psi = 0 \tag{21}$$

where $\mathcal{Q}(x)$ and $\mathcal{J}$ are assumed at this point to be arbitrary elements of the simple Lie algebra $\mathfrak{g}$, is G_R-invariant which immediately yields to the following restrictions

$$\mathrm{Ad}_C\, \mathcal{Q}(x) = \mathcal{Q}(x), \qquad \kappa(\lambda)\,\mathrm{Ad}_C\, \mathcal{J} = \lambda\mathcal{J}. \tag{22}$$

Thus, the number of the independent components of $\mathcal{Q}(x)$ is reduced and that is why G_R is called a reduction group.

In particular, let $G_R = \mathbb{Z}_h$ (h is the Coxeter number of $\mathfrak{g}$) and $\mathbb{Z}_h$ acts on $\mathfrak{g}$ by Coxeter morphisms

$$\kappa : \lambda \to \omega\lambda, \qquad \omega = \mathrm{e}^{2\mathrm{i}\pi/h}, \qquad C = \exp(\sum_k \omega^k H_k). \tag{23}$$

The symmetry conditions (22) imply that the matrices $\mathcal{Q}(x)$ and $\mathcal{J}$ have the form

$$\mathcal{Q} = \sum_{k=1}^{r} \mathcal{Q}_k H_k, \qquad \mathcal{J} = \sum_{\alpha \in \mathscr{A}} E_\alpha. \tag{24}$$

We remind that $\mathscr{A}$ stands for the set of all admissible roots (simple + minimal root) of $\mathfrak{g}$. Hence the existence of reduction determines uniquely the form of the matrices $\mathcal{Q}(x)$ and $\mathcal{J}$.

As we saw in the previous section the spectral problem for KKE was associated with the $\mathfrak{sl}(3)$ algebra and the matrices $\mathcal{Q}(x)$ and $\mathcal{J}$ have exactly the same form as shown in (24). Thus, KKE is naturally related to a $\mathbb{Z}_3$-reduced L operator associated with $\mathfrak{sl}(3)$. For the sake of convenience we shall consider the gauge equivalent system of (21) – the one which has a diagonal matrix $J \in \mathfrak{h}$ and a potential $Q(x)$ as a nondiagonal matrix with zero diagonal elements. The eigenvalues of J are the cubic roots of 1, i.e., $J = \mathrm{diag}(1, \omega, \omega^2)$, where $\omega = \mathrm{e}^{2\mathrm{i}\pi/3}$. This allows us to apply the general results to be described in the next subsection to the problem which we solve.

2.4. Caudrey-Beals-Coifman Systems

Spectral problems with Coxeter type reductions imposed on them are studied for the first time by Caudrey [3], Beals and Coifman [1, 2] in the case of the algebra $\mathfrak{sl}(n)$ (see also [9]). That is why they bear their names – **Caudrey-Beals-Coifman** (CBC) **systems**. Further generalization for an arbitrary simple Lie algebra was provided by Gerdjikov and Yanovski [6].

In case of a presence of a $\mathbb{Z}_h$ Coxeter type reduction the spectral properties of the scattering operator (17) change substancially. The continuous spectrum now is a bunch of $2h$ rays l_ν closing equal angles π/h and the Cartan element J must be complex by all means. The λ-plane is split into $2h$ regions of analyticity Ω_ν, $\nu = 1, \ldots, 2h$, meaning that in every sector there exists a fundamental analytic solution $\chi^\nu(x, \lambda)$. In each sector Ω_ν there exist equal number of discrete eigenvalues λ_{n_ν}. They are situated symmetrically. With each ray l_ν it can be associated a subset $\delta_\nu \subset \Delta$ defined by

$$\delta_\nu \equiv \{\alpha \in \Delta\,;\, \Im\lambda\alpha(J) = 0,\ \text{for all } \lambda \in l_\nu\}, \qquad \nu = 1, \ldots, 2h \qquad (25)$$

and a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$ generated by all roots of δ_ν

$$\mathfrak{g}_\nu \equiv \{E_\alpha, H_\alpha,\ \alpha \in \delta_\nu\}. \qquad (26)$$

Obviously, $\Delta = \bigcup_{\nu=1}^{h} \delta_\nu$ and if $\alpha \in \delta_\nu$ then $-\alpha \in \delta_\nu$. Moreover, the following relations hold true $\delta_\nu = \delta_{\nu+h}$. One can introduce ordering in Ω_ν by defining "positive"and "negative" roots in Ω_ν as follows

$$\Delta_\nu^\pm \equiv \{\alpha \in \Delta\,;\, \Im\lambda\alpha(J) \gtrless 0,\ \text{for all } \lambda \in \Omega_\nu\}. \qquad (27)$$

We shall use the auxiliary notation $\delta_\nu^\pm = \Delta_\nu^\pm \cap \delta_\nu$ as well. As a direct consequence of (27) one can check that the following symmetries hold

$$\Delta_{\nu+h}^\pm = \Delta_\nu^\mp, \qquad \delta_{\nu+h}^\pm = \delta_\nu^\mp. \qquad (28)$$

The Coxeter automorphism induces a natural $\mathbb{Z}_h$ grading in the Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{k=1}^{h} \mathfrak{g}^{(k)}, \qquad \mathfrak{g}^{(k)} \equiv \{X \in \mathfrak{g}\,;\, CXC^{-1} = \omega^k X\}, \qquad \omega^h = 1. \qquad (29)$$

It can be verified that the grading requirement $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}] \subset \mathfrak{g}^{(k+l)}$ is satisfied.

The fundamental analytic solutions $\chi^\nu(x, \lambda)$ with adjacent indices are interrelated via a **local Riemann-Hilbert problem**

$$\chi^\nu(x, \lambda) = \chi^{\nu-1}(x, \lambda)G^\nu(\lambda), \qquad \lambda \in l_\nu$$
$$G^\nu(\lambda) = \hat{S}_\nu^-(\lambda)S_\nu^+(\lambda) = \hat{D}_\nu^-(\lambda)\hat{T}_\nu^+(\lambda)T_\nu^-(\lambda)D_\nu^+(\lambda). \qquad (30)$$
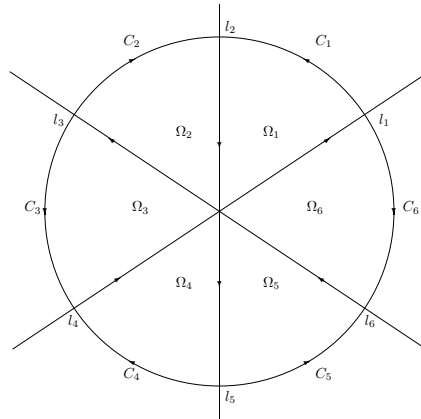
Figure 1. Contours of integration $\gamma_\nu = l_\nu \cup C_\nu \cup l_{\nu+1}$.

The matrices $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$ are the corresponding Gauss factors of the scattering matrix $T_\nu(\lambda)$. They all belong to the Lie subgroup $G^{(\nu)} \subset G$ related to the subalgebra $\mathfrak{g}^{(\nu)} \subset \mathfrak{g}$.

From now on we shall consider a L operator associated with the $\mathfrak{sl}(3)$ algebra with a $\mathbb{Z}_3$ reduction. This spectral problem was investigated by Kaup in [8]. In this case the complex λ-plane is separated into six regions by six rays as it is shown in Fig. 1. Each ray is connected with only one positive root as it is presented in Table 1. Hence, one can associate a $\mathfrak{sl}(2)$ subalgebra with each ray — this is the

Table 1

Ray l_ν	l_1, l_4	l_2, l_5	l_3, l_6
Roots of δ_ν	$\pm(e_1 - e_2)$	$\pm(e_2 - e_3)$	$\pm(e_1 - e_3)$

algebra $\{E_\alpha, E_{-\alpha}, H_\alpha\}$ generated by the positive root α. As we discussed before the algebra $\mathfrak{sl}(3)$ gets $\mathbb{Z}_3$-grading, i.e., we have

$$\mathfrak{sl}(3) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}.$$

3. Completeness Relations for Squared Solutions of a $\mathbb{Z}_3$ Reduced Scattering Problem Related to the Algebra $\mathfrak{sl}(3, \mathbb{C})$

The fundamental analytic solutions allow one to introduce the so-called squared solutions (eigenfunctions) by

$$e_\alpha^{(\nu)}(x, \lambda) = P_J(\chi^\nu E_\alpha \hat{\chi}^\nu(x, \lambda)), \qquad h_j^{(\nu)}(x, \lambda) = P_J(\chi^\nu H_j \hat{\chi}^\nu(x, \lambda)) \quad (31)$$

where P_J stands for the mapping onto the quotient space $\mathfrak{sl}(3,\mathbb{C})/\ker\mathrm{ad}_J$. Since J is diagonal the kernel of ad_J obviously coincides with the subspace of all diagonal matrices. The squared solutions occur naturally in Wronskian relations. One typical example of such an Wronskian relation is the following one

$$(\hat{\chi}^{\nu}J\chi^{\nu}(x,\lambda)-J)|_{-\infty}^{\infty}=\int_{-\infty}^{\infty}\mathrm{d}x\,\hat{\chi}^{\nu}[J,Q(x)]\chi^{\nu}(x,\lambda). \tag{32}$$

Next theorem holds true

Theorem 1. *The squared solutions* (31) *form a complete set with the following completeness relations*

$$\delta(x-y)\Pi=\frac{1}{2\pi}\sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{l_{\nu}}\mathrm{d}\lambda\left(e_{\beta_{\nu}}^{(\nu)}(x,\lambda)\otimes e_{-\beta_{\nu}}^{(\nu)}(y,\lambda)\right.$$

$$\left.-e_{-\beta_{\nu}}^{(\nu-1)}(x,\lambda)\otimes e_{\beta_{\nu}}^{(\nu-1)}(y,\lambda)\right)-\mathrm{i}\sum_{\nu=1}^{6}\sum_{n_{\nu}}\operatorname*{Res}_{\lambda=\lambda_{n_{\nu}}}G^{(\nu)}(x,y,\lambda). \tag{33}$$

where

$$\Pi=\sum_{\alpha\in\Delta^{+}}\frac{E_{\alpha}\otimes E_{-\alpha}-E_{-\alpha}\otimes E_{\alpha}}{\alpha(J)},\ G_{\beta_{\nu}}^{(\nu)}(x,y,\lambda)=e_{\beta_{\nu}}^{(\nu)}(x,\lambda)\otimes e_{-\beta_{\nu}}^{(\nu)}(y,\lambda).$$

Proof: We will derive the completeness relations (33) by simply applying Cauchy's residue theorem in calculating the expression

$$\mathcal{J}(x,y)=\sum_{\nu=1}^{6}(-1)^{\nu+1}\oint_{\gamma_{\nu}}G^{(\nu)}(x,y,\lambda)\,\mathrm{d}\lambda \tag{34}$$

where the contours γ_{ν} are shown in Fig. 1 and the Green functions $G^{(\nu)}(x,y,\lambda)$ have the form

$$G^{(\nu)}(x,y,\lambda)=\theta(y-x)\sum_{\alpha\in\Delta_{\nu}^{+}}e_{\alpha}^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda)-\theta(x-y)$$

$$\times\left[\sum_{\alpha\in\Delta_{\nu}^{-}}e_{\alpha}^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda)+\sum_{j=1}^{2}h_{j}^{(\nu)}(x,\lambda)\otimes h_{j}^{(\nu)}(y,\lambda)\right].$$

According to Cauchy's theorem $\mathcal{J}(x,y)$ is equal to the sum of all residues of the integrands, namely

$$\mathcal{J}(x,y)=2\pi\mathrm{i}\sum_{\nu=1}^{6}\sum_{n_{\nu}}\operatorname*{Res}_{\lambda=\lambda_{n_{\nu}}}G^{(\nu)}(x,y,\lambda). \tag{35}$$

On the other hand taking into account the orientation of the contours γ_ν the integrals in the expression (34) can be regrouped to obtain

$$
\mathcal{J}(x,y) = \sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{l_\nu}(G^{(\nu)}(x,y,\lambda) - G^{(\nu-1)}(x,y,\lambda))\,\mathrm{d}\lambda
$$
$$
+ \sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{C_\nu}G^{(\nu)}(x,y,\lambda)\,\mathrm{d}\lambda. \tag{36}
$$

Next important result underlies the proof of our theorem

Lemma 1. *The following equality is valid for any $\lambda \in l_\nu$*

$$
\sum_{\alpha\in\Delta}e_\alpha^{(\nu-1)}(x,\lambda)\otimes e_{-\alpha}^{(\nu-1)}(y,\lambda) + \sum_{j=1,2}h_j^{(\nu-1)}(x,\lambda)\otimes h_j^{(\nu-1)}(y,\lambda)
$$
$$
= \sum_{\alpha\in\Delta}e_\alpha^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda) + \sum_{j=1,2}h_j^{(\nu)}(x,\lambda)\otimes h_j^{(\nu)}(y,\lambda). \tag{37}
$$

Proof of Lemma 1: The proof is based on the interrelation (30), the definition of $\chi^\nu(x,\lambda)$ and the properties of the Casimir operator P (see formula (16)). $\qquad\square$

The terms corresponding to the integrals along the rays in (36) can be simplified due to the following lemma

Lemma 2. *In the integrals along the rays contribute only terms related to the roots that belong to δ_ν^+ and δ_ν^- respectively, i.e.,*

$$
G^{(\nu)}(x,y,\lambda) - G^{(\nu-1)}(x,y,\lambda)
$$
$$
= e_{\beta_\nu}^{(\nu)}(x,\lambda)\otimes e_{-\beta_\nu}^{(\nu)}(y,\lambda) - e_{-\beta_\nu}^{(\nu-1)}(x,\lambda)\otimes e_{\beta_\nu}^{(\nu-1)}(y,\lambda). \tag{38}
$$

Proof of Lemma 2: As a consequence of Lemma 1 one can verify that

$$
G^{(\nu)}(x,y,\lambda) - G^{(\nu-1)}(x,y,\lambda)
$$
$$
= \sum_{\alpha\in\Delta_\nu^+}e_\alpha^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda) - \sum_{\alpha\in\Delta_{\nu-1}^+}e_\alpha^{(\nu-1)}(x,\lambda)\otimes e_{-\alpha}^{(\nu-1)}(y,\lambda).
$$

At this point we make use of the property $\Delta_\nu^+\backslash\delta_\nu^+ = \Delta_{\nu-1}^+\backslash\delta_\nu^-$ and the fact that the sewing function $G^\nu(\lambda)$ is an element of SL(2) group related to l_ν. Then the sums in $G^{(\nu)}(x,y,\lambda)$ and in $G^{(\nu-1)}(x,y,\lambda)$ over these subsets annihilate each other and what survive are terms corresponding to the subsets δ_ν^+ and δ_ν^- respectively. $\qquad\square$

It remains to evaluate the integrals along the arcs C_ν. For that purpose we have to use the asymptotic behavior of $G^{(\nu)}(x,y,\lambda)$ as $\lambda \to \infty$. It is given by the expression

$$G^{(\nu)}(x, y, \lambda) \underset{\lambda \to \infty}{\approx} \sum_{\alpha \in \Delta_\nu^+} \mathrm{e}^{\mathrm{i}\lambda\alpha(J)(y-x)} E_\alpha \otimes E_{-\alpha} - \theta(x-y)$$

$$\times \left(\sum_{\alpha \in \Delta} \mathrm{e}^{\mathrm{i}\lambda\alpha(J)(y-x)} E_\alpha \otimes E_{-\alpha} + \sum_{j=1,2} H_j \otimes H_j \right).$$

Asymptotically $G^{(\nu)}(x, y, \lambda)$ is an entire function, hence we are allowed to deform the arcs C_ν into $\bar{l}_\nu \cup \bar{l}_{\nu+1}$. Consequently the integrals along the arcs C_ν can be rewritten in the following manner

$$\sum_{\nu=1}^6 (-1)^{\nu+1} \int_{C_\nu} G^{(\nu)}(x, y, \lambda) \, \mathrm{d}\lambda = \sum_{\nu=1}^6 (-1)^{\nu+1} \int_{l_\nu} \mathrm{d}\lambda$$

$$\times \left(\mathrm{e}^{-\mathrm{i}\lambda\beta_\nu(J)(y-x)} E_{-\beta_\nu} \otimes E_{\beta_\nu} - \mathrm{e}^{\mathrm{i}\lambda\beta_\nu(J)(y-x)} E_{\beta_\nu} \otimes E_{-\beta_\nu} \right).$$

After we combine the term associated with l_ν and that one associated with $l_{\nu+3}$ and recall the well known formula for the Fourier transform of Dirac's delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, \mathrm{e}^{\mathrm{i}\lambda x} = \delta(x)$$

we derive the result

$$2\pi\delta(x-y) \sum_{\alpha \in \Delta^+} \frac{(E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha)}{\alpha(J)}. \tag{39}$$

Thus, taking into account (35), (36), (38) and (39) we finally reach the completeness relations (33). □

Remark: All elements $a \in \mathfrak{sl}(3)$ admit a uniquely determined $\mathbb{Z}_3$ expansion, for example $Q(x) \in \mathfrak{g}^{(0)}$ while the squared solutions can be expanded as follows

$$e_\alpha^{(\nu)}(x, \lambda) = e_{\alpha,0}^{(\nu)}(x, \lambda) + e_{\alpha,1}^{(\nu)}(x, \lambda) + e_{\alpha,2}^{(\nu)}(x, \lambda), \qquad e_{\alpha,k}^{(\nu)}(x, \lambda) \in \mathfrak{g}^{(k)}.$$

Then we have completeness relations for all components $e_{\alpha,k}^{(\nu)}(x, \lambda)$.

Completeness of the squared solutions means that each function $X(x)$ with values in $\mathfrak{sl}(3, \mathbb{C})/\operatorname{ad}_J$ can be expanded over them, namely

$$X(x) = \frac{1}{2\pi} \sum_{\nu=1}^6 (-1)^{\nu+1} \int_{l_\nu} \mathrm{d}\lambda \left(X_{\beta_\nu}(\lambda) e_{-\beta_\nu}^{(\nu)}(x, \lambda) - X_{-\beta_\nu}(\lambda) e_{\beta_\nu}^{(\nu-1)}(x, \lambda) \right)$$

$$- \mathrm{i} \sum_{\nu=1}^6 \sum_{n_\nu} X_{n_\nu}$$

where the components of $X(x)$ are given by

$$X_{\beta_\nu}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}y \langle \mathrm{ad}_J \, e_{\beta_\nu}^{(\nu)}(y, \lambda), X(y) \rangle$$

$$X_{-\beta_\nu}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}y \langle \mathrm{ad}_J \, e_{-\beta_\nu}^{(\nu-1)}(y, \lambda), X(y) \rangle$$

$$X_{n_\nu} = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{tr}_1 \left(\mathrm{ad}_J \otimes \mathbb{1} \operatorname*{Res}_{\lambda=\lambda_{n_\nu}} G^{(\nu)}(x, y, \lambda) X \otimes \mathbb{1} \right).$$

Here tr_1 means taking the trace of the first multiplier in the tensor product.

4. Conclusion

We have demonstrated that the squared solutions to the scattering problem connected with the Kaup-Kupershmidt equation form a complete system in the space of functions which take values in $\mathfrak{g}/\operatorname{ad}_J$. This allows one to expand any function which belongs to this space in series over the squared solutions. As a matter of fact the squared solutions represent a generalization of the plane waves $\mathrm{e}^{\mathrm{i}kx}$ in the standard Fourier analysis. This quite general result motivates the interpretation of the inverse scattering transform as a generalization of the Fourier transform. In order to prove the completeness relations we have applied the contour integration technique (Cauchy's residue theorem) to an appropriate contour. The spectral properties of the scattering operator L affect the structure of the completeness relations themselves: there are terms associated with the continuous part of its spectrum and terms related to its discrete eigenvalues λ_{n_ν} (see (33)).

Acknowledgements

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