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IS THE MAXIMAL FUNCTION OF A LIPSCHITZ FUNCTION CONTINUOUS?

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Abstract. We examine the action of the maximal operator on Lipschitz and Hölder functions in the context of homogeneous spaces. Boundedness results are proven for spaces satisfying an annular decay property and counterexamples are given for some other spaces. The annular decay property is defined and investigated.

0. Introduction

The Hardy–Littlewood maximal operator M is bounded on $L^p(\mathbf{R}^n)$, $1 < p \le \infty$, according to the well-known theorem of Hardy, Littlewood, and Wiener. The action of M on some other Euclidean function spaces, such as BMO and rearrangement-invariant Banach function spaces, is also well-understood; see [BDS], [L] and [S]. Recently, Kinnunen [Ki] showed that M is bounded on the Sobolev space $W^{1,p}(\mathbf{R}^n)$, $1 < p \le \infty$; see also [KL]. It is well known that $W^{1,\infty}(\mathbf{R}^n)$ consists precisely of all bounded Lipschitz functions, and so Kinnunen's $p = \infty$ case says that M is bounded on this class (of course, $W^{1,p}$ spaces for $p < \infty$ are also closely related to Lipschitz spaces; see [H]).

We shall see in Section 1 that a weaker version of this endpoint result holds for any doubling measure μ on $\mathbf{R}^n$. Specifically, the maximal operator with respect to μ takes the Lipschitz space $\mathrm{Lip}_1(\mathbf{R}^n)$ to the "Hölder space" $\mathrm{Lip}_t(\mathbf{R}^n)$, for some $t > 0$ (definitions are given in Section 1). More generally, this paper is concerned with the action of M on Lipschitz or Hölder spaces over homogeneous spaces (X, d, μ), where M is now defined as a supremum of μ-averages over centered metric balls. Since the pioneering work of Coifman and Weiss [CW], it has been known that much of the theory of harmonic analysis on $\mathbf{R}^n$ carries over to the setting of homogeneous spaces. One might therefore guess that if $f \in \mathrm{Lip}_1(X)$, then $Mf \in \mathrm{Lip}_t(X)$ for some $t > 0$. In Section 1, we shall see that this is incorrect—in fact, Mf may even be discontinuous. However, we show that if μ satisfies what we call an *annular decay property*, and $0 < t \le 1$, then

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$M\colon \mathrm{Lip}_t(X) \to \mathrm{Lip}_s(X)$, for some $0 < s \le t$. Furthermore, if t is small or if μ satisfies what we call a strong annular decay property, we can take $s = t$. In Section 2, we show that if (X,d) is any of a large class of metric spaces, including all length spaces, then all doubling measures on X possess an annular decay property; by contrast, the strong annular decay property is typically valid for only a few of those measures.

1. The action of M on $\mathrm{Lip}_t(X)$

We say that (X,d,μ) is a *homogeneous space* if (X,d) is a metric space and μ is a doubling measure on X, i.e. a positive Borel measure for which there exists a constant C such that $0 < \mu\big(B(x,2r)\big) \le C\mu\big(B(x,r)\big)$ for all $x \in X$, $r > 0$; $B(x,r)$ denotes the set of all points y such that $d(x,y) < r$. We refer the reader to [CW] for an exposition of analysis on these spaces. The smallest value of C for which the doubling condition is valid is called the *doubling constant of* X, and we denote it as C_μ. If $B = B(x,r)$, we shall often write tB, $t > 0$, to denote its concentric dilate $B(x,tr)$. Note that the ball B, viewed simply as a set in a metric space, might not specify its center and radius uniquely; consequently, whenever we say that B is a ball, it is assumed that we are also specifying a center x and radius r, even if these are not explicitly given (this point is also significant in the definition of a chain space in Section 2). With this convention, the notation tB is well-defined in any metric space.

We denote by $⨍_S g\,d\mu$ the μ-average of a function g on a set S. The centered and uncentered maximal functions, Mf and $\widetilde{M}f$ respectively, of a locally integrable function $f\colon X \to \mathbf{R}$ are defined by

$$Mf(x) = \sup_{r>0} ⨍_{B(x,r)} |f|\,d\mu,$$
$$\widetilde{M}f(x) = \sup_{x\in B} ⨍_{B} |f|\,d\mu,$$

where the second supremum is taken over all balls B containing x.

For each $0 < t \le 1$, we say that the (continuous) function $f\colon X \to \mathbf{R}$ belongs to the Lipschitz class $\mathrm{Lip}_t(X)$ if f is bounded and there exists a constant C such that $|f(x) - f(y)| \le C d(x,y)^t$ for all $x, y \in X$; $\mathrm{Lip}_t(X)$ is a Banach space with norm

$$\|f\|_{\mathrm{Lip}_t(X)} = \|f\|_{L^\infty(X)} + \sup_{x\ne y} \frac{|f(x)-f(y)|}{d(x,y)^t}.$$

For $0 < t < 1$, the functions in $\mathrm{Lip}_t(X)$ are often called Hölder continuous functions (e.g. in Section 0 above), but a single name is more convenient for us as we wish to treat them collectively. We also define the related space $\mathrm{lip}_t(X)$,

$0 < t \le 1$, to consist of all (not necessarily bounded) functions f such that $|f(x) - f(y)| \le Cd(x,y)^t$ for all $x, y \in X$, and we define the associated seminorm

$$\|f\|_{\mathrm{lip}_t(X)} = \sup_{x \ne y} \frac{|f(x) - f(y)|}{d(x,y)^t}.$$

Note that $\mathrm{lip}_t(X) = \mathrm{Lip}_t(X)$ if X is bounded.

Given $0 < \delta \le 1$, and a homogeneous space (X, d, μ), we say that the measure μ (or more correctly, the space (X, d, μ)) satisfies the δ*-annular decay property* if there exists a constant $K \ge 1$ such that for all $x \in X$, $r > 0$, $0 < \varepsilon < 1$, we have

$$\mu\big(B(x,r) \setminus B\big(x, r(1-\varepsilon)\big)\big) \le K\varepsilon^\delta \mu\big(B(x,r)\big). \tag{1.1}$$

We omit the prefix "δ" in the above notation if we do not care about its value.

We use the term *strong annular decay property* as a synonym for the 1-annular decay property. In the next section, we show that for many metric spaces (X, d), any doubling measure that we put on X must satisfy an annular decay property. Typically, however, few of these measures satisfy the strong annular decay (for instance, the reader can readily verify that this is the case with $X = \mathbf{R}^n$).

We now show that spaces satisfying an annular decay property map the function spaces $\mathrm{Lip}_t(X)$ among themselves, and the same is true for the spaces $\mathrm{lip}_t(X)$ if t is small.

Theorem 1.1. *Suppose that* $0 < t, \delta \le 1$*, and that* (X, d, μ) *is a homogeneous space with the* δ*-annular decay property. Then* $M\colon \mathrm{Lip}_t(X) \to \mathrm{Lip}_s(X)$*, where* $s = \min(t, \delta)$.

Theorem 1.2. *Suppose that* $0 < t \le \delta \le 1$*, and that* (X, d, μ) *is a homogeneous space with the* δ*-annular decay property. Then* $M\colon \mathrm{lip}_t(X) \to \mathrm{lip}_t(X)$.

We omit the proof of Theorem 1.2, as it is essentially the same as the proof of the $t \le \delta$ case of Theorem 1.1. Note that if X is bounded, Theorem 1.1 implies Theorem 1.2.

Proof of Theorem 1.1. We fix $f \in \mathrm{Lip}_t(X)$, normalized so that $\|f\|_{\mathrm{Lip}_t(X)} = 1$. First, note that $\|Mf\|_{L^\infty(X)} = \|f\|_{L^\infty(X)}$, and so we only need to find an appropriate bound for differences in values of Mf. Fixing an arbitrary pair of points $x, y \in X$, we write $a = d(x,y)$. By symmetry of x, y, it suffices to find C (independent of x, y) such that $Mf(y) \ge Mf(x) - Ca^s$. We may assume that $a \le 1$, since otherwise the bound on Mf alone gives this inequality.

We choose $r > 0$ such that $Mf(x) \le \fint_{B(x,r)} |f| \, d\mu + a^s$. If $r \le a$, then $|f(x) - f(z)| \le 3^t a^t$ for $z \in B(x,r) \cup B(y,r)$, and so

$$\left| \fint_{B(x,r)} |f| \, d\mu - \fint_{B(y,r)} |f| \, d\mu \right| \le 3^t a^t \le 3^t a^s,$$

which readily gives the required inequality.

For each $0 < c < \infty$, let S_c be the class of functions $g \in L^1\big(B(x, r+2a)\big)$ satisfying

$$⨍_{B(x,r)} |g|\, d\mu - ⨍_{B(y,r+a)} |g|\, d\mu \le ca^s.$$

To finish the proof, we show that if $r > a$, then there exists such a constant c, independent of x, y, such that our function f lies in the class S_c. We claim that this will follow if there exists such a constant c such that $F \subset S_c$, where

$$F = \big\{g : \|g\|_{\mathrm{Lip}_t(B(x,r+2a))} \le 1,\ \|g\|_{L^\infty(B(x,r+2a))} \le A \equiv \min\{1, (6r)^t\}\big\}.$$

Since $\|f\|_{L^\infty(X)} \le 1$, this claim is obvious if $A = 1$. Suppose therefore that $A < 1$ and that $F \subset S_c$ for some such constant c. Letting $m = \inf_{z\in B(x,r+2a)} f(z)$ and $M = \sup_{z\in B(x,r+2a)} f(z)$, the Lipschitz estimate for f and the fact that $r > a$ imply that $f_1 = f - m$ and $f_2 = f - M$ lie in F. Furthermore if $f(z_0) \ge A$ for some $z_0 \in B(x, r+2a)$, then f and f_1 are both non-negative on $B(x, r+2a)$. Since $f_1 \in S_c$, it readily follows that $f \in S_c$. Similarly if $f(z_0) \le -A$, then f and f_2 are both non-positive on $B(x, r+2a)$, and so $f \in S_c$ because $f_1 \in S_c$. This justifies our claim.

It is left to show that $F \subset S_c$. If $g \in F$ then

$$\begin{aligned}
⨍_{B(x,r)} |g|\, d\mu - ⨍_{B(y,r+a)} |g|\, d\mu &\le \left[\frac{1}{\mu\big(B(x,r)\big)} - \frac{1}{\mu\big(B(y,r+a)\big)}\right] \cdot \int_{B(x,r)} |g| \\
&\le \left[\frac{1}{\mu\big(B(x,r)\big)} - \frac{1}{\mu\big(B(x,r+2a)\big)}\right] \cdot A\mu\big(B(x,r)\big) \\
&= A \cdot \left[\frac{\mu\big(B(x,r+2a)\big) - \mu\big(B(x,r)\big)}{\mu\big(B(x,r+2a)\big)}\right] \\
&\le AK \cdot \big(a/(r+2a)\big)^{-\delta},
\end{aligned}$$

where the last inequality follows from (1.1). To finish the proof we need only note that if $\delta \ge t$, then $AK \cdot \big(a/(r+2a)\big)^{-\delta} \le K(6r)^t \cdot \big(a/(r+2a)\big)^{-\delta} \le Ca^t$, while if $\delta < t$, the factor $A(r+2a)^{-\delta}$ is at most 1 if $r \ge 1$, and at most 6^t if $r < 1$. □

We now state a few points related to Theorems 1.1 and 1.2. The rest of this section is devoted to justifying these statements.

(A) These theorems remain true if we replace M with $\widetilde{M}$. As a very special case, we note that the case $p = \infty$ of the main result in [Ki] remains true in the uncentered case, i.e. the uncentered Hardy–Littlewood maximal operator is bounded on $\mathrm{Lip}_1(\mathbf{R}^n)$, because Lebesgue measure satisfies the strong annular decay property.

(B) The indices are sharp in the sense that s is maximal in Theorem 1.1, and the restriction $t \le \delta$ is necessary in Theorem 1.2.

(C) If no annular decay property is assumed, then Mf can fail to be continuous, even if $f \in \mathrm{Lip}_1(X)$.

Justifying (A) reduces to a routine set of adjustment to the proof of our theorems; we merely remark that in the case $r > a$, where the role of the ball $B(x,r)$ is now taken by a ball $B(z,r)$ containing x, the balls $B(y, r+a)$ and $B(x, r+2a)$ are both replaced by $B(z, r+a)$.

As for (B), it is easy to convince oneself, by considering the simple example $X = \mathbf{R}$ with Euclidean distance and Lebesgue measure attached, that M does not in general map $\mathrm{lip}_t(X)$ to $\mathrm{lip}_s(X)$ if $s \ne t$ (first note that these spaces, unlike the spaces $\mathrm{Lip}_s(X)$, are not necessarily nested). The following example of a "smooth" compactly supported function on a homogeneous space whose maximal function is Hölder of order no better than δ, shows that M does not in general map $\mathrm{lip}_t(X)$ to $\mathrm{lip}_s(X)$ for any $s > \delta$, and so the restriction $t \le \delta$ in Theorem 1.2 is necessary. This example also shows that the parameter s is maximal in Theorem 1.1 in the case $\delta < t$ is implied by $\delta \ge t$. Since it is easy to show that s is sharp in the case $\delta \ge t$, we have therefore justified (B).

Example 1.3. Fix $0 < \delta < 1$, let d be the Euclidean metric on $\mathbf{R}$, and let $d\mu = w\,dx$, where w is defined by

$$w(x) = \begin{cases} \delta|x-2|^{\delta-1}, & 0 < |x-2| < 1, \\ \delta|x+2|^{\delta-1}, & 0 < |x+2| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly $(\mathbf{R}, d, \mu)$ is a homogeneous space satisfying the δ-annular decay property. In fact, $\mu([n-\varepsilon, n]) = \mu([n, n+\varepsilon]) = \varepsilon^\delta$ for all $\varepsilon < 1$, and $n = \pm 2$. It is easy to see that one can choose a non-negative C^∞ function f with the following properties:

(i) $f(2) = 2$,

(ii) $\int_1^2 f\,d\mu = 4$,

(iii) f is supported on $[1,3]$,

(iv) $f'(x) > 0$ if $1 < x < \frac{3}{2}$ and $f'(x) < 0$ for $\frac{3}{2} < x < 3$.

Let M be the maximal operator for $(\mathbf{R}, d, \mu)$. Then $Mf \notin \mathrm{Lip}_s(\mathbf{R})$ for all $s > \delta$ because

$$Mf(\varepsilon) \ge 1 + \tfrac{1}{4}\varepsilon^\delta > 1 = Mf(0), \qquad \text{for all } 0 < \varepsilon < 1.$$

The lower bound for $Mf(\varepsilon)$ follows from the fact that

$$\fint_{B(\varepsilon, 2-\varepsilon)} f\,d\mu = \frac{4}{(4-\varepsilon^\delta)} \ge 1 + \frac{\varepsilon^\delta}{4}, \qquad \text{if } 0 < \varepsilon < 1.$$

As for $Mf(0)$, note that $\fint_{B(0,2)} f\,d\mu = 1$, while the properties of f, and the symmetric nature of μ, imply that μ-averages of f over all other balls $B(0,r)$ are strictly smaller.

Finally, the following example justifies statement (C).

Example 1.4. Let X be the subset of the complex plane consisting of the real line and all points z on the unit circle whose argument θ lies in the interval $\left[0,\frac{1}{2}\pi\right]$. We attach the Euclidean metric d and let μ be Hausdorff measure of exponent 1. Then (X,d,μ) is a homogeneous space. Let $v\colon \mathbf{R}\to[0,1]$ be any smooth function with the property that $v(t)=0$ for $t\le \sin^{-1}\pi/5$ and $v(t)=1$ for $t\ge 1/\sqrt{2} = \sin^{-1}\pi/4$. Using complex number notation, we define u on X by the formula $u(x+iy)=v(y)$. Certainly u is a very nice (Lipschitz) function, but Mu has a jump discontinuity at the origin. In fact, we claim that

$$Mu(0)\le \frac{3\pi}{20+5\pi} < 0.27 < 0.28 < \frac{\pi}{8+\pi} < \lim_{t\to 0^-} Mu(t).$$

The first of these inequalities is rather obvious: $Mu(0)$ equals the limiting average value of u over balls $B(0,t)$ as $t\to 1^+$. Here the limiting measure of these balls is $2+\frac{1}{2}\pi$, and the integral of u is at most $\frac{3}{10}\pi$ (the length of the arc on which u can be non-zero). The key observation in proving the other non-trivial inequality (the last one in the above string) is that, for all $t<0$, we can find a ball $B\bigl(t,r(t)\bigr)$ centered at t which includes points on the arc if and only if their argument exceeds $\frac{1}{4}\pi$; clearly $r(t)\to 1$ $(t\to 0^+)$. Thus we have established our claim.

The reader may wish to check that the discontinuity in the above example disappears if we simply replace the (subspace) Euclidean metric by the internal Euclidean metric (where distance between a pair of points is given as the infimum of the lengths of paths joining them). Of course, the latter metric changes our example into a length space and, as we shall see in the next section, length spaces always have an annular decay property. Note also that the same class of Lipschitz functions are given by the Euclidean metric and the internal Euclidean metric in the above example (since both metrics are equivalent); it is the change in shape of the metric balls which alone is responsible for the change in behaviour of the maximal operator.

2. Spaces that satisfy an annular decay property

So far, we have shown that the boundedness of M on Lipschitz type spaces associated with a homogeneous space (X,d,μ) is related to whether or not the space has an annular decay property. It is therefore appropriate to investigate which spaces satisfy annular decay properties, a task we now undertake.

If X is a homogeneous group, μ is Haar measure, and $d(x,y)=|x^{-1}y|$ for some homogeneous norm $|\cdot|$ on X, then μ satisfies the strong annular decay property; in fact, by normalizing μ, we get the stronger property $\mu\big(B(x,r)\big) = r^Q$, for all $x \in G$, $r > 0$, where Q is the homogeneous dimension of X. Here we are using the terminology of Folland and Stein [FS, p. 10], to which we refer the reader for an exposition of harmonic analysis on these groups. Basic examples of this type include $\mathbf{R}^n$ and the Heisenberg group $\mathbf{H}^n$.

Measures satisfying the strong annular decay property are, however, rather special—in Euclidean and many other metric spaces, it is easy to construct doubling measures that do not possess this property. Nevertheless, any doubling measure on $\mathbf{R}^n$ and $\mathbf{H}^n$ will possess the δ-annular decay property for some $\delta > 0$ (dependent only on the doubling constant). In fact, the only property of $\mathbf{R}^n$ and $\mathbf{H}^n$ that we shall need to prove this is the well-known fact that they are *length spaces*, i.e. metric spaces in which the distance between any pair of points equals the infimum of the lengths of rectifiable paths joining them (actually we prove such a result for a much more general class of spaces satisfying a certain chain condition, but length spaces form a simple and rather large subclass). Since many important homogeneous spaces are naturally defined as length spaces, Theorems 1.1 and 1.2 are therefore applicable to such spaces. An important class of examples are spaces of Carnot–Carathéodory type (where distance is given as the infimum of lengths of "subunit" paths), including those spaces associated with Hörmander or Grushin families of vector fields. The recent literature on such spaces is quite extensive; see for instance [NSW], [VSC], [BKL1], [GN], or many of the references cited therein.

We first wish to define a metric version of what is often called a chain domain (or "Boman chain domain") in the Euclidean setting. Similar definitions include, for example, the Boman chain conditions of [Bo] (for Euclidean space) and [BKL2] (for homogeneous spaces), and the $\mathscr{C}(\lambda, M)$ condition of [HK] (for metric spaces).

Let (X,d) be a metric space, and let $\alpha, \beta > 1$. A ball $B \equiv B(z,r) \subset X$ is said to be an (α,β)*-chain ball*, with respect to a "central" sub-ball $B_0 = B(z_0, r_0) \subset B$ if, for every $x \in B$, there is an integer $k = k(x) \geq 0$ and a chain of balls $B_{x,i} = B(z_{x,i}, r_{x,i})$, $0 \leq i \leq k$, with the following properties:

(i) $B_{x,0} = B_0$ and $x \in B_{x,k}$,
(ii) $B_{x,i} \cap B_{x,i+1}$ is non-empty, $0 \leq i < k$,
(iii) $x \in \alpha B_{x,i}$, $0 \leq i \leq k$,
(iv) $\beta r_{x,i} \leq r - d(z_{x,i}, z)$, $0 \leq i \leq k$.

We say that X is a (α,β)*-chain space* if every ball in X is an (α,β)-chain ball. We drop the parameters α, β in these terms if we do not care about their exact values.

Let us pause to discuss how the above definition relates to some related conditions. By comparing it with the Boman chain condition in [BKL2] and the $\mathscr{C}(\lambda, M)$ chain condition of [HK] [1], we see that the above definition is in most ways

[1] Such a comparison requires some careful notational translation; for example, the balls $B_{x,i}$ above play a similar role to the dilated balls $C_1 B_i$ in [BKL2].

less restrictive than the other definitions—we have dropped assumptions involving partial disjointness or bounded overlap of the balls in a chain, and weakened the assumption on the overlap between adjacent balls. In fact, it is easy to show that if a ball satisfies either of these other two chain conditions, then it must satisfy (i)–(iii) above, and the following weaker version of (iv):

(iv′) $\beta B_{x,i} \subset B$, $0 \le i \le k$.

Condition (iv) itself is not implied by the other chain conditions, but is necessary in order to prove that any doubling measure μ on a chain space X satisfies an annular decay property. For example, let X consist of the interval $[0,2]$ where μ is length measure and the metric d is defined by $d(x,y) = \max\{|x-y|, 1\}$. Balls in X are easily seen to satisfy the chain conditions of [BKL2] and [HK] but, given any $\alpha, \beta > 1$, the balls $B(0,r)$ are not (α,β)-chain balls when r is only slightly larger than 1 (since (iv) then forces the balls close to $x = 2$ to be very short intervals and (iii) prevents them from getting very far from 2). Considering these balls $B(0,r)$, it is clear that μ does not satisfy any annular decay property.

As shown in [BKL2] and [HK], any ball in a homogeneous space X which is a John domain must satisfy a chain condition as defined in those papers, and hence (i)–(iii) and (iv′) above. If X is also a length space, i.e. a metric space in which distance between points is the infimum of the lengths of rectifiable paths joining those points, then it follows from Theorem 3.1 and the proof of Corollary 3.2 in [BKL2] that all balls in X are John domains, and that X is a chain space. In fact, by taking a path from $x \in B(z,r)$ to z of length less than r as our John path γ_x, and then choosing a finite number of balls $B(z_{x,i}, r_{x,i})$ covering the image of γ_x, where $z_{x,i}$ lies on the image of γ_x and $r_{x,i} = \frac{1}{2}\big(r - d(z_{x,i}, z)\big)$, it is easy to see that length spaces are $(2,2)$-chain spaces. Note that, unlike John domains, chain balls do not have to be connected; for a simple example, consider the ball $B\big((0,0),1\big)$ in the space X consisting of all points in the plane whose first coordinate is not $\frac{1}{2}$ (with the Euclidean metric and Lebesgue measure attached).

A well-known covering lemma for homogeneous spaces that we shall have occasion to use below says that if a bounded open subset U of X is covered by a family of balls, then we can pick a subfamily $\{B_i\}_{i\in S}$, $S \subset \mathbf{N}$, of these balls such that the dilated balls $5B_i$ cover U. This statement follows from [CW, III.1.2]; note that the parameter k can be chosen to be 5 since d is a genuine metric. We refer to the above result as simply “the Covering Lemma” below.

We now state the main theorem of this section.

Theorem 2.1. *Suppose that the metric space (X,d) is a (α,β)-chain space, and that μ is a doubling measure on X with doubling constant C_μ. Then μ has the δ-annular decay property for some $0 < \delta \le 1$ dependent only on α, β, and C_μ.*

This theorem (and the following corollary) was proven in the very special case of Euclidean space and Lebesgue measure somewhat implicitly in [Bu, Lemma 3.3], and more explicitly in [Ko], with the minor difference that these earlier results are stated for cubes rather than balls.

Corollary 2.2. *If (X,d) is a length space, and μ is a doubling measure on X with doubling constant C_μ, then μ has the δ-annular decay property for some $0<\delta\le 1$ dependent only on C_μ.*

Proof. The fact that μ has an annular decay property follows from the above discussion. The value of δ then depends on the chain space parameters of X as well as C_μ. But, as discussed above, length spaces are $(2,2)$-chain spaces, so we are done. □

Proof of Theorem 2.1. We wish to verify an annular decay property for a fixed, but arbitrary, ball $B(z,r)$. Let $K=(\beta+1)^2/(\beta-1)^2$, $A_t=B(z,r)\setminus B(z,r-t)$, and $\delta(x)=r-d(x,z)$, where $0<t<r$ and $x\in B(z,r)$. It suffices to show that there exists a constant $C=C(\alpha,\beta,C_\mu)$ such that $\mu(A_t)\le C\mu(A_{Kt}\setminus A_t)$, for all $0<t<r/K$. If A_t is empty, there is nothing to prove, so we suppose that it is non-empty. It is clearly sufficient to prove the indicated decay for small t, and in particular for $t<\min\{r/K,(\beta-1)\delta(z_0)/\beta\}$. If $x\in A_t$, we define the ball $B_x=B(z_x,r_x)$, where $z_x=z_{x,i}, r_x=r_{x,i}$, and i is the largest index for which $t<(\beta-1)\delta(z_{x,i})/\beta$. For any $0\le i<k(x)$, chain conditions (ii) and (iv) imply that

$$\delta(z_{x,i})\le\delta(z_{x,i+1})+r_{x,i}+r_{x,i+1}\le(1+\beta^{-1})\delta(z_{x,i+1})+\beta^{-1}\delta(z_{x,i})$$

and so $\delta(z_{x,i})\le(\beta+1)\delta(z_{x,i+1})/(\beta-1)$. We therefore have

$$\beta t/(\beta-1)\le\delta(z_x)\le\beta(\beta+1)t/(\beta-1)^2.$$

Condition (iv) now ensures that $B_x\subset B(z,r-t)\setminus B(z,r-Kt)$; thus $U\equiv\bigcup_{x\in A_t}B_x\subset A_{Kt}\setminus A_t$. Consequently, $r_x\le Kt/\beta$ and, because $d(x,z_x)\ge t/(\beta-1)$, we also have $r_x\ge t/\alpha(\beta-1)$.

By the Covering Lemma, we can pick a subfamily $\{B_i\}_{i\in S}$, $S\subset\mathbf{N}$, of these balls such that the dilated balls $5B_i$ cover U. By (iii) and the upper and bounds for r_x, $x\in A_t$, we see that $\bigcup_{i\in S}K'B_i\supset A_t$, where $K'=5+K\alpha^2(\beta-1)/\beta$. Letting N be any integer greater than $\log_2 K'$, we therefore have

$$\mu(A_t)\le\sum_{i\in S}\mu(K'B_i)\le C_\mu^N\sum_{i\in S}\mu(B_i)\le C_\mu^N\mu(U)\le C_\mu^N\mu(A_{Kt}\setminus A_t),$$

and so we are done. □

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