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ON COMPACTNESS OF EMBEDDING FOR SOBOLEV SPACES DEFINED ON METRIC SPACES

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Abstract. We generalize the classical Rellich–Kondrachov compactness theorem for Sobolev spaces defined on metric spaces.

1. Introduction

In the classical setting Sobolev spaces were defined on subdomains of $\mathbf{R}^n$ or on Riemannian manifolds. It seemed to be essential to define Sobolev spaces on a sufficiently smooth object, as the classical definition requires L^p-summability of the gradient.

However, the theory has been recently extended to the setting of metric spaces. Particularly many authors deal with Sobolev spaces associated with a family of vector fields. This leads to Sobolev-type inequalities on balls with respect to the Carnot–Carathéodory metric; see the works of Capogna, Coulhon, Danielli, Franchi, Gallot, Gutiérrez, Jerison, Garofalo, Lanconelli, Lu, Nhieu, Rothschild, Saloff-Coste, Stein, Wheeden, Varopoulos, [2], [3], [7], [10], [11], [13], [8], [9], [14], [16], [25], [28], [29], [34], [35], [37] and many others. There is, however, a much more general approach to Sobolev inequalities. Biroli and Mosco, [1] and Sturm, [36] investigate inequalities for Dirichlet forms on metric spaces, while Hajłasz, [17], defines Sobolev spaces on an arbitrary metric space equipped with a locally finite Borel measure. Franchi Lu and Wheeden [12] deal with representation formulas in metric spaces, and Hajłasz and Koskela, [20], [21] give an approach to Sobolev inequalities on metric spaces, different from that of Hajłasz, [17]. The approach of Hajłasz to Sobolev spaces has been employed by Hajłasz and Kinnunen, [19], Hajłasz and Martio, [22], Heinonen and Koskela, [24], Kinnunen and Martio, [26] and Koskela and MacManus, [27].

There are quite a lot of papers concerned with the Sobolev type inequalities in metric setting and a few results concerning compact embedding. To our knowledge, the compact embedding theorems for vector fields are obtained in Danielli, [4], Garofalo and Lanconelli, [15], Garofalo and Nhieu, [16], Lu, [30], Manfredini, [31], and Rothschild and Stein, [34]. Recently Hajłasz and Koskela, [21] obtained a

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more general result, namely, a compact embedding theorem in a general setting of metric spaces with doubling measure.

In this paper we establish a new criterion for the relative compactness in L^p and then apply it to a new compact embedding theorem for Sobolev spaces of Hajłasz, [17]. In all the above papers except [17] it is essential to assume that the measure with which the space is equipped satisfies a doubling condition (see below), while in [17] the condition is weaker, a lower boundary of the growth of the measure of a ball. The condition made in our paper concerning the measure is weaker than the doubling condition.

The approach to Sobolev spaces given in [17] covers the case of vector fields; see Franchi, Lu and Wheeden, [12] and Hajłasz and Koskela, [21]. In particular, our result covers most of the above-mentioned compact embedding theorems. It also deals with compactness of embedding for weighted Sobolev spaces considered by Heinonen, Kilpeläinen and Martio, [23].

Although the setting of our result is slightly different from that of Hajłasz and Koskela, [21], it is almost equivalent in the case when the measure is doubling; see [21]. The result of Hajłasz and Koskela was obtained independently of ours.

Now let us recall the definition of the Sobolev space. First start with the classical definition. If $\Omega\subset\mathbf{R}^n$ is an open set and $1\le p<\infty$, we define $W^{1,p}(\Omega)$ as the closure of $C^\infty(\Omega)$ in the norm $\|f\|_{1,p}=\|f\|_{L^p(\Omega)}+\|\nabla f\|_{L^p(\Omega)}$.

If X is a metric space, d is the metric and μ a Borel measure on X finite on bounded sets Hajłasz, [17] defines the Sobolev space $W^{1,p}(X,d,\mu)$ for $1\le p<\infty$ as follows: $f\in W^{1,p}(X,d,\mu)$ if and only if $f\in L^p(X,\mu)$ and there exists a function $0<g\in L^p(X,\mu)$ such that

$$|f(x)-f(y)|\le d(x,y)\big(g(x)+g(y)\big) \tag{1}$$

almost everywhere, which means that there exists a set $E\subset X$ with $\mu(E)=0$ such that the inequality (1) holds for all $x,y\in X\setminus E$. The space is equipped with the norm $\|f\|_{W^{1,p}(X,d,\mu)}=\|f\|_{L^p(X,\mu)}+\inf_g\|f\|_{L^p(X,\mu)}$, the infimum being over all functions g that satisfy inequality (1).

Hajłasz, [17], proved that if $1<p\le\infty$, $X=\Omega\subset\mathbf{R}^n$ is a bounded domain with sufficiently regular boundary, say Lipschitz boundary, the metric is the Euclidean metric and the measure is the Lebesgue measure, the above definition is equivalent to the classical definition of the Sobolev space $W^{1,p}(\Omega)$; see also [22]. If $p=1$, the equivalence fails, [18]. Then he proved that, in the metric setting, the lower boundary for the measure of the ball

$$\mu\big(B(x,r)\big)\ge Cr^s \tag{2}$$

for all $x\in X$ and $r\le\operatorname{diam}X$ implies the Sobolev embedding theorem with s playing the role of the dimension of the space.

In this paper we prove that slightly different condition from (2) implies compactness of embedding. In particular, our condition is satisfied when the measure is doubling, i.e., $\mu\big(B(x,2r)\big) \le C\mu\big(B(x,r)\big)$.

In what follows the average value of the function over the set A will be denoted by $f_A = \mu(A)^{-1}\int_A f\,d\mu = \fint_A f\,d\mu$.

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2. The main results

There are two main results obtained here. The first result is a criterion for the relative compactness in $L^p(X,\mu)$.

Theorem 1. *Let X be a metric space equipped with a finite Borel measure μ, such that, for any $r>0$, $h(r)=\inf\big\{\mu\big(B(x,r)\big) : x\in X\big\}>0$. Then every bounded sequence $\{f_n\}\subset L^p(X,\mu)$, $1\le p<\infty$ such that*

$$\sup_n \int_X |f_n(x)-(f_n)_{B(x,r)}|^p\,d\mu(x) \overset{r\to 0}{\longrightarrow} 0, \tag{3}$$

is relatively compact in $L^p(X,\mu)$.

The second result is a compact embedding theorem for Sobolev spaces on metric spaces. As we will see, the theorem is a fairly elementary consequence of the above criterion for compactness. We believe that the above result may be useful for proving compactness in other situations.

Theorem 2. *Let X be a metric space equipped with a finite Borel measure μ, such that, for any $r>0$, $h(r)=\inf\big\{\mu\big(B(x,r)\big) : x\in X\big\}>0$. Assume that there exists a function $N(r)$ such that $r^pN(r)\to 0$, as $r\to 0$, and one of the following conditions is satisfied*:

1. *Every ball $B(x,r)$ can be covered by $N(r)$ balls with radii $\frac{1}{2}r$ and centers in $B(x,r)$.*
2. *For every $x\in X$ and $r>0$ we have*

$$\mu\big(B(x,2r)\big)\le N(r)\mu\big(B(x,r)\big).$$

Then any sequence $\{f_n\}$, bounded in $W^{1,p}(X,d,\mu)$, $1\le p<\infty$, is relatively compact in $L^p(X,\mu)$.

The following corollary directly applies to [22].

Corollary 1. *Let $X\subset \mathbf{R}^n$ be a compact set, $d(x,y)=|x-y|^\lambda$ for some $0<\lambda\le 1$, and let μ be an arbitrary finite Borel measure, supported on X by the property that $h(r)=\inf\big\{\mu\big(B(x,r)\big) : x\in X\big\}>0$ for every $r>0$. Then the embedding $W^{1,p}(X,d,\mu)\subset L^p(X,\mu)$ is compact for every $p\ge 1$.*

Proof. Obviously one can take a constant function $N(r)$ with the property 1 in Theorem 2. □

Proof of Theorem 1. We start with recalling two known facts (see e.g. [5, Corollary 11 and Theorem 12 of Section IV.8], [33, p. 20]).

Theorem 3 (Hahn–Saks–Vitali.) *Let X be a measurable space equipped with a finite measure μ, $1 \le p < \infty$, and let $f_n, f \in L^p(X,\mu)$. Then $f_n \to f$ in $L^p(X,\mu)$ if and only if the following two conditions are satisfied*:

1. *All the functions $|f_n|^p$ are equi-integrable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\mu(A) < \delta \qquad \Longrightarrow \qquad \sup_n \int_A |f_n(y)|^p \, d\mu(y) < \varepsilon;$$

2. *f_n converges to f in measure.*

Theorem 4 (Dunford–Pettis). *Let X be a measurable space equipped with a finite measure μ and let $f_n \in L^1(X,\mu)$. Then $\{f_n\}$ is weakly relatively compact in $L^1(X,\mu)$ if and only if $\{|f_n|\}$ is equi-integrable.*

Now we can return to the proof of Theorem 1. We will check conditions 1 and 2 of Theorem 3 for a subsequence of $\{f_n\}$, starting with condition 1. Let $A \subset X$ be a measurable subset. Given $r > 0$, we have

$$\begin{aligned}\left(\int_A |f_n|^p \, d\mu\right)^{1/p} &\le \left(\int_A |f_n(x) - (f_n)_{B(x,r)}|^p \, d\mu(x)\right)^{1/p} \\ &\quad + \left(\int_A |(f_n)_{B(x,r)}|^p \, d\mu(x)\right)^{1/p} \\ &\le \left(\int_A |f_n(x) - (f_n)_{B(x,r)}|^p \, d\mu(x)\right)^{1/p} \\ &\quad + \mu(A)^{1/p} h(r)^{-1/p} \|f_n\|_{L^p(X,\mu)}.\end{aligned}$$

By (3), for every $\varepsilon > 0$ we can find $r > 0$ (r does not depend on n) such that the first expression on the right-hand side is less than $\frac{1}{2}\varepsilon$. For that fixed $r > 0$, the second expression is less than $\frac{1}{2}\varepsilon$, provided $\mu(A)$ is sufficiently small. This ends the proof of 1. We are left with 2.

First, note that $\{f_n\}$ is weakly relatively compact in $L^p(X,\mu)$. Indeed, for $p > 1$ the weak compactness follows from the reflexivity of $L^p(X,\mu)$, while for $p = 1$ the weak compactness follows from the equi-integrability of the family $\{|f_n|\}$ just proved and from Theorem 4. Thus, we can choose a subsequence (still denoted by $\{f_n\}$) and $f \in L^p(X,\mu)$ such that $f_n \rightharpoonup f$ weakly in $L^p(X,\mu)$. It remains to show that $f_n \to f$ in measure, which means that for every $\varepsilon > 0$

$$\mathscr{L}(n) = \mu\{x \in X : |f_n(x) - f(x)| > \varepsilon\} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Fix $\varepsilon > 0$. Obviously,

$$|f_n(x) - f(x)| \leq |f_n(x) - (f_n)_{B(x,r)}| + |(f_n)_{B(x,r)} - f_{B(x,r)}| + |f(x) - f_{B(x,r)}|.$$

Hence

$$\mathscr{L}(n) \leq \mu\big(A_1(n,r)\big) + \mu\big(A_2(n,r)\big) + \mu\big(A_3(n,r)\big),$$

where

$$\begin{aligned} A_1(n,r) &= \{x \in X : |f_n(x) - (f_n)_{B(x,r)}| \geq \tfrac{1}{3}\varepsilon\}, \\ A_2(n,r) &= \{x \in X : |(f_n)_{B(x,r)} - f_{B(x,r)}| \geq \tfrac{1}{3}\varepsilon\}, \\ A_3(r) &= \{x \in X : |f(x) - f_{B(x,r)}| \geq \tfrac{1}{3}\varepsilon\}. \end{aligned}$$

It follows from (3) and from Chebyschev's inequality that

$$\sup_n \mu\big(A_1(n,r)\big) \leq \Big(\frac{3}{\varepsilon}\Big)^p \sup_n \int_X |f_n(x) - (f_n)_{B(x,r)}|^p \, d\mu(x) \overset{r\to 0}{\longrightarrow} 0.$$

By the same argument

$$\mu\big(A_3(r)\big) \overset{r\to 0}{\longrightarrow} 0,$$

provided we prove that

$$\int_X |f(x) - f_{B(x,r)}|^p \, d\mu(x) \overset{r\to 0}{\longrightarrow} 0. \tag{4}$$

Assume for a moment that we have established (4), and we show how to complete the proof of the theorem.

Since $f_n \rightharpoonup f$ weakly in $L^p(X,\mu)$, we obtain $(f_n)_{B(x,r)} \to f_{B(x,r)}$ for all $x \in X$ and all $r > 0$. In particular, $\mu\big(A_2(n,r)\big) \to 0$ as $n \to \infty$. Hence the above estimates for $\mu\big(A_i(n,r)\big)$ imply that $\mathscr{L}(n) \to 0$ as $n \to \infty$.

Now it remains to prove (4). It follows from Banach–Steinhaus' theorem, from (3) and the fact that

$$f_n(x) - (f_n)_{B(x,r)} \rightharpoonup f(x) - (f)_{B(x,r)} \tag{5}$$

weakly in $L^p(X,\mu)$ (as a function of the variable x). Note that since $(f_n)_{B(x,r)} \to (f)_{B(x,r)}$ pointwise and $|(f_n)_{B(x,r)}| \leq h(r)^{-1/p} \sup_n \|f_n\|_{L^p(X,\mu)}$, we even have $(f_n)_{B(x,r)} \to (f)_{B(x,r)}$ strongly in $L^p(X,\mu)$, so (5) follows. The proof of Theorem 1 is complete. □

In the proof of Theorem 2 we will need the following lemma.

Lemma 1. *Let X be a metric space equipped with a finite Borel measure μ, $1 \le p < \infty$. Given a locally integrable function g on X, denote $g_r(x) = g_{B(x,r)}$. Assume that there exists a function $N(r)$ which satisfies either condition 1 or condition 2 of Theorem 2. Then*

$$\|g_r\|^p_{L^p(X,\mu)} \le N(r)\|g\|^p_{L^p(X,r)}.$$

Proof. Applying Hölder's inequality and then Fubini's theorem we have

$$\int_X |g_r|^p \, d\mu \le \int_X \fint_{B(x,r)} |g(y)|^p \, d\mu(y) \, d\mu(x) = \int_X |g(y)|^p \left(\int_{B(y,r)} \frac{d\mu(x)}{\mu\big(B(x,r)\big)} \right) d\mu(y).$$

Now it suffices to show that for all y

$$\int_{B(y,r)} \frac{d\mu(x)}{\mu\big(B(x,r)\big)} \le N(r).$$

Suppose first that condition 1 is satisfied. Let $\{B_i\}$ be a covering of $B(y,r)$ by $N(r)$ balls with radii $\frac{1}{2}r$ and centers in $B(y,r)$. We have

$$\int_{B(y,r)} \frac{d\mu(x)}{\mu\big(B(x,r)\big)} \le \sum_{i=1}^{N(r)} \int_{B_i} \frac{d\mu(x)}{\mu\big(B(x,r)\big)} \le \sum_{i=1}^{N(r)} \int_{B_i} \frac{d\mu(x)}{\mu(B_i)} = N(r).$$

Suppose now that condition 2 is satisfied. For $x \in B(y,r)$, we have $B(y,r) \subset B(x,2r)$, and

$$\mu\big(B(y,r)\big) \le \mu\big(B(x,2r)\big) \le N(r)\mu\big(B(x,r)\big).$$

This implies

$$\int_{B(y,r)} \frac{d\mu(x)}{\mu\big(B(x,r)\big)} \le N(r) \int_{(B,r)} \frac{d\mu(x)}{\mu\big(B(y,r)\big)} = N(r).$$

This ends the proof of the lemma. □

Proof of Theorem 2. It follows from definition (1) that $f \in W^{1,p}(X,d,\mu)$ satisfies

$$|f(x) - f_{B(x,r)}| \le r\left(g(x) + \fint_{B(x,r)} g(y) \, d\mu(y) \right)$$

almost everywhere, and hence by Lemma 1

$$\left(\int_X |f(x) - f_{B(x,r)}|^p \, d\mu(x) \right)^{1/p} \le r\|g\|_{L^p(X,\mu)} + rN(r)^{1/p}\|g\|_{L^p(X,\mu)} \xrightarrow{r \to 0} 0.$$

Thus Theorem 2 follows directly from Theorem 1. □

3. Some remarks

We start with a discussion of the connections between conditions 1 and 2 in the formulation of Theorem 2. Let $N_1(x,r)$ be the smallest number of balls $B(y,\frac{1}{2}r)$, $y \in B(x,r)$ covering $B(x,r)$. Define

$$N_1(r) = \sup\{N_1(x,r) : x \in X\}.$$

Denote by $N_2(r)$ the smallest constant such that for every $x \in X$

$$\mu\big(B(x,2r)\big) \le N_2(r)\mu\big(B(x,r)\big). \tag{6}$$

Remark 1. Conditions $\mu(X) < \infty$ and $h(r) = \inf\big\{\mu\big(B(x,r)\big) : x \in X\big\} > 0$ imply that the diameter of X is finite.

Moreover, the number of disjoint balls of fixed radius r, contained in X, is bounded from above by $\mu(X)/h(r)$. This implies that the minimal number of balls with radii r covering X does not exceed $\mu(X)/h(\frac{1}{2}r)$. Indeed, take the maximal family $\{B(x_i,\frac{1}{2}r)\}$ of pairwise disjoint balls. Then the number of balls in that family does not exceed $\mu(X)/h(\frac{1}{2}r)$ and $X \subset \bigcup_i B(x_i,r)$. By the same argument

$$N_1(x,r) \le \frac{\mu\big(B(x,2r)\big)}{h(\frac{1}{4}r)}.$$

Remark 2. Since the definition of $N_1(r)$ has a purely geometric nature, it is not possible to estimate $N_2(r)$ in terms of $N_1(r)$. However, it is possible to estimate $N_1(r)$ in terms of $N_2(r)$. Condition (6) gives the upper bound for the maximal number $N(r)$ of pairwise disjoint balls $B(x_i,\frac{1}{4}r)$ with centers in $B(x,r)$ in terms of $N_2(r)$. $N_1(r) \le N(r)$ since $B(x,r) \subset \bigcup_{i=1}^{N(r)} B(x_i,\frac{1}{2}r)$. See also Volberg and Konyagin [38] for deep related results.

Remark 3. The situation we deal with in Theorem 2 is more general than the situation investigated by Hajłasz in [17]. If we take for example $X = [0,a]$ and any measure μ with property $\mu(X) < \infty$, and $h(r) = \inf\big\{\mu\big(B(x,r)\big) : x \in X\big\} > 0$, the assumption of Theorem 1 is satisfied, so that the compactness property holds (even if the measure μ does not satisfy property 2 in the assumptions of Theorem 2). On the other hand, it is easy to show an example of a metric space X with measure μ, which satisfies condition 2 in Theorem 2, with $r^pN(r) \to 0$ as $r \to 0$, but is not s-regular in the sense of [17]. Let for example $X = [0, 1/(2\mathrm{e}^2)]$, and $\mu = \rho\,dx$, with $\rho = (\mathrm{e}^{-1/2(\log r)^2})' = -(\log r)/r\,\mathrm{e}^{-1/2(\log r)^2}$. It is easy to calculate that

$$\mu\big(B(x,2r)\big) \le \frac{C}{r^2}\mu\big(B(x,r)\big)$$

for some $C > 0$ and all $x \in X$, in particular, $r^pN(r) \to 0$ as $r \to 0$ for all $p > 2$, while

$$\mu\big(B(0,r)\big) = \mathrm{e}^{-1/2(\log r)^2}$$

can never exceed Cr^s for given $C > 0$, $s > 0$, and all r.

Remark 4. Assume that the embedding $W^{1,p}(X,d,\mu) \subset L^q(X,\mu)$ is compact. Then the set $\{|f|^q : \|f\|_{W^{1,q}} \le 1\}$ is relatively compact in $L^1(X,\mu)$, so by La Valleé-Pousin's theorem (see e.g. [33, p. 19], [32, p. 176]) there is a smooth increasing convex function $g\colon [0,\infty) \to [0,\infty)$, $g(t)/t \to \infty$ as $t \to \infty$ such that

$$\sup_{\|f\|_{W^{1,p}} \le 1} \int_X g(|f|^p)\, d\mu < \infty.$$

In fact this means that the space $W^{1,p}(X,d,\mu)$ is embedded in an Orlicz space which is smaller than $L^1(X,\mu)$. Thus the compactness of embedding implies that one can obtain an embedding in a better space. For the best embedding theorem in the case where the measure satisfies the growth condition $\mu\big(B(x,r)\big) \ge Cr^s$, see [17].

Remark 5. Theorem 1 can be used to obtain compactness results for weighted Sobolev spaces defined on subsets of $\mathbf{R}^n$. Let $f \in C^1(\mathbf{R}^n)$. By Taylor's formula, we have

$$f(y) - f(x) = \int_0^1 \big\langle \nabla f\big(x + \tau(y-x)\big), y - x \big\rangle\, d\tau.$$

Hence, if $|y-x| \le r$,

$$|f(x) - f(y)| \le r \int_0^1 \big|\nabla f\big(x + \tau(y-x)\big)\big|\, d\tau.$$

Now we average the above inequality over a ball $B(x,r)$, with respect to the measure $d\mu(y)$, and obtain

$$|f(x) - f_{B(x,r)}| \le r T_r(|\nabla f|)(x), \tag{7}$$

where

$$T_r g(x) = \int_0^1 ⨍_{B(x,r)} g\big(x + \tau(y-x)\big)\, d\mu(y)\, d\tau. \tag{8}$$

Hence

$$\int_{\mathbf{R}^n} |f(x) - f_{B(x,r)}|^p\, d\mu(x) \le r^p \int_{\mathbf{R}^n} \big(T_r|\nabla f|(x)\big)^p\, d\mu(x). \tag{9}$$

The weighted Sobolev space $W^{1,p}(\mathbf{R}^n,\mu)$ is defined as the completion of $C^1(\mathbf{R}^n)$ in the norm

$$\|f\|_{W^{1,p}(\mathbf{R}^n,\mu)} = \|f\|_{L^p(\mathbf{R}^n,\mu)} + \|\nabla f\|_{L^p(\mathbf{R}^n,\mu)}. \tag{10}$$

Assume that $\Omega \subset \mathbf{R}^n$ has the property that $\mu(\Omega) < \infty$ and $h(r) = \inf\big\{\mu\big(B(x,r)\big) : x \in \Omega\big\} > 0$ for every $r > 0$. If one can prove the inequality

$$\|T_r g\|^p_{L^p(\mathbf{R}^n,\mu)} \le N(r)\|g\|^p_{L^p(\mathbf{R}^n,\mu)}, \tag{11}$$

with $r^p N(r) \to 0$ as $r \to 0$, Theorem 1 and (9) imply that every bounded sequence in $W^{1,p}(\mathbf{R}^n,\mu)$ is relatively compact in $L^p(\Omega,\mu)$.

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