REMOVABLE SINGULARITIES FOR HÖLDER CONTINUOUS QUASIREGULAR MAPPINGS IN THE PLANE

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Abstract. We give necessary conditions for a set E to be removable for Hölder continuous quasiregular mappings in the plane. We also obtain some removability results for Hölder continuous mappings of finite distortion.

1. Introduction

Let $\alpha \in (0,1]$. A function $f\colon \mathbf{C} \to \mathbf{C}$ is said to be (locally) α-Hölder continuous, that is, $f \in \mathrm{Lip}_\alpha(\mathbf{C})$, if

$$(1) \qquad |f(z) - f(w)| \leq C\,|z - w|^\alpha$$

whenever $z, w \in \mathbf{C}$ are such that $|z-w| < 1$. A set $E \subset \mathbf{C}$ is said to be *removable* for α-Hölder continuous analytic functions if every function $f \in \mathrm{Lip}_\alpha(\mathbf{C})$, holomorphic on $\mathbf{C} \setminus E$, is actually an entire function. It turns out that there is a characterization of these sets E in terms of Hausdorff measures. For $\alpha \in (0,1)$, Dolzenko [6] proved that a set E is removable for α-Hölder continuous analytic functions if and only if $\mathscr{H}^{1+\alpha}(E) = 0$. In particular, for any set E with positive $(1+\alpha)$-dimensional Hausdorff measure, there is a function $f \in \mathrm{Lip}_\alpha(\mathbf{C})$, holomorphic on $\mathbf{C} \setminus E$, which is not entire. For $\alpha = 1$, we deal with the class of Lipschitz continuous analytic functions. Although the same characterization holds, a more involved argument, due to Uy [15], is needed to show that sets of positive area are not removable.

We can ask ourselves which is the corresponding situation in the more general setting of K-quasiregular mappings. Given a domain $\Omega \subset \mathbf{C}$ and $K \geq 1$, we say that a mapping $f\colon \Omega \to \mathbf{C}$ is K-quasiregular in Ω if f is a $W^{1,2}_{\mathrm{loc}}(\Omega)$ solution of the Beltrami equation,

$$\overline{\partial} f(z) = \mu(z)\,\partial f(z)$$

for almost every $z \in \Omega$, where μ, the Beltrami coefficient, is a measurable function such that $|\mu(z)| \leq \frac{K-1}{K+1}$ at almost every $z \in \Omega$. If f is a homeomorphism onto the image set, then we say that f is K-quasiconformal. When $\mu = 0$, we recover the classes of analytic functions and conformal mappings on Ω, respectively.

We say that a compact set E is *removable* for Lip_α K-quasiregular mappings if every α-Hölder continuous function $f\colon \mathbf{C} \to \mathbf{C}$, K-quasiregular on $\mathbf{C} \setminus E$, is actually K-quasiregular on the whole plane. The purpose of this paper is to give sufficient conditions for a compact set $E \subset \mathbf{C}$ to be removable for Lip_α K-quasiregular mappings. In fact, K-quasiregular mappings may also be defined in $\mathbf{R}^n$, so that this problem makes sense even in higher dimensions. In this paper, we restrict ourselves to the planar case.

Some results are already known related to this subject. For instance, Koskela and Martio [13] showed that if $d = \frac{1}{K}\left(1 + \frac{\alpha}{K}\right)$, then every compact set $E \subset \mathbf{C}$ with $\mathscr{H}^d(E) = 0$ is removable for Lip_α K-quasiregular mappings. Moreover, they also gave some sufficient conditions for removability in $\mathbf{R}^n$. In particular, they proved that if $\mathscr{H}^\lambda(E) = 0$, $\lambda = \min\{1, n\alpha\}$, then E is removable for K-quasiregular mappings in Lip_α. On the other hand, while dealing with second order quasilinear elliptic equations, Kilpeläinen and Zhong [12] showed that $\mathscr{H}^{\alpha(n-1)}(E) = 0$ is enough. In this paper, we show that compact sets $E \subset \mathbf{C}$ of zero d-dimensional Hausdorff measure, $d = \frac{2}{K+1}(1 + \alpha K)$, are removable for α-Hölder continuous planar K-quasiregular mappings. This index was suggested to us by Astala.

Unfortunately, our method does not work in $\mathbf{R}^n$, since there is no factorization theorem when $n > 2$. However, our result may be rewritten in the planar class of finite distortion mappings. Recall that $f\colon \Omega \subset \mathbf{C} \to \mathbf{C}$ is said to be a mapping of finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $Jf \in L^1_{\mathrm{loc}}(\Omega)$ and

$$|Df(z)|^2 \leq K(z)\, Jf(z)$$

at almost every $z \in \Omega$. Here K is a measurable function, $K(z) \in (1, \infty)$ at almost every $z \in \Omega$. When $n = 2$, this is equivalent to $|\mu(z)| < 1$ for almost every $z \in \Omega$. If $K \in L^\infty$, we get the class of bounded distortion (or quasiregular) functions. During the last years, there has been deep progresses in the knowledge of these mappings. Questions like the removability of singularities for bounded mappings of finite distortion have been studied (see, for instance, [3], [7], [9] or [11]). In this planar setting, relatively weak assumptions on μ are sufficient to ensure that some basic properties of the quasiregular case also hold. For instance, continuity, discreteness and openness, as soon as the existence of normalized solutions and factorization theorems are established. Althoug the natural regularity for these mappings is the Orlicz-Sobolev space $W^{1,P}_{\mathrm{loc}}(\mathbf{C})$, $P(t) = \frac{t^2}{\log(e+t)}$ (which is larger than the usual $W^{1,2}_{\mathrm{loc}}(\mathbf{C})$), we can study the removability problem for Lip_α mappings of finite distortion (see Corollaries 3 and 4), which appears as a limiting situation of the quasiregular case.

2. Sufficient conditions for Lip_α removability

We say that a K-quasiconformal mapping $\phi\colon \mathbf{C} \to \mathbf{C}$ is *principal* if both $\overline{\partial}\phi$ and $\partial\phi - 1$ are $L^p(\mathbf{C})$ functions, for some $p > 2$. Then, for compactly supported μ there exist exactly one principal K-quasiconformal mapping ϕ such that $\overline{\partial}\phi = \mu\,\partial\phi$ and $\phi(0) = 0$, which we will refer to as the *principal solution* of this Beltrami equation.

We denote by A the planar Lebesgue measure, while $\mathscr{H}^d$ stands for Hausdorff d-dimensional measure. Given a disk D of radius r, we will denote by λD the disk concentric with D whose radius is λr.

We start with an auxiliary result.

Lemma 1. *Suppose that μ_n, μ are measurable functions, compactly supported on* $\mathbf{D}$*, and assume that* $\|\mu_n\|_\infty, \|\mu\|_\infty \leq \frac{K-1}{K+1}$*. Let* $\phi_n, \phi\colon \mathbf{C} \to \mathbf{C}$ *be, respectively, the principal solutions to the corresponding Beltrami equations. If* $\mu_n \to \mu$ *at almost every point, then:*

1. $\lim\limits_n J\phi_n = J\phi$*, with convergence in* L^p_{loc}*, for every* $p \in (1, \frac{K}{K-1})$*.*
2. $\lim\limits_n \phi_n = \phi$ *uniformly on compact sets.*
3. $\lim\limits_n \phi_n^{-1} = \phi^{-1}$ *uniformly on compact sets.*

Proof. Fix $p \in (1, \frac{K}{K-1})$. The operator $\mathscr{C}$ defined by

$$\mathscr{C}f(z) = \frac{-1}{\pi} \int_{\mathbf{C}} f(w) \left(\frac{1}{w-z} - \frac{1}{w} \right) dA(w)$$

maps functions $f \in L^{2p}(\mathbf{C})$ to $\mathscr{C}f \in W^{1,2p}(\mathbf{C}) \subset \mathrm{Lip}_{1-1/p}(\mathbf{C})$, and

$$\frac{|\mathscr{C}f(z) - \mathscr{C}f(w)|}{|z-w|^{1-1/p}} \leq C \|f\|_{2p}$$

for any $z, w \in \mathbf{C}$. On the other hand, since μ_n, μ are compactly supported on $\mathbf{D}$, there are $L^{2p}(\mathbf{C})$ functions h_n, h such that

$$\phi_n(z) = z + \mathscr{C}h_n(z)$$

$$\phi(z) = z + \mathscr{C}h(z)$$

Actually, h_n and h can be obtained, respectively, as $L^{2p}(\mathbf{C})$ solutions of the equations $h_n = \mu_n \mathscr{B}(h_n) + \mu_n$ and $h = \mu \mathscr{B}(h) + \mu$, where $\mathscr{B}$ denotes the Beurling transform, $\mathscr{B}f = p.v. \frac{-1}{\pi z^2} * f$. A computation shows that

$$(I - \mu_n \mathscr{B})(h - h_n) = (\mu - \mu_n)(1 + \mathscr{B}h)$$

Thus, as $I - \mu_n \mathscr{B}$ is a bilipschitz homeomorphism in $L^{2p}(\mathbf{C})$ whose norm depends only on $\|\mu_n\| \leq \frac{K-1}{K+1}$ and p [4, Theorem 1],

$$\|h - h_n\|_{2p} \leq C(K,p)\|(\mu - \mu_n)(1 + \mathscr{B}h)\|_{2p}$$

and hence $\|h - h_n\|_{2p} \to 0$ as $n \to \infty$. It then follows that $\|D\phi_n - D\phi\|_{2p} \to 0$, and hence $J\phi_n$ converges in L^p_{loc} to $J\phi$. Furthermore, we also have

$$\|\phi - \phi_n\|_{\mathrm{Lip}_{1-1/p}(\mathbf{C})} = \|\mathscr{C}(h - h_n)\|_{\mathrm{Lip}_{1-1/p}(\mathbf{C})} \leq \|\mathscr{C}\| \cdot \|h - h_n\|_{2p}$$

and in particular $\phi_n \to \phi$ uniformly on compact sets.

Finally, we note that principal K-quasiconformal mappings are locally Hölder-continuous with exponent $1/K$, with a constant that depends only on K. Thus, if

E is a compact set,

$$
\begin{aligned}
\sup_{z \in E} |\phi^{-1}(z) - \phi_n^{-1}(z)| &= \sup_{z \in E} |\phi_n^{-1} \circ \phi_n \circ \phi^{-1}(z) - \phi_n^{-1}(z)| \\
&\leq C(K) \cdot \sup_{z \in E} |\phi_n \circ \phi^{-1}(z) - z|^{1/K} \\
&= C(K) \cdot \sup_{w \in \phi(E)} |\phi_n(w) - \phi(w)|^{1/K}
\end{aligned}
$$

Hence $\phi_n^{-1} \to \phi^{-1}$ uniformly on compact sets. $\qquad\square$

Theorem 2. *Let $E \subset \mathbf{C}$ be a compact set, $\alpha \in (0,1]$ and $K \geq 1$. Set $d = \frac{2}{K+1}(1 + \alpha K)$. If*

$$
\mathscr{H}^d(E) = 0
$$

then, E is removable for K-quasiregular mappings in Lip_α.

Proof. Assume that $f \colon \mathbf{C} \to \mathbf{C}$ is a $\mathrm{Lip}_\alpha(\mathbf{C})$ mapping, K-quasiregular in $\mathbf{C} \setminus E$. We will see that f is in fact K-quasiregular in the whole plane $\mathbf{C}$.

Let $\mu = \frac{\overline{\partial} f}{\partial f}$ be the Beltrami coefficient of f. Obviously, there is no restriction if we assume that $E \subset \mathbf{D}$ and that μ is supported on $\mathbf{D}$. Let ϕ be the principal solution to $\overline{\partial}\phi = \mu \, \partial\phi$. Then the function $F = f \circ \phi^{-1}$ is holomorphic on $\mathbf{C} \setminus \phi(E)$ and Hölder continuous on $\mathbf{C}$ with exponent α/K. In particular, $\overline{\partial} F$ defines a distribution compactly supported on $\phi(E)$. Indeed, since F is continuous, it is an $L^2_{\mathrm{loc}}(\mathbf{C})$ function, so that its distributional derivatives define continuous linear functionals on compactly supported $W^{1,2}(\mathbf{C})$ functions. From Weyl's lemma, it is enough to show that $\overline{\partial} F = 0$.

For each $\varepsilon > 0$ we can consider a finite family of disjoint disks $D_j = D(z_j, r_j)$ such that $\Omega_\varepsilon = \bigcup_j D_j$ covers E and

$$
\sum_{j=1}^{n} \mathrm{diam}(D_j)^d \leq \varepsilon
$$

We define $\mu_\varepsilon = \mu \, \chi_{\mathbf{C} \setminus \Omega_\varepsilon}$. If ϕ_ε is the principal solution to $\overline{\partial}\phi_\varepsilon = \mu_\varepsilon \, \partial\phi_\varepsilon$, then by Lemma 1 the functions $F_\varepsilon = f \circ \phi_\varepsilon^{-1}$ converge uniformly to F as $\varepsilon \to 0$, because $f \in \mathrm{Lip}_\alpha(\mathbf{C})$. Thus, for any compactly supported C^∞ function φ,

$$
\begin{aligned}
\langle \overline{\partial} F, \varphi \rangle &= \langle \overline{\partial}(F - F_\varepsilon), \varphi \rangle + \langle \overline{\partial} F_\varepsilon, \varphi \rangle \\
&= -\langle (F - F_\varepsilon), \overline{\partial}\varphi \rangle - \langle \overline{\partial} F_\varepsilon, \varphi \rangle
\end{aligned}
$$

Here the first term converges to 0 as $\varepsilon \to 0$. Thus, we just have to care about the second term.

Consider a partition of unity ψ_j subordinated to the covering D_j, that is, each ψ_j is a C^∞ function, compactly supported on $2D_j$, $|D\psi_j| \leq \frac{C}{\mathrm{diam}(D_j)}$, and $\sum_{j=1}^{n} \psi_j = 1$ on Ω_ε. We now define $\varphi_j = \psi_j \circ \phi_\varepsilon^{-1}$. Then, each φ_j is in $W^{1,2}(\mathbf{C})$, has compact

support in $\phi_\varepsilon(2D_j)$, and $\sum_{j=1}^n \varphi_j = 1$ on $\phi_\varepsilon(\Omega_\varepsilon)$. Thus, for any constants c_j we have

$$(2) \quad \begin{aligned} \langle \overline{\partial} F_\varepsilon, \varphi \rangle &= \langle \overline{\partial} F_\varepsilon, \varphi \sum_{j=1}^n \varphi_j \rangle = \sum_{j=1}^n \langle \overline{\partial} F_\varepsilon, \varphi\, \varphi_j \rangle = \sum_{j=1}^n \langle \overline{\partial} (F_\varepsilon - c_j), \varphi\, \varphi_j \rangle \\ &= -\sum_{j=1}^n \langle (F_\varepsilon - c_j), \overline{\partial}\varphi\, \varphi_j \rangle - \sum_{j=1}^n \langle (F_\varepsilon - c_j), \varphi\, \overline{\partial}\varphi_j \rangle = I + II \end{aligned}$$

and now we have just to bound both sums independently.

On one hand,

$$\left| \sum_{j=1}^n \langle (F_\varepsilon - c_j), \overline{\partial}\varphi\, \varphi_j \rangle \right| = \left| \sum_{j=1}^n \int_{\phi_\varepsilon(2D_j)} (F_\varepsilon(z) - c_j)\, \overline{\partial}\varphi(z)\, \varphi_j(z)\, dA(z) \right|$$

$$\leq \sum_{j=1}^n \int_{\phi_\varepsilon(2D_j)} |F_\varepsilon(z) - c_j|\, |\overline{\partial}\varphi(z)|\, |\varphi_j(z)|\, dA(z)$$

$$\leq \|D\varphi\|_\infty \sum_{j=1}^n \int_{2D_j} |f(w) - c_j|\, J\phi_\varepsilon(w)\, dA(w).$$

Now observe [2] that for any $p \in (1, \frac{K}{K-1})$, $J\phi_\varepsilon \in L^p_{\mathrm{loc}}(\mathbf{C})$. Using this, together with the fact that $f \in \mathrm{Lip}_\alpha$, we obtain for I the bound

$$|I| \leq C\, \|D\varphi\|_\infty \sum_{j=1}^n \left(\int_{2D_j} J\phi_\varepsilon(w)^p\, dA(w) \right)^{1/p} \mathrm{diam}(2D_j)^{\alpha + 2/q}$$

$$\leq C\, \|D\varphi\|_\infty \left(\sum_{j=1}^n \int_{2D_j} J\phi_\varepsilon(w)^p\, dA(w) \right)^{1/p} \left(\sum_{j=1}^n \mathrm{diam}(2D_j)^{q\alpha + 2} \right)^{1/q}$$

where $q = \frac{p}{p-1}$. But both sums converge to 0 as ε tends to 0, so that we get $I \to 0$. On the other hand, an analogous computation for II shows that

$$\left| \sum_{j=1}^n \langle (F_\varepsilon - c_j), \varphi\, \overline{\partial}\varphi_j \rangle \right| = \left| \sum_{j=1}^n \int_{\phi_\varepsilon(2D_j)} (F_\varepsilon(z) - c_j)\, \varphi(z)\, \overline{\partial}\varphi_j(z)\, dA(z) \right|$$

$$\leq \sum_{j=1}^n \int_{\phi_\varepsilon(2D_j)} |F_\varepsilon(z) - c_j|\, |\varphi(z)|\, |\overline{\partial}\varphi_j(z)|\, dA(z)$$

$$\leq \|\varphi\|_\infty \sum_{j=1}^n \int_{2D_j} |f(w) - c_j|\, |\overline{\partial}\varphi_j(\phi_\varepsilon(w))|\, J\phi_\varepsilon(w)\, dA(w).$$

The chain rule gives

$$|D\varphi_j(\phi_\varepsilon)| \leq |D\psi_j|\, |D\phi_\varepsilon^{-1}(\phi_\varepsilon)|.$$

Thus, we have

$$|II| \leq \|\varphi\|_\infty \sum_{j=1}^{n} \left(\int_{2D_j} |D\phi_\varepsilon^{-1}(\phi_\varepsilon(w))| \, J\phi_\varepsilon(w) \, dA(w) \right) \operatorname{diam}(2D_j)^{\alpha-1}$$

$$= \|\varphi\|_\infty \sum_{j=1}^{n} \left(\int_{\phi_\varepsilon(2D_j)} |D\phi_\varepsilon^{-1}(z)| \, dA(z) \right) \operatorname{diam}(2D_j)^{\alpha-1}$$

$$\leq \|\varphi\|_\infty \sum_{j=1}^{n} \left(\int_{2D_j} |D\phi_\varepsilon(w)| \, dA(w) \right) \operatorname{diam}(2D_j)^{\alpha-1}.$$

But $J\phi_\varepsilon^{1/2}$ defines a doubling measure, so that

$$\int_{2D_j} |D\phi_\varepsilon(w)| \, dA(w) \leq C \int_{D_j} J\phi_\varepsilon(z)^{1/2} \, dA(z)$$

where C depends only on K. On the other hand, ϕ_ε is conformal in $\Omega_\varepsilon = \cup_j D_j$. Hence, from the improved integrability results for K-quasiconformal mappings [5], we have $J\phi_\varepsilon \in L^{\frac{K}{K-1}}(\Omega_\varepsilon)$ and

$$\int_{\Omega_\varepsilon} J\phi_\varepsilon(w)^{\frac{K}{K-1}} \, dA(w) \leq 1.$$

Hence, if we choose $p = \frac{2K}{K-1}$ and $q = \frac{2K}{K+1}$, then $q(\alpha - 1) + 2 = d$ and we get

$$|II| \leq C \, \|\varphi\|_\infty \sum_{j=1}^{n} \left(\int_{D_j} J\phi_\varepsilon(w)^{1/2} \, dA(w) \right) \operatorname{diam}(D_j)^{\alpha-1}$$

$$\leq C \, \|\varphi\|_\infty \sum_{j=1}^{n} \left(\int_{D_j} J\phi_\varepsilon(w)^{p/2} \, dA(w) \right)^{1/p} \operatorname{diam}(D_j)^{\alpha-1+2/q}$$

$$\leq C \, \|\varphi\|_\infty \left(\int_{\Omega_\varepsilon} J\phi_\varepsilon(w)^{p/2} \, dA(w) \right)^{1/p} \left(\sum_{j=1}^{n} \operatorname{diam}(D_j)^{q(\alpha-1)+2} \right)^{1/q}.$$

But here the integral remains bounded, while the sum converges to 0 as $\varepsilon \to 0$. $\quad\square$

We wish to emphasize that the improved integrability properties for K-quasi-conformal mappings [5] is precisely what allows us to jump from sets of dimension strictly smaller than d to sets of null d-dimensional Hausdorff measure. Notice that in the particular case $\alpha = 1$, if $\mathscr{H}^2(E) > 0$ then there exists a nonconstant $\operatorname{Lip}_1(\mathbf{C})$ function f which is holomorphic (and thus K-quasiregular, for any K) on $\mathbf{C} \setminus E$ (e.g. [15]). As we said in the introduction, it is also well known that compact sets E with $\mathscr{H}^{1+\alpha}(E) > 0$ are not removable for analytic functions in $\operatorname{Lip}_\alpha$, for any $\alpha \in (0,1)$.

Note that Theorem 2 may be stated in a more general sense. Assume that f is K-quasiregular in $\mathbf{C} \setminus E$ and, instead of inequality (1), suppose that

$$|f(z) - f(w)| \leq \omega(|z - w|)$$

for any $z, w \in \mathbf{C}$ with $|z - w| < 1$, where $\omega(r) = \omega_f(r)$ is any bounded continuous non-decreasing function $\omega \colon [0, 1) \to [0, \infty)$ such that $\omega(0) = 0$. If $\mathscr{H}^h(E) = 0$ for $h(t) = t^{\frac{2}{K+1}} \omega(2t)^{\frac{2K}{K+1}}$, then f is in fact K-quasiregular on the whole plane. As particular cases, $\omega(r) = C\, r^\alpha$ gives the previous Theorem, while if $f \in \mathrm{lip}_\alpha(\mathbf{C})$, that is, if $\frac{\omega_f(r)}{r^\alpha} \to 0$ as $r \to 0$, then $\mathscr{M}_*^d(E) = 0$ is enough (here $\mathscr{M}_*^d$ denotes the lower d-dimensional Hausdorff content). Finally, if we only assume f to be continuous, then our argument shows that the condition $\mathscr{M}_*^{\frac{2}{K+1}}(E) = 0$ suffices.

The same procedure also gives removability results in the more general setting of finite distortion mappings. More precisely, subexponentially integrable distortion mappings are locally in $W^{1,p}$ for any $p < 2$. In this context, we obtain the following.

Corollary 3. *Suppose that* $f \in \mathrm{Lip}_\alpha(\mathbf{C})$, *and let* $E \subset \mathbf{D}$ *be a compact set. Assume that* f *is a mapping of finite distortion on* $\mathbf{C} \setminus E$, *with distortion function* K *such that* $K(z) = 1$ *whenever* $|z| > 1$, *and*

$$(3) \qquad\qquad \exp\left(\frac{K}{\log(e + K)}\right) \in L^p(\mathbf{D})$$

for some $p > 0$. *If* $\dim(E) < 2\alpha$, *then* f *is a mapping of finite distortion on* $\mathbf{C}$.

Actually, here one can replace condition (3) by the more general condition given by

$$\exp\left(\mathscr{A}(K)\right) \in L^p(\mathbf{D})$$

where $\mathscr{A} \colon [1, \infty) \to [1, \infty)$ is any smoothly increasing function, such that $\lim_{t \to \infty} t\, \mathscr{A}'(t) = \infty$, $t \mapsto e^{\mathscr{A}(t)}$ is convex, and $\int_1^\infty \mathscr{A}(t)/t^2\, dt = \infty$ [10]. On the other hand, if we ask K to be in a stronger space, then one can use the sharp integrability results in [7] and obtain bigger removable singularities.

Corollary 4. *There exists a positive constant* $p_0 > 0$ *with the following property. Let* $f \in \mathrm{Lip}_\alpha(\mathbf{C})$, *and* $E \subset \mathbf{D}$ *be a compact set. Assume that* f *is a mapping of finite distortion on* $\mathbf{C} \setminus E$, *with distortion function* K *such that* $K(z) = 1$ *whenever* $|z| > 1$, *and*

$$\exp(K) \in L^p(\mathbf{D})$$

for some $p > p_0$. *If* $\mathscr{H}^{2\alpha}(E)$ *is* σ-*finite, then* f *is a mapping of finite distortion on* $\mathbf{C}$.

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