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ALEKSANDROV–CLARK MEASURES AND SEMIGROUPS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper we prove a formula describing the infinitesimal generator of a continuous semigroup (φ_t) of holomorphic self-maps of the unit disc with respect to a boundary regular fixed point. The result is based on Aleksandrov–Clark measures techniques. In particular we prove that the Aleksandrov–Clark measure of (φ_t) at a boundary regular fixed point is differentiable (in the weak*-topology) with respect to t.

1. Introduction

The aim of the present note is to study the incremental ratio of Aleksandrov–Clark measures (sometimes called spectral measures) of continuous semigroups of the unit disc at boundary regular fixed points, obtaining a measure-theoretic generalization of the well renowned Berkson–Porta formula at the Denjoy–Wolff point.

To state our results, we briefly recall the notion of Aleksandrov–Clark measures and semigroups as needed for our aims (for details on Aleksandrov–Clark measures we refer the reader to the recent surveys [8], [10], [11] and the references therein; while we refer to [1] and [12] for more about iteration theory and semigroups).

Let $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ be the unit disc. Let $f : \mathbf{D} \to \mathbf{D}$ be holomorphic. Fix $\tau \in \partial\mathbf{D}$ and consider the positive harmonic function $\mathsf{Re}\,(\tau + f(z))/(\tau - f(z))$. Then

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there exists a non-negative and finite Borel measure $\mu_{f,\tau}$ (called the *Aleksandrov–Clark measure* of f at τ) on $\partial\mathbf{D}$ such that

$$\operatorname{Re}\frac{\tau+f(z)}{\tau-f(z)}=\int_{\partial\mathbf{D}}P(\zeta,z)d\mu_{f,\tau}(\zeta)\quad\text{for all } z\in\mathbf{D}, \tag{1.1}$$

where $P(\zeta,z)=\frac{1-|z|^2}{|z-\zeta|^2}$ is the Poisson kernel.

We recall that a point $\zeta\in\partial\mathbf{D}$ is said to be a boundary contact point for $f\colon\mathbf{D}\to\mathbf{D}$ if $\lim_{r\to1}f(r\zeta)=\tau\in\partial\mathbf{D}$. In such a case, as customary, we write $f^*(\zeta):=\tau$. It is a remarkable fact that the *angular derivative* at a boundary contact point ζ always exists (possibly infinity). Namely, the following non-tangential (or angular) limit exists in the Riemann sphere:

$$f'(\zeta):=\angle\lim_{z\to\zeta}\frac{f(z)-f^*(\zeta)}{z-\zeta}.$$

The modulus of $f'(\zeta)$ is known as the boundary dilatation coefficient of f at τ. A point $\tau\in\partial\mathbf{D}$ is a *boundary regular fixed point*, BRFP for short, for f if $f^*(\tau)=\tau$ and $f'(\tau)$ is finite. If τ is a BRFP for f then the classical Julia–Wolff–Carathéodory theorem (see, e.g., [1, Prop. 1.2.8, Thm. 1.2.7]) asserts that the non-tangential limit $\angle\lim_{z\to\tau}f'(z)=f'(\tau)$ and $f'(\tau)\in(0,+\infty)$.

In 1929, Nevanlinna obtained a very deep relationship between angular derivatives and Aleksandrov–Clark measures. Namely, (see, e.g., [1, p. 61], [11, Thm. 3.1]) he proved

Theorem 1.1. (Nevanlinna) *Let $f\colon\mathbf{D}\to\mathbf{D}$ be a holomorphic self-map and $\zeta\in\partial\mathbf{D}$. Then ζ is a boundary contact point of f with $f'(\zeta)\in\mathbf{C}$ if and only if for some $\tau\in\partial\mathbf{D}$ the Aleksandrov–Clark measure $\mu_{f,\tau}$ has an atom at ζ (that is, $\mu_{f,\tau}(\{\zeta\})>0$). In this case, it follows $f^*(\zeta)=\tau$ and $\mu_{f,\tau}(\{\zeta\})=1/|f'(\zeta)|$.*

A (continuous) semigroup (φ_t) of holomorphic self-maps is a continuous homomorphism from the additive semigroup of non-negative real numbers to the composition semigroup of all holomorphic self-maps of $\mathbf{D}$ endowed with the compact-open topology. It is well known after the basic work of Berkson and Porta [2] that in fact the dependence of the semigroup (φ_t) on the parameter t is real-analytic and there exists a holomorphic vector field $G\colon\mathbf{D}\to\mathbf{C}$, called the *infinitesimal generator of the semigroup*, such that

$$\frac{\partial\varphi_t}{\partial t}(z)=G(\varphi_t(z)) \tag{1.2}$$

for all $z\in\mathbf{D}$.

A point $\tau\in\partial\mathbf{D}$ is a *boundary regular fixed point*, BRFP for short, for the semigroup (φ_t) provided it is a BRFP for φ_t for all $t\geq0$ (and this is the case if and only if τ is a BRFP for φ_t for some $t>0$, see [3]). According to [4, Thm. 1], the point $\tau\in\partial\mathbf{D}$ is a BRFP for (φ_t) if and only if $G(\tau)=0$ as non-tangential limit and the non-tangential limit $\angle\lim_{z\to\tau}G'(z)=\lambda$ exists finitely. Moreover, if τ is a BRFP for (φ_t) then $\varphi_t'(\tau)=e^{\lambda t}$ and $\lambda\in\mathbf{R}$. It is well known that if (φ_t) has no fixed point in $\mathbf{D}$ then there exists a unique boundary regular fixed point $\tau \in \partial\mathbf{D}$, called the *Denjoy–Wolff point* of the semigroup, such that $\varphi_t(z) \to \tau$ as $t \to \infty$ for all $z \in \mathbf{D}$.

In the rest of the paper we will denote by dm the Lebesgue measure on $\partial\mathbf{D}$ normalized so that $m(\partial\mathbf{D}) = 1$ and by δ_ξ the Dirac atomic measure concentrated at $\xi \in \partial\mathbf{D}$.

Let (φ_t) be a semigroup and $\tau \in \partial\mathbf{D}$. We will denote by $\mu_{t,\tau}$ the Aleksandrov–Clark measure of φ_t at τ. It can be checked (test with Poisson kernels and use the density of their span in $C(\partial\mathbf{D})$) that $\{\mu_{t,\tau}\}$ is continuous (in the weak*-topology) with respect to t. We will prove that it is actually differentiable at 0. Indeed, our first result is the following (note that $\mu_{0,\tau} = \delta_\tau$):

Proposition 1.2. *Let (φ_t) be a continuous semigroup of holomorphic self-maps of the unit disc $\mathbf{D}$. Let $\tau \in \partial\mathbf{D}$ be a boundary regular fixed point for (φ_t) with boundary dilatation coefficients $(e^{\lambda t})$. Then there exists a positive measure μ on $\partial\mathbf{D}$ such that it has no atom at the point τ and*

$$\frac{\mu_{t,\tau} - \delta_\tau}{t} \xrightarrow{w^*} -\lambda\delta_\tau + \mu, \quad \text{as } t \to 0. \tag{1.3}$$

The measure μ in (1.3) is strictly related to the infinitesimal generator of the semigroup, see Proposition 4.1. From such a formula we will obtain our main result:

Theorem 1.3. *Let (φ_t) be a continuous semigroup of holomorphic self-maps of the unit disc $\mathbf{D}$ and let G be its infinitesimal generator. Let $\tau \in \partial\mathbf{D}$ be a boundary regular fixed point for (φ_t) with boundary dilatation coefficients $(e^{\lambda t})$. Then there exists a unique $p\colon \mathbf{D} \to \mathbf{C}$ holomorphic, with $\mathsf{Re}\, p \geq 0$ and $\angle\lim_{z\to\tau}(z-\tau)p(z) = 0$ such that*

$$G(z) = (\bar{\tau}z - 1)(z - \tau)\left[p(z) - \frac{\lambda}{2}\frac{\tau + z}{\tau - z}\right] \quad \text{for all } z \in \mathbf{D}. \tag{1.4}$$

Conversely, given $p\colon \mathbf{D} \to \mathbf{C}$ holomorphic with $\mathsf{Re}\, p \geq 0$ and $\angle\lim_{z\to\tau}(z-\tau)p(z) = 0$, $\tau \in \partial\mathbf{D}$ and $\lambda \in \mathbf{R}$, the function G defined as in (1.4) *is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc for which τ is a boundary regular fixed point with boundary dilatation coefficients $(e^{\lambda t})$.*

In particular, τ is the Denjoy–Wolff point of (φ_t) if and only if $\lambda \leq 0$ and, if this is the case,

$$\mathsf{Re}\left(\frac{G(z)}{(\bar{\tau}z - 1)(z - \tau)}\right) \geq 0 \quad \text{for all } z \in \mathbf{D},$$

recovering in this way the celebrated Berkson–Porta representation formula when τ belongs to $\partial\mathbf{D}$ [2]. We also note that the above λ in formula (1.4) is clearly unique bearing in mind the "conversely" part of Theorem 1.3.

It is worth pointing out that there is a very simple connection between the function p and the measure μ appearing in Theorem 1.3 and Proposition 1.2. Namely, for every $z \in \mathbf{D}$,

$$\mathsf{Re}\, p(z) = \frac{1}{2} \int_{\partial \mathbf{D}} P(\xi, z)\, d\mu(\xi).$$

The plan of the paper is the following. In the second section we compute the singular part of Aleksandrov–Clark measures of N-to-1 mappings (in particular for $N = 1$, univalent maps). In the third section we will use such a computation to prove Proposition 1.2 and Theorem 1.3. In the final section we discuss some consequences of our results.

2. Singular parts of Aleksandrov–Clark measures for N-to-1 mappings

Nieminen and Saksman [9, p. 3186] already remarked that for holomorphic N-to-1 self-maps of the unit disc the singular part of the corresponding Aleksandrov–Clark measures is discrete. Using this fact and Theorem 1.1, it can be deduced the following proposition.

Firstly, we introduce some terminology. Given a positive Borel measure ϱ on $\partial \mathbf{D}$ we will write $\varrho = \varrho^s + \varrho^a$ for its Lebesgue decomposition in the singular part ϱ^s and the absolutely continuous part ϱ^a with respect to the Lebesgue measure.

Proposition 2.1. *Let $f\colon \mathbf{D} \to \mathbf{D}$ be a N-to-1 ($N \geq 1$) holomorphic map and let $\tau \in \partial \mathbf{D}$. Then there exist $0 \leq m \leq N$ and $\zeta_1, \dots, \zeta_m \in \partial \mathbf{D}$ such that $f^*(\zeta_j) = \tau$, the non-tangential limit $f'(\zeta_j)$ of f' at ζ_j exists finitely for $j = 1, \dots, m$ and*

$$\mu^s_{f,\tau} = \sum_{k=1}^{m} \frac{1}{|f'(\zeta_k)|} \delta_{\zeta_k}. \tag{2.1}$$

Moreover, if $x \in \partial \mathbf{D} \setminus \{\zeta_1, \dots, \zeta_m\}$ is such that $f^*(x) = \tau$ then $\limsup_{z \to x} |f'(z)| = \infty$.

Corollary 2.2. *Let $f\colon \mathbf{D} \to \mathbf{D}$ be a univalent map and let $\tau \in \partial \mathbf{D}$. Then either $\mu^s_{f,\tau} = 0$ or there exists a unique point $\zeta \in \partial \mathbf{D}$ such that $f^*(\zeta) = \tau$, the non-tangential limit $f'(\zeta)$ of f' at ζ exists finitely and*

$$\mu^s_{f,\tau} = \frac{1}{|f'(\zeta)|} \delta_{\zeta}. \tag{2.2}$$

Moreover, if $x \in \partial \mathbf{D} \setminus \{\zeta\}$ is such that $f^*(x) = \tau$ then $\limsup_{z \to x} |f'(z)| = \infty$.

Remark 2.3. Corollary 2.2 implies in particular that for a univalent self-map f of the unit disc and any point $\tau \in \partial \mathbf{D}$ there exists at most one point $x \in \partial \mathbf{D}$ such that $f^*(x) = \tau$ and the non-tangential limit of f' exists finitely at x. This latter fact can also be proved directly, see [5, Lemma 8.2].

Corollary 2.4. *Let (φ_t) be a continuous semigroup of holomorphic self-maps of $\mathbf{D}$. Suppose that $\tau \in \partial \mathbf{D}$ is a BRFP for (φ_t) with boundary dilatation coefficients $(e^{\lambda t})$. Then*

$$\mu^s_{t,\tau} = e^{-\lambda t} \delta_{\tau}. \tag{2.3}$$

Proof. For every $t \geq 0$ the map φ_t is univalent (see, e.g., [1] or [12]). Therefore by Corollary 2.2 it follows that $\mu^s_{t,\tau} = \frac{1}{|\varphi'_t(\tau)|}\delta_\tau$ and since $\varphi'_t(\tau) = e^{\lambda t}$, we are done. □

Remark 2.5. Bearing in mind Proposition 2.1, we see that for an arbitrary holomorphic self-map of the unit disk f, given $\tau \in \partial\mathbf{D}$ and $\zeta_1, \dots, \zeta_n$ different points in $\partial\mathbf{D}$ such that $f^*(\zeta_j) = \tau$ with $f'(\zeta_j) \in \mathbf{C}$ for all $j = 1, \dots, n$, then

$$\sum_{j=1}^{n} \frac{1}{|f'(\zeta_j)|}\delta_{\zeta_j} \leq \mu^s_{f,\tau}.$$

In particular, we have

$$\sum_{j=1}^{n} \frac{1}{|f'(\zeta_j)|} \leq \|\mu_{f,\tau}\| = \int_{\partial\mathbf{D}} d\mu_{f,\tau} = \mathsf{Re}\,\frac{\tau + f(0)}{\tau - f(0)}. \tag{2.4}$$

Moreover, equality holds in (2.4) if and only if $\sum_{j=1}^{n} \frac{1}{|f'(\zeta_j)|}\delta_{\zeta_j} = \mu_{f,\tau}$ if and only if

$$\begin{aligned}\mathsf{Re}\,\frac{\tau + f(z)}{\tau - f(z)} &= \int_{\partial\mathbf{D}} P(\zeta, z) d\mu_{f,\tau}(\zeta) = \sum_{j=1}^{n} \frac{1}{|f'(\zeta_j)|} \int_{\partial\mathbf{D}} P(\zeta, z) d\delta_{\zeta_j}(\zeta)\\ &= \mathsf{Re} \sum_{j=1}^{n} \frac{1}{|f'(\zeta_j)|} \frac{\zeta_j + z}{\zeta_j - z} \quad \text{for all } z \in \mathbf{D},\end{aligned}$$

namely, if and only if f is a finite Blaschke product of order n. Inequality (2.4) was obtained in [5, Thm 8.1] by Cowen and Pommerenke with complete different techniques.

3. Differentiability of Aleksandrov–Clark measures and the representation formula

First of all we prove Proposition 1.2.

Proof of Proposition 1.2. For the sake of simplicity, let us denote by $\mu_t := \mu_{t,\tau}$ the Aleksandrov–Clark measure of φ_t at τ. Moreover, for $t \geq 0$ we define

$$\sigma_t := \frac{\mu_t - \delta_\tau}{t}.$$

Let $\sigma_t = \sigma^s_t + \sigma^a_t$ be the Lebesgue decomposition of σ_t with respect to the Lebesgue measure. By Corollary 2.4 it follows:

$$\sigma^s_t = \frac{e^{-\lambda t} - 1}{t}\delta_\tau. \tag{3.1}$$

Taking the limit as $t \to 0$, we have

$$\sigma^s_t \xrightarrow{w^*} -\lambda\delta_\tau.$$

Now we examine the absolutely continuous part σ_t^a. Since $\sigma_t^a \geq 0$ we have

$$\begin{aligned}
\|\sigma_t^a\| &= \int_{\partial\mathbf{D}} d\sigma_t^a = \int_{\partial\mathbf{D}} d\sigma_t - \int_{\partial\mathbf{D}} d\sigma_t^s \overset{(3.1)}{=} \int_{\partial\mathbf{D}} d\sigma_t - \frac{e^{-\lambda t}-1}{t} \\
&= \frac{1}{t}\int_{\partial\mathbf{D}} d\mu_t - \frac{1}{t} - \frac{e^{-\lambda t}-1}{t} \overset{(1.1)}{=} \frac{1}{t}\,\mathsf{Re}\left[\frac{\tau+\varphi_t(0)}{\tau-\varphi_t(0)} - 1\right] - \frac{e^{-\lambda t}-1}{t} \\
&= \mathsf{Re}\left[\frac{2\varphi_t(0)}{t}\frac{1}{\tau-\varphi_t(0)}\right] - \frac{e^{-\lambda t}-1}{t},
\end{aligned}$$

and then, taking the limit for $t \to 0$ and by (1.2), we obtain

$$\lim_{t\to 0}\|\sigma_t^a\| = 2\,\mathsf{Re}\,[\overline{\tau}G(0)] + \lambda. \tag{3.2}$$

This implies that $\{\|\sigma_t\|\}$ is uniformly bounded for $t << 1$. Since the ball in the weak*-topology of measures on $\partial\mathbf{D}$ is compact and metrizable, the net $\{\sigma_t\}$ is sequentially compact. Now, by (1.1),

$$\begin{aligned}
\int_{\partial\mathbf{D}} P(\zeta,z)\,d\sigma_t(\zeta) &= \frac{1}{t}\,\mathsf{Re}\left[\frac{\tau+\varphi_t(z)}{\tau-\varphi_t(z)} - \frac{\tau+z}{\tau-z}\right] \\
&= 2\,\mathsf{Re}\left[\frac{\varphi_t(z)-z}{t}\cdot\frac{\tau}{(\tau-\varphi_t(z))(\tau-z)}\right],
\end{aligned}$$

and (1.2) yields

$$\lim_{t\to 0}\int_{\partial\mathbf{D}} P(\zeta,z)\,d\sigma_t(\zeta) = 2\,\mathsf{Re}\left[\frac{G(z)\tau}{(\tau-z)^2}\right]. \tag{3.3}$$

This implies that given two accumulation points σ and σ' of $\{\sigma_t\}$ we have

$$\int_{\partial\mathbf{D}} P(\zeta,z)\,d\sigma(\zeta) = \int_{\partial\mathbf{D}} P(\zeta,z)\,d\sigma'(\zeta).$$

Hence $\sigma = \sigma'$ (see, e.g., [7, p. 10]). Therefore the net $\{\sigma_t\}$ is actually weak*-convergent for $t \to 0$.

Finally, denote by μ the limit of $\{\sigma_t^a\}$. Since σ_t^a is a positive measure for all $t \geq 0$, so is μ. Notice that

$$\begin{aligned}
\int_{\partial\mathbf{D}} P(\xi,z)\,d\mu(\xi) &= \lim_{t\to 0}\int_{\partial\mathbf{D}} P(\xi,z)\,d\sigma_t(\xi) - \lim_{t\to 0}\int_{\partial\mathbf{D}} P(\xi,z)\,d\sigma_t^s(\xi) \\
&= 2\,\mathsf{Re}\left[\frac{G(z)\tau}{(z-\tau)^2}\right] + \lambda P(\tau,z).
\end{aligned}$$

Let us show now that μ has no atom at τ. Consider $f_r(\xi) = \frac{(1-r)^2}{|\xi - r\tau|^2}$, for $r \in (0,1)$ and $\xi \in \partial\mathbf{D}$. Trivially, every f_r is uniformly bounded by one on the boundary of the unit disk. Moreover, the pointwise limit of f_r, as r goes to 1, is exactly the characteristic function at τ. Hence, by the Lebesgue dominated convergence theorem, we deduce that

$$\mu(\{\tau\}) = \lim_{r\to 1}\int_{\partial\mathbb{D}} \frac{(1-r)^2}{|\xi - r\tau|^2} d\mu(\xi) = \frac{1}{2}\lim_{r\to 1}(1-r)\int_{\partial\mathbb{D}} P(\xi, r\tau) d\mu(\xi).$$

Therefore,

$$\mu(\{\tau\}) = \lim_{r\to 1} \mathsf{Re}\left[\frac{G(r\tau)}{\tau - r\tau}\right] + \lambda.$$

Since τ is a boundary regular fixed point, by [4, Theorem 1], we know that

$$\angle\lim_{z\to\tau}\frac{G(z)}{z-\tau} = \lambda.$$

Hence, $\mu(\{\tau\}) = 0$ as wanted. □

Now we are in the good shape to prove our representation formula.

Proof of Theorem 1.3. We retain the same notations as in the proof of Proposition 1.2.

Since

$$\mathsf{Re}\,\frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} = \int_{\partial\mathbf{D}} \mathsf{Re}\,\frac{\zeta + z}{\zeta - z}\, d\mu_t(\zeta) \quad \text{for all } z \in \mathbf{D},$$

and $\frac{\tau+\varphi_t(z)}{\tau-\varphi_t(z)}$ and $\int_{\partial\mathbf{D}} \frac{\zeta+z}{\zeta-z}\, d\mu_t(\zeta)$ are analytic functions, it follows

$$\frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} = \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z}\, d\mu_t(\zeta) + i\,\mathsf{Im}\left(\frac{\tau + \varphi_t(0)}{\tau - \varphi_t(0)}\right) \quad \text{for all } z \in \mathbf{D}.$$

Hence

$$\frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} - \frac{\tau + z}{\tau - z} = \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z}\, d(\mu_t - \delta_\tau)(\zeta) + i\,\mathsf{Im}\left(\frac{\tau + \varphi_t(0)}{\tau - \varphi_t(0)} - 1\right) \quad \text{for all } z \in \mathbf{D}.$$

After some computations and dividing by t we obtain

$$\frac{\varphi_t(z) - z}{t} = \frac{(1 - \varphi_t(z)\bar\tau)(\tau - z)}{2}\left[\int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z}\, d\sigma_t(\zeta) + 2i\,\mathsf{Im}\left(\frac{\varphi_t(0)}{t}\frac{1}{\tau - \varphi_t(0)}\right)\right]$$

for all $z \in \mathbf{D}$. Now, by Proposition 1.2, passing to the limit as t goes to 0, we deduce

$$G(z) = \frac{(1 - z\bar\tau)(\tau - z)}{2}\left[-\lambda\frac{\tau + z}{\tau - z} + \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) + 2i\,\mathsf{Im}\left(\frac{G(0)}{\tau}\right)\right].$$

Setting $p(z) := \frac{1}{2}\int_{\partial\mathbf{D}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) + i\,\mathsf{Im}\left(\frac{G(0)}{\tau}\right)$, we obtain (1.4).

Moreover, since τ is a boundary regular fixed point, by [4, Theorem 1] it follows $\angle\lim_{z\to\tau}\frac{G(z)}{z-\tau} = \lambda$. Then an easy computation shows that $\angle\lim_{z\to\tau}(z-\tau)p(z) = 0$.

In order to prove uniqueness, assume that $q\colon \mathbf{D} \to \mathbf{C}$ is holomorphic and $\gamma \in \mathbf{C}$ are such that $\mathsf{Re}\, q \geq 0$, $\angle\lim_{z\to\tau}(z-\tau)q(z) = 0$, and

$$G(z) = (\bar\tau z - 1)(z - \tau)\left[q(z) - \frac{\gamma}{2}\frac{\tau + z}{\tau - z}\right] \quad \text{for all } z \in \mathbf{D}.$$

Then, by [4, Theorem 1],

$$\lambda = \angle \lim_{z\to\tau} \frac{G(z)}{z-\tau} = \angle \lim_{z\to\tau} \bar{\tau}(z-\tau)\left[q(z) - \frac{\gamma}{2}\frac{\tau+z}{\tau-z}\right] = \gamma.$$

From this, it follows immediately that $p = q$.

Now we prove the converse: let $p\colon \mathbf{D} \to \mathbf{C}$ holomorphic with $\operatorname{Re} p \geq 0$, $\tau \in \partial\mathbf{D}$ and $\lambda \in \mathbf{R}$, and let G be defined by (1.4). We want to prove that G is an infinitesimal generator of some continuous semigroup of holomorphic self-maps of the unit disc.

Since $\operatorname{Re} p \geq 0$, by [1, Theorem 1.4.19], the function

$$H_1(z) := (\bar{\tau}z - 1)(z-\tau)p(z) \quad \text{for all } z \in \mathbf{D}$$

is the infinitesimal generator of a continuous semigroup of holomorphic functions with Denjoy–Wolff point τ. By [4, Theorem 1], $\angle \lim_{z\to\tau} H_1(z) = 0$ and

$$\angle \lim_{z\to\tau} \frac{H_1(z)}{z-\tau} = \beta$$

for some $\beta \leq 0$. Our hypothesis that $\angle \lim_{z\to\tau}(z-\tau)p(z) = 0$ implies that actually $\beta = 0$. Therefore the semigroup associated to H_1 has Denjoy–Wolff point τ with boundary dilatation coefficient 1 for all $t \geq 0$. In particular, if $\lambda = 0$ we are done.

Assume $\lambda \neq 0$. Then $H_2\colon \mathbf{D} \to \mathbf{C}$ defined by $H_2(z) := \frac{\lambda}{2}(\bar{\tau}z - 1)(z+\tau)$ is also the infinitesimal generator of a semigroup of linear fractional maps (in fact of hyperbolic automorphisms) with fixed points τ and $-\tau$ (see, [1, Corollary 1.4.16]). Since the set of infinitesimal generators is a real convex cone (see, e.g., [1, Corollary 1.4.15]), it follows that $G(z) = H_1(z) + H_2(z)$ is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc. Moreover,

$$\angle \lim_{z\to\tau} G(z) = \angle \lim_{z\to\tau} H_1(z) + \angle \lim_{z\to\tau} H_2(z) = 0$$

and

$$\angle \lim_{z\to\tau} \frac{G(z)}{z-\tau} = \angle \lim_{z\to\tau} \frac{H_1(z)}{z-\tau} + \angle \lim_{z\to\tau} \frac{H_2(z)}{z-\tau} = \beta + \lambda = \lambda.$$

Therefore, by [4, Theorem 1], τ is a boundary regular fixed point of the semigroup with boundary dilatation coefficients $(e^{t\lambda})$. □

4. Final Remarks

1. The measure μ in formula (1.3) is strictly related to the infinitesimal generator G of (φ_t). In fact, from classical measure theory, if μ_r for $r \in (0,1)$ is the measure defined by $\mu_r = \rho_r dm$ with density $\rho_r(\xi) := \int_{\partial\mathbf{D}} P(\zeta, r\xi)\, d\mu(\zeta)$, $\xi \in \partial\mathbf{D}$, then

(1) $\lim_{r\to1} \mu_r(\xi) = \mu^a(\xi)$ for m-almost every $\xi \in \partial\mathbf{D}$ and

(2) $\mu_r \xrightarrow{w^*} \mu$.

From (1.3) and (3.3) it follows that

$$\mu_r(\xi) = 2\operatorname{Re}\left[\frac{G(r\xi)\tau}{(\tau - r\xi)^2}\right] dm(\xi) + \lambda \operatorname{Re}\left[\frac{\tau + r\xi}{\tau - r\xi}\right] dm(\xi).$$

Thus, from (1) and (2) above we obtain

Proposition 4.1. *Let (φ_t) be a continuous semigroup of holomorphic self-maps of the unit disc* $\mathbf{D}$ *with infinitesimal generator* G. *Let* $\tau \in \partial\mathbf{D}$ *be a boundary regular fixed point for* (φ_t). *Let* μ *be the positive measure defined in* (1.3). *Then*

a) $\operatorname{Re}\left[\frac{G^*(\xi)\tau}{(\tau-\xi)^2}\right] \in L^1(\partial\mathbf{D}, m)$ *and* $\mu^a(\xi) = 2\operatorname{Re}\left[\frac{G^*(\xi)\tau}{(\tau-\xi)^2}\right] dm(\xi)$.

b) $\int_{\partial\mathbf{D}} f(\zeta)\, d\mu(\zeta) = \lim_{r\to 1} \int_{\partial\mathbf{D}} f(\zeta)\left(2\operatorname{Re}\left[\frac{G(r\zeta)\tau}{(\tau-r\zeta)^2}\right] + \lambda \operatorname{Re}\left[\frac{\tau+r\xi}{\tau-r\xi}\right]\right) dm(\zeta)$ *for all* $f \in C(\partial\mathbf{D})$.

2. From the proof of Theorem 1.3 it follows that the condition $\angle\lim_{z\to\tau}(z-\tau)p(z) = 0$ is not necessary in order to show that (1.4) defines an infinitesimal generator, namely, what we really proved is:

Proposition 4.2. *Let* $p\colon \mathbf{D} \to \mathbf{C}$ *holomorphic with* $\operatorname{Re} p \geq 0$, $\tau \in \partial\mathbf{D}$ *and* $\lambda \in \mathbf{R}$. *Then* $\angle\lim_{z\to\tau}(z-\tau)p(z) = \beta\tau$ *exists for some* $\beta \leq 0$ *and the function* G *defined as in* (1.4) *is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc for which* τ *is a boundary regular fixed point with boundary dilatation coefficients* $(e^{(\beta+\lambda)t})$.

3. Theorem 1.3 shows that given a semigroup of holomorphic functions (φ_t) with a boundary regular fixed point τ, its infinitesimal generator G is the sum of the infinitesimal generator of a semigroup of parabolic holomorphic maps with Denjoy–Wolff point at τ (namely, $H_1(z) = (\bar\tau z - 1)(z-\tau)p(z))$ plus, if $\lambda \neq 0$, the infinitesimal generator of a group of hyperbolic automorphisms of the unit disc (namely, $H_2(z) = \frac{\lambda}{2}(\bar\tau z^2 - \tau))$ with a fixed point at τ. Notice that τ is the Denjoy–Wolff point for (φ_t) if and only if it is the Denjoy–Wolff point for the group of hyperbolic automorphisms.

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