Annals of Functional Analysis
Ann. Funct. Anal. 2 (2011), no. 1, 19–30
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

PROPERTIES OF THE SLANT WEIGHTED TOEPLITZ OPERATOR

SUBHASH CHANDER ARORA[1] AND RITU KATHURIA[2*]

Communicated by Y. Seo

ABSTRACT. If $\beta = \langle \beta_n \rangle_{n \in Z}$ is a sequence of positive numbers, then a slant weighted Toeplitz operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi = WM_\phi$ where M_ϕ is the multiplication operator on $L^2(\beta)$. When the sequence $\beta \equiv 1$, this operator reduces to the ordinary slant Toeplitz operator given by M.C. Ho in 1996. In this paper, we study some algebraic properties of the slant weighted Toeplitz operator. We also obtain its matrix characterization and discuss the adjoint of this operator.

1. INTRODUCTION AND PRELIMINARIES

Toeplitz operators arise in plenty of applications like prediction theory, wavelet analysis and solution of differential equations. These operators were introduced by O. Toeplitz [7] in the year 1911. Subsequently many mathematicians like Devinatz [10], Abrahmse [3], Brown and Halmos [4] came up with different generalisations of Toeplitz operators. In 1995, Ho [2] introduced the class of slant Toeplitz operator having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators. Villemoes [8] associated the Besov regularity of solutions of the refinement equation with the spectral radius of an associated slant Toeplitz operator and Goodman, Micchelli and Ward [9] showed the connection between their spectral radii and conditions for the solutions of certain differential equations being in Lipschitz classes.

Date: Received: 22 November 2010; Accepted: 6 March 2011.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47B37, Secondary 47B35.

Key words and phrases. Weighted Toeplitz operator, slant weighted Toeplitz operator, weighted shift, weighted multiplication operator.

However these studies were made in the context of the usual Hardy spaces H^2 and H^p and the Lorentz spaces L^2 and L^p. Meanwhile, the notion of the weighted sequence spaces $H^2(\beta)$, $L^2(\beta)$ and their generalisations came up. Shields [1] made a systematic study of the shift operator and the multiplication operator on $L^2(\beta)$. Lauric [6] studied the Toeplitz operators on $H^2(\beta)$. Motivated by the increasing popularity of the spaces $L^2(\beta)$, $H^2(\beta)$ and the multidirectional applications of the slant Toeplitz operators, we introduced [5] the notion of slant weighted Toeplitz operators. In this paper we further investigate the properties of these operators. The study of weighted Toeplitz operators and that of slant weighted Toeplitz operators is supposed to be meaningful not only to specialists in the theory of Toeplitz operators, but would also be of interest to physicists, probabilists and computer scientists. We begin with the following preliminaries:

Let $\beta = \{\beta_n\}_{n\in Z}$ be a sequence of positive numbers such that $\beta_0 = 1$, $0 < \dfrac{\beta_n}{\beta_{n+1}} \le 1$ for $n \ge 0$ and $0 < \dfrac{\beta_n}{\beta_{n-1}} \le 1$ for $n \le 0$. Also let $\dfrac{\beta_{2n}}{\beta_n}$ be bounded. Consider the spaces [6], [1].

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

and

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [6] with an orthonormal basis given by $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k\in Z}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2.$$

Further, $H^2(\beta)$ is a subspace [6] of $L^2(\beta)$. Now, let

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta) \text{ and } \exists\, c \in \mathbb{R} \right.$$
$$\left. \text{such that } \|\phi f\|_\beta \le c\|f\|_\beta \text{ for all } f \in L^2(\beta) \right\}.$$

Then, $L^\infty(\beta)$ is a Banach space [6] with respect to the norm defined by

$$\|\phi\|_\infty = \inf\{c | \|\phi f\|_\beta \le c\|f\|_\beta \text{ for all } f \in L^2(\beta)\}.$$

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$.

Let $\phi \in L^\infty(\beta)$, then the weighted multiplication operator [1] with symbol ϕ, that is, $M_\phi : L^2(\beta) \to L^2(\beta)$ is given by

$$M_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z).$$

If we put $\phi_1(z) = z$, then $M_{\phi_1} = M_z$ is the operator defined as $M_z e_k(z) = w_k e_{k+1}(z)$, where $w_k = \dfrac{\beta_{k+1}}{\beta_k}$ for all $k \in Z$, and is known as a weighted shift [1].

Further, the weighted Toeplitz operator T_ϕ [6] on $H^2(\beta)$ is defined as

$$T_\phi(f) = P(\phi f).$$

This mapping is well defined, for, if $f \in H^2(\beta) \subset L^2(\beta)$, then by definition, $\phi f \in L^2(\beta)$ and hence $P(\phi f) \in H^2(\beta)$. The matrix of T_ϕ is :

$$\begin{bmatrix} a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \dots & \dots \\ a_1\frac{\beta_1}{\beta_0} & a_0\frac{\beta_1}{\beta_1} & a_{-1}\frac{\beta_1}{\beta_2} & \dots & \dots \\ a_2\frac{\beta_2}{\beta_0} & a_1\frac{\beta_2}{\beta_1} & a_0\frac{\beta_2}{\beta_2} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Hence the effect of T_ϕ on the orthonormal basis can be described by

$$T_\phi e_k(z) = \frac{1}{\beta_k}\sum_{n=0}^{\infty} a_{n-k}\beta_n e_n(z).$$

2. Slant weighted Toeplitz operator

Let $\phi \in L^\infty(\beta)$. Then the slant weighted Toeplitz operator A_ϕ, introduced in [5] is an operator on $L^2(\beta)$ defined as $A_\phi : L^2(\beta) \to L^2(\beta)$ such that

$$A_\phi e_k(z) = \frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_{2n-k}\beta_n e_n(z).$$

If $W : L^2(\beta) \to L^2(\beta)$ such that

$$We_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z)$$

and

$$We_{2n-1}(z) = 0 \quad \text{for all } n \in Z,$$

then an alternate definition of A_ϕ [5] is given by

$$A_\phi(f) = WM_\phi(f) = W(\phi f) \quad \text{for all} \quad f \in L^2(\beta).$$

The matrix of W is

$$\left[\begin{array}{c|cccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \vdots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & 0 & \dots \\ \vdots & 0 & 0 & \frac{\beta_1}{\beta_2} & 0 & 0 & \dots \\ \vdots & 0 & 0 & 0 & 0 & \frac{\beta_2}{\beta_4} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right].$$

Also, $\|W\| = \sup \left| \dfrac{\beta_n}{\beta_{2n}} \right| \leq 1$. The adjoint of W is given by

$$W^* e_n(z) = \frac{\beta_n}{\beta_{2n}} e_{2n}(z), \quad n \in Z.$$

Theorem 2.1. *W does not commute with M_z.*

Proof. $A_\phi = W M_\phi$ and when $\phi = 1$,

$$A_1 = W M_1 = W.$$

But since A_1 is a slant weighted Toeplitz operator, it must satisfy the characterization [5]

$$\begin{aligned} & M_z A_1 = A_1 M_{z^2} \\ \Rightarrow \quad & M_z W = W M_{z^2}. \end{aligned}$$

Hence W does not commute with M_z. □

Theorem 2.2. *The mapping $\phi \to A_\phi$ is linear and one-to-one.*

Proof.

$$\begin{aligned} A_{(\alpha\phi+\beta\psi)} &= W M_{(\alpha\phi+\beta\psi)} \\ &= \alpha W M_\phi + \beta W M_\psi \\ &= \alpha A_\phi + \beta A_\psi \end{aligned}$$

Hence the mapping is linear.

For one-one ness, let $A_\phi = A_\psi$ where $\phi, \psi \in L^\infty(\beta)$. Then

$$\begin{aligned} & A_{\phi-\psi} = 0 \\ \Rightarrow & A(\phi-\psi) e_n(z) = 0 \quad \text{for all } \; n \in Z \\ \Rightarrow & W M_{(\phi-\psi)} e_n(z) = 0 \quad \text{for all } \; n \in Z \\ \Rightarrow & W(\phi-\psi) e_n(z) = 0 \quad \text{for all } \; n \in Z\,. \end{aligned}$$

On taking $n = 1$,

$$\begin{aligned} & W(\phi-\psi) e_1(z) = 0 \\ \Rightarrow & \phi - \psi = 0 \text{ or } \; \phi - \psi \text{ has only even coefficients.} \end{aligned}$$

On taking $n = 2$,

$$\begin{aligned} & W(\phi-\psi) e_2(z) = 0 \\ \Rightarrow & \phi - \psi = 0 \text{ or } \; \phi - \psi \text{ has only odd coefficients.} \end{aligned}$$

Hence we conclude that $\phi - \psi = 0$. □

Theorem 2.3. *$W(\phi(z^2)) = \phi(z)$ for all $\phi \in L^2(\beta)$.*

Proof. Let $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$ be in $L^2(\beta)$. Then

$$\begin{aligned} W(\phi(z^2)) &= W\left(\sum a_n z^{2n}\right) \\ &= W \sum a_n \beta_{2n} e_{2n}(z) \\ &= \sum a_n \beta_n e_n(z) \\ &= \sum a_n z^n = \phi(z). \end{aligned}$$

□

Lemma 2.4. *If $f(z)$ is an $L^2(\beta)$ function, then $f(z^2)$ is also an $L^2(\beta)$ function if $\frac{\beta_{2n}}{\beta_n} < M < \infty$ for all n.*

Proof. Let $f(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n$ be an $L^2(\beta)$ function.

Then

$$\|f(z)\|_\beta^2 = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty$$

Also, then

$$f(z^2) = \sum_{n=-\infty}^{\infty} \alpha_n z^{2n} = \sum_{n=-\infty}^{\infty} \alpha_n \beta_{2n} e_{2n}(z)\,.$$

Hence

$$\begin{aligned} \|f(z^2)\|_\beta^2 &= \sum |\alpha_n|^2 \beta_{2n}^2 \\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 \times \frac{\beta_{2n}^2}{\beta_n^2} \\ &\leq M^2 \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty. \end{aligned}$$

Therefore $f(z^2)$ is also an $L^2(\beta)$ function. □.

Theorem 2.5. *Let $\frac{\beta_{2n}}{\beta_n} < M < \infty$ for all n. Then*

(i) *$W^* f \in L^2(\beta)$ if $f \in L^2(\beta)$.*

(ii) *$WW^* f(z) = g(z)$ where $g(z) = \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^n$ and $g \in L^2(\beta)$.*

(iii) *$W^* W f(z) = h(z^2)$ where $h(z) = \sum a_{2n} \frac{\beta_n^2}{\beta_{2n}^2} z^n$ and $h \in L^2(\beta)$.*

Proof. (i)

$$
\begin{aligned}
W^*f(z) &= W^*\left(\sum a_n z^n\right) \\
&= W^*\sum a_n\beta_n e_n(z) \\
&= \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^{2n} \\
&= \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} (z^2)^n
\end{aligned}
$$

Hence,

$$W^*f(z) = g(z^2)$$

where $g(z) = \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^n$.

Now clearly $g(z) \in L^2(\beta)$. Further from Lemma 2.5, $g(z^2) \in L^2(\beta)$. Hence $W^*f \in L^2(\beta)$.

$$
\text{(ii)} \qquad
\begin{aligned}
WW^*f(z) &= W\sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^{2n} \\
&= W\sum a_n \frac{\beta_n^2}{\beta_{2n}} e_{2n}(z) \\
&= \sum a_n \frac{\beta_n^3}{\beta_{2n}^2} e_n(z) \\
&= \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^n
\end{aligned}
$$

Thus $WW^*f(z) = g(z)$ where $g(z) = \sum a_n \frac{\beta_n^2}{\beta_{2n}^2} z^n$.

$$
\text{(iii)} \qquad
\begin{aligned}
W^*Wf(z) &= W^*\left(W\sum a_n z^n\right) \\
&= W^*\left(\sum a_{2n} z^n\right) \\
&= W^*\left(\sum a_{2n}\beta_n e_n(z)\right) \\
&= \sum a_{2n}\beta_n \frac{\beta_n}{\beta_{2n}} e_{2n}(z) \\
&= \sum a_{2n} \frac{\beta_n^2}{\beta_{2n}^2} z^{2n} \\
&= \sum a_{2n} \frac{\beta_n^2}{\beta_{2n}^2} (z^2)^n
\end{aligned}
$$

Hence $W^*Wf(z) = h(z^2)$ where $h(z) = \sum a_{2n} \frac{\beta_n^2}{\beta_{2n}^2} z^n$, $n \in Z$. □

3. Slant weighted Toeplitz matrix

Definition 3.1. Let $w_n = \dfrac{\beta_{n+1}}{\beta_n}$ for all $n \in Z$. Then the slant weighted Toeplitz matrix corresponding to the weight sequence $\langle w_n \rangle$ is a bilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that

$$\lambda_{i+1,j+2} = \frac{w_i}{w_j w_{j+1}} \lambda_{ij}.$$

It has been proved [5] that A is a slant weighted Toeplitz operator if and only if $M_z A = A M_{z^2}$ where M_z is the weighted shift. We now give another characterization of the slant weighted Toeplitz operator in terms of the matrix defined above.

Theorem 3.2. *A necessary and sufficient condition that an operator A on $L^2(\beta)$ be a slant weighted Toeplitz operator is that its matrix with respect to the orthonormal basis $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k \in Z}$ is a slant weighted Toeplitz matrix.*

Proof. Let A_ϕ be a slant weighted Toeplitz operator. Then its matrix $\langle \lambda_{ij} \rangle$ is given by

$$\begin{aligned} \lambda_{ij} &= \langle A_\phi e_j(z), e_i(z) \rangle \\ &= a_{2i-j} \frac{\beta_i}{\beta_j} \ . \end{aligned}$$

Also,

$$\begin{aligned} \lambda_{i+1,j+2} &= a_{2i-j} \frac{\beta_{i+1}}{\beta_{j+2}} \\ &= \frac{w_i}{w_j w_{j+1}} \lambda_{ij} \end{aligned}$$

where $w_n = \dfrac{\beta_{n+1}}{\beta_n}$ for every $n \in Z$. Thus the matrix of A_ϕ is a slant weighted Toeplitz matrix.

Conversely, let the matrix $\langle \lambda_{ij} \rangle$ of an operator A on $L^2(\beta)$ be a slant weighted Toeplitz matrix. Then, for all $i, j \in Z$,

$$\begin{aligned} \langle A e_j(z), e_i(z) \rangle = \lambda_{ij} &= \frac{w_j w_{j+1}}{w_i} \lambda_{i+1,j+2} \\ &= \frac{w_j w_{j+1}}{w_i} \langle A e_{j+2}(z), e_{i+1}(z) \rangle. \end{aligned}$$

Now,

$$\begin{aligned} \langle M_z A e_j, e_i \rangle &= \langle A e_j, M_z^* e_i \rangle \\ &= \langle A e_j, w_{i-1} e_{i-1} \rangle \\ &= w_{i-1} \langle A e_j, e_{i-1} \rangle \\ &= w_{i-1} \frac{w_j w_{j+1}}{w_{i-1}} \langle A e_{j+2}, e_i \rangle \\ &= \langle A M_{z^2} e_j(z), e_i(z) \rangle. \end{aligned}$$

Hence $M_zA = AM_{z^2}$.

Thus A is a slant weighted Toeplitz operator. □

Theorem 3.3. (i) *The sum of two slant weighted Toeplitz operators is a slant weighted Toeplitz operator.*

(ii) *If M_ϕ is a weighted multiplication operator and A_ψ is a slant weighted Toeplitz operator for ϕ, ψ in $L^\infty(\beta)$, then $M_\phi A_\psi$ is a slant weighted Toeplitz operator.*

(iii) *If $\phi \in L^\infty(\beta)$, then $A_{\phi(z^2)} = M_{\phi(z)}W$.*

Proof. (i) Let A_{ϕ_1} and A_{ϕ_2} be two slant weighted Toeplitz operators. Then

$$\begin{aligned}(A_{\phi_1} + A_{\phi_2}) &= (WM_{\phi_1} + WM_{\phi_2})\\ &= W(M_{\phi_1} + M_{\phi_2})\\ &= W(M_{\phi_1+\phi_2})\\ &= (A_{\phi_1+\phi_2}).\end{aligned}$$

(ii) Consider

$$\begin{aligned}M_zM_\phi A_\psi &= M_\phi M_z A_\psi\\ &= M_\phi A_\psi M_{z^2}\end{aligned}$$

Hence $M_\phi A_\psi$ is a slant weighted Toeplitz operator

(iii) We know that $M_zW = WM_{z^2}$. We prove by induction on n that

$$M_{z^n}W = WM_{z^{2n}}$$

suppose the result is true for $n = m$.

Then we have $M_{z^m}W = WM_{z^{2m}}$.

Now

$$\begin{aligned}M_{z^{m+1}W} &= M_zM_{z^m}W\\ &= M_zWM_{z^{2m}}\\ &= WM_{z^2}M_{z^{2m}}\\ &= WM_{z^{2(m+1)}}\ .\end{aligned}$$

Thus $M_{z^n}W = WM_{z^{2n}}$ for all positive n.

For $n = 0$, the result is clear.

For $n = -1$, and odd j, $M_{z^n}We_j(z) = 0 = WM_{z^{2n}}e_j(z)$.

For $n = -1$, and even $j = 2k$ we get

$$\begin{aligned}M_{z^n}We_j(z) &= M_{z^{-1}}We_{2k}(z)\\ &= M_{z^{-1}}\frac{\beta_k}{\beta_{2k}}e_k(z)\\ &= \frac{\beta_k}{\beta_{2k}}\frac{\beta_{k-1}}{\beta_k}e_{k-1}(z)\\ &= \frac{\beta_{k-1}}{\beta_{2k}}e_{k-1}(z)\,. \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} WM_{z^{2n}}e_{j(z)} &= WM_{z^{-2}}e_{2k}(z) \\ &= \frac{\beta_{2(k-1)}}{\beta_{2k}} We_{2(k-1)}(z) \\ &= \frac{\beta_{k-1}}{\beta_{2k}} e_{k-1}(z)\,. \end{aligned} \tag{3.2}$$

From equations (3.1) and (3.2) we get that $M_{z^n}W = WM_{z^{2n}}$ for $n = -1$.

Further, using induction we can extend this result to all negative integers n.

Consequently we get that $M_{z^n}W = WM_{z^{2n}}$ for all $n \in \mathbb{Z}$. This implies further that

$$M_{\phi(z)}W = WM_{\phi(z^2)} \quad \text{for all} \;\; \phi = \sum_{n=-\infty}^{\infty} a_n z^n\,.$$

Finally, we get that

$$\begin{aligned} A_{\phi(z^2)} &= WM_{\phi(z^2)} \\ &= M_{\phi(z)}W\,. \qquad \square \end{aligned}$$

Theorem 3.4. *WA_ϕ is a slant weighted Toeplitz operator if and only if $\phi = 0$.*

Proof.

$$\begin{aligned} & \langle WA_\phi e_j(z), e_i(z)\rangle = \frac{w_j w_{j+1}}{w_i} \langle WA_\phi e_{j+2}(z), e_{i+1}(z)\rangle \\ \Rightarrow\quad & \langle A_\phi e_j(z), W^* e_i(z)\rangle = \frac{w_j w_{j+1}}{w_i} \langle A_\phi e_{j+2}(z), W^* e_{i+1}(z)\rangle \\ \Rightarrow\quad & \left\langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{2n-j}\beta_n e_n(z), \frac{\beta_i}{\beta_{2i}} e_{2i}(z) \right\rangle \\ & \qquad = \frac{w_j w_{j+1}}{w_i} \left\langle \frac{1}{\beta_{j+2}} \sum a_{2n-j-2}\beta_n e_n(z), \frac{\beta_{i+1}}{\beta_{2i+2}} e_{2i+2}(z) \right\rangle \\ \Rightarrow\quad & \frac{\beta_i}{\beta_{2i}} a_{4i-j}\beta_{2i} = \frac{\beta_i}{\beta_{i+1}} \frac{\beta_{i+1}}{\beta_{2i+2}} \{a_{4i-j+2}\beta_{2i+2}\} \\ \Rightarrow\quad & a_{4i-j} = a_{4i-j+2} \quad \text{for all} \;\; i, j \in Z. \end{aligned}$$

Putting $i = 0$ we get,

$$a_{-j} = a_{-j+2}.$$

Hence $a_0 = a_{2n}$ and $a_1 = a_{2n-1}$ for all $n \in Z$. Now, since $\sum |a_n|^2 \beta_n^2 < \infty$, hence $\lim_{n\to\infty} a_n\beta_n = 0$ But β_n's are positive.

Hence $\lim_{n\to\infty} a_n = 0$.

$$\begin{aligned} \Rightarrow\quad & a_0 = a_1 = 0 \\ \Rightarrow\quad & a_n = 0 \quad \text{for all} \;\; n \in Z. \end{aligned}$$

Therefore $\phi = 0$. The converse is obvious. $\square$

Theorem 3.5. *$A_\phi A_\psi$ is not a slant weighted Toeplitz operator in general.*

Proof. Let $\langle\lambda_{ij}\rangle$ and $\langle\delta_{ij}\rangle$ be the matrices of A_ϕ and A_ψ respectively and let $\langle\gamma_{ij}\rangle$ be the matrix of the product $A_\phi A_\psi$. Further let $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\psi = \sum_{n=-\infty}^{\infty} b_n z^n$. Now, [4]

$$\begin{aligned}\gamma_{ij} &= \sum_{k=-\infty}^{\infty} \lambda_{ik}\delta_{kj} \\ &= \sum_{k=-\infty}^{\infty} a_{2i-k}\frac{\beta_i}{\beta_k} b_{2k-j}\frac{\beta_k}{\beta_j} \\ &= \frac{\beta_i}{\beta_j}\sum_{k=-\infty}^{\infty} a_{2i-k}b_{2k-j}\ .\end{aligned}$$

Similarly,

$$\begin{aligned}\gamma_{i+1,j+2} &= \frac{\beta_{i+1}}{\beta_{j+2}}\sum_{k=-\infty}^{\infty} a_{2i+2-k}b_{2k-j-2} \quad \text{take } t = k-2 \\ &= \frac{\beta_{i+1}}{\beta_{j+2}}\sum_{t=-\infty}^{\infty} a_{2i-t}b_{2t-j+2}\end{aligned}$$

Hence,

$$\gamma_{i+1,j+2} \neq \frac{w_i}{w_j w_{j+1}}\gamma_{i,j}\ .$$

Hence by matrix characterization we conclude that the product is not a slant weighted Toeplitz operator. □

Next we obtain a condition for the commutativity of the product of two slant weighted Toeplitz operators.

Theorem 3.6. *$A_\phi A_\psi = A_\psi A_\phi$ if and only if $\phi(z^2)\psi(z) = \psi(z^2)\phi(z)$.*

Proof. Let A_ϕ and A_ψ be two slant weighted Toeplitz operators. Then

$$\begin{aligned}A_{\phi(z)}A_{\psi(z)} &= WM_{\phi(z)}WM_{\psi(z)} \\ &= WWM_{\phi(z^2)}M_{\psi(z)} \\ &= WWM_{\phi(z^2)\psi(z)} \\ &= WA_{\phi(z^2)\psi(z)}\ .\end{aligned}$$

On the other hand

$$\begin{aligned}A_{\psi(z)}A_{\phi(z)} &= WM_{\psi(z)}WM_{\phi(z)} \\ &= WWM_{\psi(z^2)}M_{\phi(z)} \\ &= WWM_{\psi(z^2)\phi(z)} \\ &= WA_{\psi(z^2)\phi(z)}\ .\end{aligned}$$

Hence $A_\phi A_\psi = A_\phi A_\phi$ if and only if $\phi(z^2)\psi(z) = \psi(z^2)\phi(z)$. □

Now we give a necessary and sufficient condition for $A_\phi A_\psi$ to be a slant weighted Toeplitz operator.

Theorem 3.7. *$A_\phi A_\psi$ is a slant weighted Toeplitz operator if and only if $A_\phi A_\psi = 0$.*

Proof.

$$\begin{aligned} A_\phi A_\psi &= WM_\phi WM_\psi \\ &= WWM_{\phi(z^2)\psi(z)} \\ &= WA_{\phi(z^2)\psi(z)} \end{aligned}$$

Therefore by Theorem 3.4 we get that $WA_{\phi(z^2)\psi(z)}$ is a slant weighted Toeplitz operator if and only if $\phi(z^2)\cdot\psi(z) = 0$ if and only if

$$A_\phi A_\psi = 0. \qquad \square$$

4. The adjoint of slant weighted Toeplitz operator

Given the slant weighted Toeplitz operator A_ϕ, we now prove some results for A_ϕ^*.

Theorem 4.1. *A_ϕ^* is not a slant weighted Toeplitz operator in general.*

Proof. The matrix of A_ϕ^* is given by

$$\left[\begin{array}{c|cccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \vdots & \bar{a}_0\dfrac{\beta_0}{\beta_0} & \bar{a}_2\dfrac{\beta_1}{\beta_0} & \bar{a}_4\dfrac{\beta_2}{\beta_0} & \bar{a}_6\dfrac{\beta_3}{\beta_0} & \cdots & & & \\ \vdots & \bar{a}_{-1}\dfrac{\beta_0}{\beta_1} & \bar{a}_1\dfrac{\beta_1}{\beta_1} & \bar{a}_3\dfrac{\beta_2}{\beta_1} & \bar{a}_5\dfrac{\beta_3}{\beta_1} & \cdots & & & \\ \vdots & \bar{a}_{-2}\dfrac{\beta_0}{\beta_2} & \bar{a}_0\dfrac{\beta_1}{\beta_2} & \bar{a}_2\dfrac{\beta_2}{\beta_2} & \bar{a}_4\dfrac{\beta_3}{\beta_2} & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right]$$

Since the above matrix does not satisfy the characterization given in Theorem 3.2, A_ϕ^* is not a slant weighted Toeplitz operator. $\square$

Theorem 4.2. *A_ϕ^* is a slant weighted Toeplitz operator if and only if $\phi = 0$.*

Proof. If A_ϕ^* is a slant weighted Toeplitz operator, then for all $i, j \in Z$, we have,

$$\langle A_\phi^* e_j(z), e_i(z)\rangle = \langle A_\phi^* e_{j+2}(z), e_{i+1}(z)\rangle \frac{w_j}{w_i} w_{j+1}$$

Hence

$$\left\langle \beta_j \sum_{k=-\infty}^{\infty} \bar{a}_{2j-k}\frac{e_k(z)}{\beta_k}, e_i(z)\right\rangle = \left\langle \beta_{j+2} \sum_{k=-\infty}^{\infty} \bar{a}_{2(j+2)-k}\frac{e_k(z)}{\beta_k}, e_{i+1}(z)\right\rangle w_j \cdot \frac{w_{j+1}}{w_i}.$$

Therefore $\bar{a}_{2j-i} = \bar{a}_{2j+3-i}\left(\frac{w_j w_{j+1}}{w_i}\right)^2$ for all $i, j \in Z$. Putting $j = 0$, we get $\bar{a}_{-i} = \frac{w_1^2}{w_i^2}\bar{a}_{-i+3}$ for all $i \in Z$.

But $\lim_{n\to\infty} \bar{a}_n = 0$ as shown in Theorem 3.4. Hence $a_n = 0$ for all $n \in Z$. So $\phi = 0$. □

Corollary 4.3. *There is no non-zero self adjoint slant weighted Toeplitz operator.*

5. Compactness

Theorem 5.1. *A_ϕ is compact if and only if $\phi = 0$.*

Proof. Let A_ϕ be compact.
$\Leftrightarrow$ WM_ϕ is compact
$\Leftrightarrow$ M_ϕ is compact
$\Leftrightarrow$ $\phi = 0$ □

Acknowledgement

The authors would like to thank the referee for the valuable suggestions that improved the original version of the paper.

References

1. A.L. Shields, *Weighted Shift Operators and Analytic Function Theory*, Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc. Providence, R.I., 1974.
2. M.C. Ho, *Properties of slant Toeplitz operators*, Univ. Math. J. **45** (1996), no. 3, 843–862.
3. M.B. Abrahamse, *Toeplitz operators in multiply connected domain*, Bull. Amer. Math. Soc. **77** (1971), 441–454.
4. A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1964), 89–102.
5. S.C. Arora and Ritu Kathuria, *Slant weighted Toeplitz operator*, Int. J. Pure Appl. Math. **62** (2010), no. 4, 433–442.
6. V. Lauric, *On a weighted Toeplitz operator and its commutant*, Int. J. Math. Math. Sci. **6** (2005), 823–835.
7. O. Toeplitz, *Zur theorie der quadratishen und bilinearan Formen von unendlichvielen*, Math. Ann. **70** (1911), 351–376.
8. L. Villemoes, *Wavelet analysis of refinement equations*, SIAM J. Math. Anal. **25** (1994), 1433–1460.
9. T. Goodman, C. Micchelli and J. Ward *Spectral Radius Formula for Subdivision Operators*, Recent Advances in Wavelet Analysis, ed. L. Schumaker and G. Webb, Academic PRess (1994), 335–360.
10. A. Devinatz, *Toeplitz operator on H^2 spaces*, Trans. Amer. Math. Soc. **112** (1964), 304–317.

[1] Professor (Retired), Department of Mathematics, University of Delhi, Delhi 110 007, India.
E-mail address: scaroradu@gmail.com

[2] Department of Mathematics, Motilal Nehru College, University of Delhi, Benito Juarez Marg,New Delhi 110 021, India.
E-mail address: ritu.kathuria@yahoo.ac.in