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A GENERAL ITERATIVE ALGORITHM FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Let E be a real q-uniformly smooth Banach space whose duality map is weakly sequentially continuous. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $A : E \to E$ be an η-strongly accretive map which is also κ-Lipschitzian. Let $f : E \to E$ be a contraction map with coefficient $0 < \alpha < 1$. Let a sequence $\{y_n\}$ be defined iteratively by $y_0 \in E,\ y_{n+1} = \alpha_n \gamma f(y_n) + (I - \alpha_n \mu A)Ty_n, n \geq 0$, where $\{\alpha_n\},\ \gamma$ and μ satisfy some appropriate conditions. Then, we prove that $\{y_n\}$ converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle(\gamma f - \mu A)x^*, j(y - x^*)\rangle \leq 0,\ \forall\ y \in F(T)$. Convergence of the correspondent implicit scheme is also proved without the assumption that E has weakly sequentially continuous duality map. Our results are applicable in l_p spaces, $1 < p < \infty$.

1. Introduction

Let E be a real Banach space and E^* be the dual space of E. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a gauge function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let φ be a gauge function, a generalized duality mapping with respect to φ, $J_\varphi : E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\},$$

where $\langle ., .\rangle$ denotes the duality pairing between element of E and that of E^*. If $\varphi(t) = t$, then J_φ is simply called the normalized duality mapping and is denoted by J. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi(x)$.
If however $\varphi(t) = t^{q-1}$, for some $q > 1$, then J_φ is still called the generalized

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duality mapping and is denoted by J_q (see, for example [9, 10]).

The space E is said to have weakly (sequentially) continuous duality map if there exists a gauge function φ such that J_φ is singled valued and (sequentially) continous from E with weak topology to E^* with weak* topology. It is well known that all l_p spaces, $(1 < p < \infty)$ have weakly sequentially continuous duality mappings. It is well known (see, for example, [16]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single valued.

A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if $\forall x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \tag{1.1}$$

where $D(A)$ denotes the domain of A. A is called $\eta-$*strongly accretive* if $\forall x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ and $\eta \in (0,1)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \eta\|x-y\|^q. \tag{1.2}$$

A is $\kappa-$*Lipschitzian* if for some $\kappa > 0$, $\|A(x) - A(y)\| \leq \kappa\|x-y\|\ \forall\ x, y \in D(A)$. A mapping $T : E \to E$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x-y||,\ \forall x, y \in E.$$

A point $x \in E$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in E : Tx = x\}$. In Hilbert spaces, accretive operators are called *monotone* where inequalities (1.1) and (1.2) hold with j_q replaced by the identity map on H.

Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n,\ n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Xu [12] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.3) strongly converges to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (I-f)x^*, x - x^*\rangle \geq 0,\ \text{for } x \in F(T).$$

In [13], it is proved, under some conditions on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrary,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n,\ n \geq 0, \tag{1.4}$$

converges strongly to $x^* \in F(T)$ which is the unique solution of the minimization problem

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where A is a strongly positive bounded linear operator. That is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}||x||^2,\ \forall x \in H.$$

Combining the iterative method (1.3) and (1.4), Marino and Xu [7] consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,\ n \geq 0. \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $x^* \in F(T)$ which solves the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0,\ x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

Let K be a nonempty, closed and convex subset of a real Hilbert space H. The variational inequality problem: Find a point $x^* \in K$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0,\ \forall y \in K$$

is equivalent to the following fixed point equation

$$x^* = P_K(x^* - \delta Ax^*), \tag{1.6}$$

where $\delta > 0$ is an arbitrary fixed constant, A is a nonlinear operator on K and P_K is the *nearest point projection map* from H onto K, i.e., $P_K x = y$ where $\|x - y\| = \inf_{u \in K} \|x - u\|$ for $x \in H$. Consequently, under appropriate conditions on A and δ, fixed point methods can be used to find or approximate a solution of the variational inequality. Considerable efforts have been devoted to this problem (see, for example, [14, 17] and the references contained therein). For instance, if A is strongly monotone and Lipschitz then, a mapping $B : H \to H$ defined by $Bx = P_K(x - \delta Ax)$, $x \in H$ with $\delta > 0$ sufficiently small is a strict contraction. Hence, the *Picard iteration,* $x_0 \in H$, $x_{n+1} = Bx_n,\ n \geq 0$ of the classical Banach contraction mapping principle converges to the unique solution of the variational inequality. It has been observed that the projection operator P_K in the fixed point formulation (1.6) may make the computation of the iterates difficult due to possible complexity of the convex set K. In order to reduce the possible difficulty with the use of P_K, Yamada [17] introduced the following hybrid descent method for solving the variational inequality:

$$x_{n+1} = Tx_n - \lambda_n \mu A(Tx_n),\ n \geq 0, \tag{1.7}$$

where T is a nonexpansive mapping, A is an $\eta-$strongly monotone and $\kappa-$Lipschitz operator with $\eta > 0,\ \kappa > 0,\ 0 < \mu < \frac{2\eta}{\kappa^2}$. He proved that if $\{\lambda_n\}$ satisfies appropriate conditions then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality

$$\langle Ax^*, x - x^* \rangle \geq 0,\ x \in F(T).$$

Very recently, Tian [6] combined the Yamada's method (1.7) with the iterative method (1.5) and introduced the following general iterative method in Hilbert spaces:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n,\ n \geq 0. \tag{1.8}$$

Then, he proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality

$$\langle (\gamma f - \mu A)x^*, x - x^* \rangle \leq 0,\ x \in F(T).$$

We remark immediately here that the results of Tian [6] improved the results of Yamada [17], Moudafi [5], Xu [12] and Marino and Xu [13] in Hilbert spaces.

In this paper, motivated and inspired by the above research results, our purpose is to extend the result of Tian [6] to q-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Thus, our results are applicable in l_p spaces, $1 < p < \infty$. Furthermore, our results extend the results of Moudafi [5], Xu [12] and Marino and Xu [13] to Banach spaces much more general than Hilbert.

2. Preliminaries

Let E be a real Banach space. Let K be a nonempty closed convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := sup\{\|x-y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := sup\{\|x - y\| : y \in K\}$ and let $r(K) := inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (see, for example, [1]) is defined by $N(E) := inf\Big\{\frac{d(K)}{r(K)} : K$ is a closed convex and bounded subset of E with $d(K) > 0\Big\}$. A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, for example, [2, 4]).

Let μ be a linear continuous functional on ℓ^∞ and let $a = (a_1, a_2, \dots) \in \ell^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional μ such that $||\mu|| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, \dots) \in \ell^\infty$ is called a *Banach limit*. It is known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \dots) \in \ell^\infty$ (see, for example, [2, 3]).

Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| \ + \|x-y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}.$$

The space E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$. For some positive constant q, E is called $q-$*uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q,\ t > 0$. It is known that

$$L_p(or\ l_p) \text{ spaces are } \begin{cases} 2- \text{ uniformly smooth, if, } 2 \leq p < \infty \\ p- \text{ uniformly smooth, if, } 1 < p \leq 2. \end{cases}$$

It is well known that if E is smooth then the duality mapping is singled-valued, and if E is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

We shall make use of the following well known results.

Lemma 2.1. *Let E be a real normed space. Then*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle,$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$.

Lemma 2.2. (Xu, [15]) *Let E be a real q-uniformly smooth Banach space for some $q > 1$, then there exists some positive constant d_q such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q \ \forall\ x, y \in E \text{ and } j_q(x) \in J_q(x).$$

Lemma 2.3. (Xu, [11]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup\ \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.4. (Lim and Xu, [4]) *Suppose E is a Banach space with uniform normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is uniformly $k-$Lipschitzian mapping with $k < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset C of K with the following property (P) :*

$$(P)\ x \in C \text{ implies } \omega_w(x) \subset C,$$

where $\omega_w(x)$ is the $\omega-$limit set of T at x, i.e., the set

$$\{y \in E : y = \text{weak} - \lim_j T^{n_j}x \text{ for some } n_j \to \infty\}.$$

Then, T has a fixed point in C.

Lemma 2.5. (Jung, [8]) *Let C be a nonempty, closed and convex subset of a reflexive Banach space E which satisfies Opial's condition and suppose $T : C \to E$ is nonexpansive. Then $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x,\ x_n - Tx_n \to 0$ implies that $x = Tx$.*

Lemma 2.6. *Let E be a real Banach space, $f : E \to E$ a contraction with coefficient $0 < \alpha < 1$, and $A : E \to E$ a κ-Lipschitzian and η-strongly accretive operator with $\kappa > 0,\ \eta \in (0,1)$. Then for $\gamma \in (0, \frac{\mu\eta}{\alpha})$,*

$$\langle(\mu A - \gamma f)x - (\mu A - \gamma f)y, j(x-y)\rangle \geq (\mu\eta - \gamma\alpha)||x-y||^2,\ \forall x, y \in E.$$

That is, $\mu A - \gamma f$ is strongly accretive with coefficient $\mu\eta - \gamma\alpha$.

3. Main Results

We begin with the following lemma.

Lemma 3.1. *Let E be a q-uniformly smooth real Banach space with constant $d_q, q > 1$. Let $f : E \to E$ be a contraction mapping with constant of contraction $\alpha \in (0,1)$. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $A : E \to E$ be an η-strongly accretive mapping which is also κ-Lipschitzian. Let $\mu \in \Big(0, \min\Big\{1, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{q-1}}\Big\}\Big)$ and $\tau := \mu\big(\eta - \frac{\mu^{q-1}d_q\kappa^q}{q}\big)$. For each $t \in (0,1)$ and $\gamma \in (0, \frac{\tau}{\alpha})$ define a map $T_t : E \to E$ by*

$$T_t x = t\gamma f(x) + (I - t\mu A)Tx,\ x \in E.$$

Then, T_t is a strict contraction. Furthermore

$$||T_t x - T_t y|| \leq [1 - t(\tau - \gamma\alpha)]||x - y||.$$

Proof. Without loss of generality, assume $\eta < \frac{1}{q}$. Then, as $\mu < (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}$, we have $0 < q\eta - \mu^{q-1}d_q\kappa^q$. Furthermore, from $\eta < \frac{1}{q}$ we have $q\eta - \mu^{q-1}d_q\kappa^q < 1$ so that $0 < q\eta - \mu^{q-1}d_q\kappa^q < 1$. Also as $\mu < 1$ and $t \in (0,1)$ we obtained that $0 < t\mu(q\eta - \mu^{q-1}d_q\kappa^q) < 1$.

For each $t \in (0,1)$, define $S_t x = (I - t\mu A)Tx,\ x \in E$, then for $x, y \in K$

$$\begin{aligned}
||S_t x - S_t y||^q &= ||(I - t\mu A)Tx - (I - \mu A)Ty||^q \\
&= ||(Tx - Ty) - t\mu(A(Tx) - A(Ty))||^q \\
&\leq ||Tx - Ty||^q - qt\mu\langle A(Tx) - A(Ty), j_q(Tx - Ty)\rangle \\
&\quad + t^q\mu^q d_q||A(Tx) - A(Ty)||^q \\
&\leq ||Tx - Ty||^q - qt\mu\eta||Tx - Ty||^q \\
&\quad + t^q\mu^q\kappa^q d_q||Tx - Ty||^q \\
&\leq [1 - t\mu(q\eta - t^{q-1}\mu^{q-1}\kappa^q d_q)]||x - y||^q \\
&\leq \Big[1 - qt\mu\big(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q}\big)\Big]||x - y||^q \\
&\leq \Big[1 - t\mu\big(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q}\big)\Big]^q||x - y||^q \\
&= (1 - t\tau)^q||x - y||^q. \qquad (3.1)
\end{aligned}$$

It then follows from (3.1) that,

$$||S_t x - S_t y|| \leq (1 - t\tau)||x - y||.$$

Using the fact that $T_t x = t\gamma f(x) + S_t x,\ x \in E$, we obtain for all $x, y \in E$ that

$$\begin{aligned}
||T_t x - T_t y|| &= ||t\gamma(f(x) - f(y)) + (S_t x - S_t y)|| \\
&\leq t\gamma||f(x) - f(y)|| + ||S_t x - S_t y|| \\
&\leq t\gamma\alpha||x - y|| + (1 - t\tau)||x - y|| \\
&= [1 - t(\tau - \gamma\alpha)]||x - y||.
\end{aligned}$$

Therefore

$$||T_tx - T_ty|| \leq [1 - t(\tau - \gamma\alpha)]||x - y||,$$

which implies that T_t is a strict contraction. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_t of T_t in E. That is,

$$x_t = t\gamma f(x_t) + (I - t\mu A)Tx_t. \tag{3.2}$$

□

Proposition 3.2. *Let $\{x_t\}$ be defined by (3.2), then*

(i) *$\{x_t\}$ is bounded for $t \in (0, \frac{1}{\tau})$.*
(ii) *$\lim\limits_{t\to 0}||x_t - Tx_t|| = 0$.*

Proof. (i) For any $p \in F(T)$, we have

$$\begin{aligned} ||x_t - p|| &= ||(I - t\mu A)Tx_t - (I - t\mu A)p + t(\gamma f(x_t) - \mu A(p))|| \\ &\leq (1 - t\tau)||x_t - p|| + t\gamma\alpha||x_t - p|| + t||\gamma f(p) - \mu A(p)|| \\ &= [1 - t(\tau - \gamma\alpha)]||x_t - p|| + t||\gamma f(p) - \mu A(p)||. \end{aligned}$$

Therefore,

$$||x_t - p|| \leq \frac{1}{\tau - \gamma\alpha}||\gamma f(p) - \mu A(p)||.$$

Hence, $\{x_t\}$ is bounded. Furthermore $\{f(x_t)\}$ and $\{A(Tx_t)\}$ are also bounded.
(ii) From (3.2), we have

$$||x_t - Tx_t|| = t||\gamma f(x_t) - \mu A(Tx_t)|| \to 0 \text{ as } t \to 0. \tag{3.3}$$

□

Next, we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\}$ be a sequence in (0,1) such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.3), we obtain that

$$||x_n - Tx_n|| \to 0 \text{ as } n \to \infty.$$

Theorem 3.3. *Assume that $\{x_t\}$ is defined by (3.2), then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality problem:*

$$\langle(\mu A - \gamma f)\tilde{x}, j(\tilde{x} - z)\rangle \leq 0, \;\; z \in F(T). \tag{3.4}$$

Proof. By Lemma 2.6, $(\mu A - \gamma f)$ is strongly accretive, so the variational inequality (3.4) has a unique solution in $F(T)$. Below we use $x^* \in F(T)$ to denote the unique solution of (3.4).

We next prove that $x_t \to x^*$ $(t \to 0)$. Now, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(x) := \mu_n||x_n - x||^2, \;\; \forall x \in E,$$

where μ_n is a Banach limit for each n. Then, $\phi(x) \to \infty$ as $||x|| \to \infty$, ϕ is continuous and convex, so as E is reflexive, it follows that there exits $y^* \in E$ such that $\phi(y^*) = \min\limits_{u\in E}\phi(u)$. Hence, the set

$$K^* := \{x \in E : \phi(x) = \min_{u\in E}\phi(u)\} \neq \emptyset.$$

We now show that T has a fixed point in K^*. We shall make use of Lemma 2.4. If x is in K^* and $y := \omega - \lim_j T^{m_j}x$, then from the weak lower semi-continuity of ϕ (since ϕ is lower semi-continuous and convex) and $\lim\limits_{n\to\infty}||x_n - Tx_n|| = 0$, we have (since $\lim\limits_{n\to\infty}||x_n - Tx_n|| = 0$ implies $\lim\limits_{n\to\infty}||x_n - T^m x_n|| = 0,\ m \geq 1$, this is easily proved by induction),

$$\begin{aligned}
\phi(y) &\leq \liminf_{j\to\infty}\phi\Big(T^{m_j}x\Big) \leq \limsup_{m\to\infty}\phi\Big(T^m x\Big)\\
&= \limsup_{m\to\infty}\Big(\mu_n||x_n - T^m x||^2\Big)\\
&= \limsup_{m\to\infty}\Big(\mu_n||x_n - T^m x_n + T^m x_n - T^m x||^2\Big)\\
&\leq \limsup_{m\to\infty}\Big(\mu_n||T^m x_n - T^m x||^2\Big) \leq \limsup_{m\to\infty}\Big(\mu_n||x_n - x||^2\Big) = \phi(x)\\
&= \min_{u\in E}\phi(u).
\end{aligned}$$

So, $y \in K^*$. By Lemma 2.4, T has a fixed point in K^* and so $K^* \cap F(T) \neq \emptyset$. Now let $y \in K^* \cap F(T)$. Then, it follows that $\phi(y) \leq \phi(y + t(\gamma f - \mu A)y)$ and using Lemma 2.1, we obtain that

$$||x_n - y - t(\gamma f - \mu A)y||^2 \leq ||x_n - y||^2 - 2t\langle(\gamma f - \mu A)y, j(x_n - y - t(\gamma f - \mu A)y)\rangle.$$

This implies that $\mu_n\langle(\gamma f - \mu A)y, j(x_n - y - t(\gamma f - \mu A)y)\rangle \leq 0$. Moreover, $\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle = \mu_n\langle(\gamma f - \mu A)y, j(x_n - y) - j(x_n - y + t(\mu A - \gamma f)y)\rangle + \mu_n\langle(\gamma f - \mu A)y, j(x_n - y + t(\mu A - \gamma f)y)\rangle \leq \mu_n\langle(\gamma f - \mu A)y, j(x_n - y) - j(x_n - y + t(\mu A - \gamma f)y)\rangle$.

Since j is norm-to-norm uniformly continuous on bounded subsets of E, we obtain as $t \to 0$ that

$$\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle \leq 0.$$

Now, using (3.2), we have

$$\begin{aligned}
||x_n - y||^2 &= t_n\langle\gamma f(x_n) - \mu Ay, j(x_n - y)\rangle + \langle(I - t_n\mu A)(Tx_n - y), j(x_n - y)\rangle\\
&= t_n\langle\gamma f(x_n) - \mu Ay, j(x_n - y)\rangle + \langle(I - \mu A)Tx_n - (I - \mu A)y, j(x_n - y)\rangle\\
&\leq [1 - t_n(\tau - \gamma\alpha)]||x_n - y||^2 + t_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle.
\end{aligned}$$

So,

$$||x_n - y||^2 \leq \frac{1}{\tau - \gamma\alpha}\langle(\gamma f - \mu A)y, j(x_n - y)\rangle.$$

Again, taking Banach limit, we obtain

$$\mu_n||x_n - y||^2 \leq \frac{1}{\tau - \gamma\alpha}\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle \leq 0,$$

which implies that $\mu_n||x_n - y||^2 = 0$. Hence, there exists a subsequence of $\{x_n\}_{n=1}^\infty$ which we still denoted by $\{x_n\}_{n=1}^\infty$ such that $\lim\limits_{n\to\infty} x_n = y$. We now show that y solves the variational inequality (3.4). Since

$$x_t = t\gamma f(x_t) + (I - t\mu A)Tx_t,$$

we can derive that

$$(\mu A - \gamma f)(x_t) = -\frac{1}{t}(I - T)x_t + \mu(Ax_t - ATx_t).$$

It follows that for $z \in F(T)$,

$$\begin{aligned}\langle (\mu A - \gamma f)(x_t), j(x_t - z)\rangle &= -\frac{1}{t}\langle (I-T)x_t - (I-T)z, j(x_t - z)\rangle \\ &\quad + \mu\langle (Ax_t - ATx_t), j(x_t - z)\rangle \\ &\leq \mu\langle (Ax_t - ATx_t), j(x_t - z)\rangle. \end{aligned} \tag{3.5}$$

Since T is nonexpansive, then, $I - T$ is accretive, which implies, $\langle (I - T)x_t - (I - T)z, j(x_t - z)\rangle \geq 0$. Now replacing t in (3.5) with t_n and letting $n \to \infty$, noticing that $(Ax_{t_n} - ATx_{t_n}) \to (Ay - Ay)$ we obtain

$$\langle (\mu A - \gamma f)y, j(y - z)\rangle \leq 0,$$

since $z \in F(T)$ is arbitrary, we get $y = x^*$.
Assume now that there exists another subsequence $\{x_m\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{m\to\infty} x_m = u^*$. Then, since $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = 0$, we have that $u^* \in F(T)$. Repeating the argument above with y replaced by u^* we will get that u^* solves the variational inequality (3.4), and so by uniqueness, we obtain $x^* = y = u^*$. This complete the proof. □

Theorem 3.4. *Let E be a real q-uniformly smooth Banach space with whose duality map is weakly sequentially continuous. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $A : E \to E$ be an η-strongly accretive map which is also κ-Lipschitzian. Let $f : E \to E$ be a contraction map with coefficient $0 < \alpha < 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in [0,1] satisfying:*
$(C1)$ $\lim \alpha_n = 0$,
$(C2)$ $\sum \alpha_n = \infty$ *and*
$(C3)$ $\sum |\alpha_{n+1} - \alpha_n| < \infty$.
Let μ, γ and τ be as in Lemma 3.1. Define a sequence $\{y_n\}_{n=1}^{\infty}$ iteratively in E by $y_0 \in E$,

$$y_{n+1} = \alpha_n \gamma f(y_n) + (I - \alpha_n \mu A)Ty_n. \tag{3.6}$$

Then, $\{y_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which is also a solution to the following variational inequality

$$\langle (\gamma f - \mu A)x^*, j(y - x^*)\rangle \leq 0, \ \forall \ y \in F(T). \tag{3.7}$$

Proof. Since the mapping $T : E \to E$ is nonexpansive, then from Theorem 3.3, the variational inequality (3.7) has a unique solution x^* in $F(T)$. Furthermore, the sequence $\{y_n\}$ satisfies

$$||y_n - x^*|| \leq \max\left\{||y_0 - x^*||, \frac{||\gamma f(x^*) - \mu Ax^*||}{\tau - \gamma\alpha}\right\}, \ \forall n \geq 0.$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$, from the recursion formula (3.6), we have

$$\begin{aligned}
||y_{k+1} - x^*|| &= ||\alpha_k \gamma f(y_k) + (I - \alpha_k \mu A)Ty_k - x^*|| \\
&= ||\alpha_k(\gamma f(y_k) - \mu A x^*) + (I - \alpha_k \mu A)Ty_k - (I - \alpha_k \mu A)x^*|| \\
&\leq [1 - \alpha_k(\tau - \gamma\alpha)]||y_k - x^*|| + \alpha_k(\tau - \gamma\alpha)\frac{||\gamma f(x^*) - \mu A x^*||}{\tau - \gamma\alpha} \\
&\leq \max\left\{||y_k - x^*||, \frac{||\gamma f(x^*) - \mu A x^*||}{\tau - \gamma\alpha}\right\}
\end{aligned}$$

and the claim follows by induction. Thus, the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded and so are $\{f(y_n)\}_{n=1}^{\infty}$ and $\{Ty_n\}_{n=1}^{\infty}$. Also from (3.6), we have

$$\begin{aligned}
||y_{n+1} - y_n|| &= ||\alpha_n \gamma(f(y_n) - f(y_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(y_{n-1}) \\
&\quad +(I - \mu\alpha_n A)Ty_n - (I - \mu\alpha_n A)Ty_{n-1} + \mu(\alpha_n - \alpha_{n-1})ATy_{n-1}|| \\
&\leq (1 - \alpha_n(\tau - \gamma\alpha))||y_n - y_{n-1}|| + M|\alpha_n - \alpha_{n-1}|,
\end{aligned}$$

for some $M > 0$. By Lemma 2.3, we have $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$. Furthermore, we obtain

$$\begin{aligned}
||y_n - Ty_n|| &\leq ||y_n - y_{n+1}|| + ||y_{n+1} - Ty_n|| \\
&= ||y_n - y_{n+1}|| + \alpha_n||\gamma f(y_n) - \mu ATy_n|| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.8}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\mathop{limsup}_{n\to\infty}\langle(\gamma f - \mu A)x^*, j(y_n - x^*)\rangle = \mathop{lim}_{j\to\infty}\langle(\gamma f - \mu A)x^*, j(y_{n_j} - x^*)\rangle.$$

Assume also $y_{n_j} \rightharpoonup z$ as $j \to \infty$, for some $z \in E$. Then, using this, (3.8) and the demiclosedness of $(I-T)$ at zero, we have $z \in F(T)$. Since j is weakly sequentially continuous, we have

$$\begin{aligned}
\mathop{limsup}_{n\to\infty}\langle(\gamma f - \mu A)x^*, j(y_n - x^*)\rangle &= \mathop{lim}_{j\to\infty}\langle(\gamma f - \mu A)x^*, j(y_{n_j} - x^*)\rangle \\
&= \langle(\gamma f - \mu A)x^*, j(z - x^*)\rangle \leq 0.
\end{aligned}$$

Finally, we show that $y_n \to x^*$. From the recursion formula (3.6), let

$$T_n y_n := \alpha_n \gamma f(y_n) + (I - \alpha_n \mu A)Ty_n,$$

and from Lemma 3.1, we have

$$\begin{aligned}
||y_{n+1} - x^*||^2 &= ||T_n y_n - T_n x^* + T_n x^* - x^*||^2 \\
&= ||T_n y_n - T_n x^* + \alpha_n(\gamma f - \mu A)x^*||^2 \\
&\leq ||T_n y_n - T_n x^*||^2 + 2\alpha_n\langle(\gamma f - \mu A)x^*, j(y_{n+1} - x^*)\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma\alpha)]||y_n - x^*||^2 \\
&\quad +2\alpha_n(\tau - \gamma\alpha)\frac{\langle(\gamma f - \mu A)x^*, j(y_{n+1} - x^*)\rangle}{\tau - \gamma\alpha}
\end{aligned}$$

and by Lemma 2.3 we have that $y_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

We have the following corollaries.

Corollary 3.5. *Let $E = l_p$ space, $(1 < p < \infty)$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in E$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n.$$

Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality

$$\langle(\gamma f - \mu A)x^*, y - x^*\rangle \leq 0, \ \forall \ y \in F(T). \tag{3.9}$$

Corollary 3.6. (Tian [6]) *Let $E = H$ be a real Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n.$$

Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality (3.9)

Corollary 3.7. (Marino and Xu [7]) *Let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n.$$

Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality (3.9).

Corollary 3.8. (Xu [12]) *Let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \ n \geq 0.$$

Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$.

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