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SINGULAR VALUE AND ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR OPERATORS

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ABSTRACT. A singular value inequality for sums and products of Hilbert space operators is given. This inequality generalizes several recent singular value inequalities, and includes that if A, B, and X are positive operators on a complex Hilbert space H, then

$$s_j\left(A^{1/2}XB^{1/2}\right) \leq \frac{1}{2}\|X\|\ s_j\left(A+B\right),\ \ j=1,2,\cdots,$$

which is equivalent to

$$s_j\left(A^{1/2}XA^{1/2}-B^{1/2}XB^{1/2}\right) \leq \|X\|\, s_j\left(A\oplus B\right),\ \ j=1,2,\cdots.$$

Other singular value inequalities for sums and products of operators are presented. Related arithmetic–geometric mean inequalities are also discussed.

1. INTRODUCTION

Let H be a complex Hilbert space and let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. A norm $|||.|||$ on $B(H)$ is said to be unitarily invariant if it satisfies the invariance property $|||UAV||| = |||A|||$ for all A and for all unitary operators U and V. For a compact operator $A \in B(H)$, let $s_1(A) \geq s_2(A) \geq \ldots$ denote the singular values of A, i.e. the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated counting multiplicities, so it is convenience to let $||A|| = s_1(A)$ denote the usual operator norm. We say that the family $\{s_j(A),\ j = 1, 2, \ldots\}$ is weakly majorized by $\{s_j(B),\ j = 1, 2, \ldots\}$, denoted by $s(A) \prec_w s(B)$, if we have $\sum_{j=1}^{k} s_i(A) \leq \sum_{j=1}^{k} s_i(B)$ for all $k \geq 1$. Note that the Fan dominance theorem [5] asserts that

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$s(A) \prec_w s(B)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms. The direct sum of of two operators A and B in $B(H)$, denoted by $A \oplus B$, is the block-diagonal operator matrix defined on $H \oplus H$ by $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For the theory of unitarily invariant norms we refer to [5, 16]. A detailed study for singular values and majorization can be found in [2, 10].

The classical arithmetic–geometric mean inequality for positive numbers a and b could be written as

$$ab \leq \frac{a^2 + b^2}{2}. \tag{1.1}$$

This inequality is important in functional analysis, matrix theory, electrical networks, etc. Several unitarily invariant norm and singular value inequalities of the arithmetic–geometric mean type for matrices and Hilbert space operators have been obtained. These forms can be found in [4, 13, 15] and references therein. Related inequalities for sums of operators have been given in [14].

The first matrix version of the arithmetic–geometric mean inequality for singular values, which is related to inequality (1.1), was proved in [7]. It was shown that if A, B are positive $n \times n$ matrices, then

$$s_j(AB) \leq \frac{1}{2} s_j\left(A^2 + B^2\right), \;\; j = 1, 2, \cdots, n, \tag{1.2}$$

and consequently,

$$|||AB||| \leq \frac{1}{2} \left|\left|\left|A^2 + B^2\right|\right|\right|. \tag{1.3}$$

These attractive inequalities seem to have a considerable interest, they were discussed and generalized in various directions. An operator version of inequality (1.2) asserts that if $A, B \in B(H)$, then

$$s_j(AB^*) \leq \frac{1}{2} s_j(A^*A + B^*B), \;\; j = 1, 2, \cdots, \tag{1.4}$$

while the operator version of inequality (1.3) asserts that if $A, B \in B(H)$ are positive, then

$$\left|\left|\left|A^{1/2} B^{1/2}\right|\right|\right| \leq \frac{1}{2} |||A + B|||.$$

Bhatia and Davis [6] and Kittaneh [11] generalized inequality (1.3) for positive matrices A, B and any matrix X to get

$$|||AXB||| \leq \frac{1}{2} \left|\left|\left|A^2 X + XB^2\right|\right|\right|,$$

or equivalently

$$\left|\left|\left|A^{1/2} X B^{1/2}\right|\right|\right| \leq \frac{1}{2} |||AX + XB|||. \tag{1.5}$$

On the other hand Zhan has proved in [15] a new equivalent form of inequality (1.2), that is

$$s_j(A - B) \leq s_j(A \oplus B), \;\; j = 1, 2, \cdots, n. \tag{1.6}$$

New arithmetic–geometric mean inequalities for sums and products of operators have been proved. It was shown in [9] that if $A_i \in B(H)$ $(i = 1, \ldots, 4)$, then

$$\sqrt{2}s_j\left(|A_1A_2^* + A_3A_4^*|^{1/2}\right) \leq s_j\left(\left[\begin{array}{cc} A_1 & A_3 \\ A_2 & A_4 \end{array}\right]\right), \ \ j = 1, 2, \cdots . \tag{1.7}$$

Also in [8], for positive $n \times n$ matrices A and B the inequality

$$|||AB||| \leq \frac{1}{4}\left|\left|\left|(A+B)^2\right|\right|\right|$$

was shown to hold for every unitarily invariant norm. It should be mentioned here that Ando [3] obtained an extension of the arithmetic–geometric mean inequality, he proved that

$$s_j\left(AB\right) \leq s_j\left(\frac{A^p}{p} + \frac{B^q}{q}\right), \ \ j = 1, 2, \cdots , n,$$

where A, B are positive matrices and p, q are positive real numbers such that $1/p + 1/q = 1$, which implies that

$$|||AB||| \leq \left|\left|\left|\frac{A^p}{p} + \frac{B^q}{q}\right|\right|\right|.$$

Moreover, Kosaki [12] showed that the inequality

$$|||AXB||| \leq \frac{|||A^pX|||}{p} + \frac{|||XB^q|||}{q} \tag{1.8}$$

holds for positive matrices A, B, X, and for positive real numbers p, q such that $1/p + 1/q = 1$. An equivalent form of inequality (1.8) and related Hölder-type norm inequalities can be found in [1].

In this article, we present singular value inequalities for sums and products of operators that generalize (1.2), (1.6), and (1.7). Our analysis is based on majorization of singular values and the matrix arithmetic–geometric mean inequality. Relations between the different forms of the arithmetic–geometric mean inequality for operators are also obtained.

2. Main Results

In this section, we establish a singular value inequality for Hilbert space operators which yields well known and new arithmetic–geometric mean inequalities as special cases. To prove our generalized inequality, we need the following basic lemmas. The first lemma, which can be found in [5], contains a relation between singular values and usual operator norm.

Lemma 2.1. *Let A and B be operators in $B(H)$. Then*

$$s_j\left(AB\right) \leq \|A\| \ \ s_j\left(B\right), \ \ j = 1, 2, \cdots .$$

The second lemma, which can be found in [5], concerned singular value majorization for product of operators.

Lemma 2.2. *Let A and B be operators in $B(H)$. Then*

$$s\left(AB\right) \prec_w s\left(A\right)\ s\left(B\right)$$

The following lemma can be easily proved, it can be found in [5] and it will be helpful in our work.

Lemma 2.3. *Let A be self-adjoint operator. Then*

$$\pm A \leq |A|. \tag{2.1}$$

Using Lemma 2.1 and inequality (1.4) we are able to get the following inequality for singular values.

Theorem 2.4. *Let A, B, and X be operators in $B(H)$, such that X is positive. Then*

$$s_j\left(AXB^*\right) \leq \frac{1}{2}\ \|X\|\, s_j\left(A^*A + B^*B\right),\ \ j = 1, 2, \cdots. \tag{2.2}$$

Proof. For $j = 1, 2, \cdots$, we have

$$\begin{aligned} 2s_j\left(AXB^*\right) &= 2s_j\left(AX^{1/2}X^{1/2}B^*\right) \\ &\leq s_j\left(\left|AX^{1/2}\right|^2 + \left|BX^{1/2}\right|^2\right) \\ &= s_j\left(X^{1/2}A^*AX^{1/2} + X^{1/2}B^*BX^{1/2}\right) \\ &= s_j\left(X^{1/2}\left(A^*A + B^*B\right)X^{1/2}\right) \\ &\leq \left\|X^{1/2}\right\|\ s_j\left(A^*A + B^*B\right)\left\|X^{1/2}\right\| \\ &= \|X\|\, s_j\left(A^*A + B^*B\right). \end{aligned}$$

This completes the proof. □

Remark 2.5. An equivalent form of inequality (2.2) can be stated as follows: Let A, B, and X be positive operators in $B\left(H\right)$. Then

$$s_j\left(A^{1/2}XB^{1/2}\right) \leq \frac{1}{2}\|X\|\ s_j\left(A + B\right),\ \ j = 1, 2, \cdots. \tag{2.3}$$

Also, a related inequality that can be proved by using Lemma 2.2 and inequality (1.4) asserts that

$$2s\left(AXB^*\right) \prec_w\ s\left(X\right)s\left(A^*A + B^*B\right).$$

Note that inequality (2.2) implies (1.4), while inequality (2.3) implies (1.2). Moreover, an inequality related to (1.5), that follows from inequality (2.3) and the Fan dominance theorem says that

$$\left|\left|\left|A^{1/2}XB^{1/2}\right|\right|\right| \leq \frac{1}{2}\|X\| \,.\, |||A + B|||\,. \tag{2.4}$$

Our main result of this paper, which leads to a generalization of (1.7), is given in the following theorem.

Theorem 2.6. *Let A_i, B_i, $X_i \in B(H)$ $(i = 1, 2, \cdots, n)$, such that X_i is positive. Then*

$$2s_j\left(\sum_{i=1}^{n} A_i X_i B_i^*\right) \le \left(\max_{i=1,2,\cdots,n} \|X_i\|\right) s_j^2\left(\left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \end{array}\right]\right), \quad j = 1, 2, \cdots. \tag{2.5}$$

Proof. On $\oplus_n H$, let

$$A = \left[\begin{array}{cccc} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right], \ B = \left[\begin{array}{cccc} B_1 & B_2 & \dots & B_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right],$$

$$X = \left[\begin{array}{cccc} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_n \end{array}\right]. \text{ Then } AXB^* = \left[\begin{array}{cccc} \sum_{i=1}^{n} A_i X_i B_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right],$$

and

$$A^*A + B^*B = \left[\begin{array}{cccc} A_1^*A_1 + B_1^*B_1 & A_1^*A_2 + B_1^*B_2 & \dots & A_1^*A_n + B_1^*B_n \\ A_2^*A_1 + B_2^*B_1 & A_2^*A_2 + B_2^*B_2 & \dots & A_2^*A_n + B_2^*B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^*A_1 + B_n^*B_1 & A_n^*A_2 + B_n^*B_2 & \dots & A_n^*A_n + B_n^*B_n \end{array}\right]$$

$$= \left[\begin{array}{cc} A_1^* & B_1^* \\ A_2^* & B_2^* \\ \vdots & \vdots \\ A_n^* & B_n^* \end{array}\right] \left[\begin{array}{cccc} A_1 & A_2 & \dots & A_n \\ B_1 & B_2 & \dots & B_n \end{array}\right] = \left|\left[\begin{array}{cccc} A_1 & A_2 & \dots & A_n \\ B_1 & B_2 & \dots & B_n \end{array}\right]\right|^2.$$

It follows from inequality (2.2), for $j = 1, 2, \cdots$, that

$$2s_j\left(\sum_{i=1}^{n} A_i X_i B_i^* \oplus 0 \oplus \ldots \oplus 0\right) \le$$
$$\|X_1 \oplus X_2 \oplus \ldots \oplus X_n\| \, s_j\left(\left|\left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \end{array}\right]\right|^2\right)$$

and so

$$2s_j\left(\sum_{i=1}^{n} A_i X_i B_i^*\right) \le \left(\max_{i=1,2,\cdots,n} \|X_i\|\right) s_j^2\left(\left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \end{array}\right]\right),$$

for $j = 1, 2, \cdots$, as required. □

Inequality (2.5) includes several singular value inequalities as special cases. Samples of inequalities are demonstrated below.

Corollary 2.7. *Let A_i, B_i, $X_i \in B(H)$ such that X_i is positive $(i = 1, 2)$. Then*

$$2s_j\left(A_1X_1B_1^* + A_2X_2B_2^*\right) \le \max\{\|X_1\|, \|X_2\|\} \ s_j^2\left(\left[\begin{array}{cc} A_1 & A_2 \\ B_1 & B_2 \end{array}\right]\right), \quad j = 1, 2, \cdots. \tag{2.6}$$

In particular,

$$2s_j\left(AXB^*+BXA^*\right)\leq\|X\|\ \ s_j^2\left(\left[\begin{array}{cc}A & B\\ B & A\end{array}\right]\right),\ \ j=1,2,\cdots. \tag{2.7}$$

Proof. Inequality (2.6) follows by letting $n=2$ in inequality (2.5), while the particular case follows by letting $A_1=B_2=A$, $A_2=B_1=B$, and $X_1=X_2=X$ in inequality (2.6). □

Remark 2.8. Choosing $X=I$ in inequality (2.5) implies the following generalization of inequality (1.7) to $n-$tuple of operators.

$$\sqrt{2}s_j^{1/2}\left(\sum_{i=1}^{n}A_iB_i^*\right)\leq s_j\left(\left[\begin{array}{cccc}A_1 & A_2 & \cdots & A_n\\ B_1 & B_2 & \cdots & B_n\end{array}\right]\right),\ \ j=1,2,\cdots.$$

On the other hand using the weak majorization $s\left(A+B\right)\prec_w s\left(A\right)+s\left(B\right)$, gives a related result that is

$$2s_j\left(\sum_{i=1}^{n}A_iB_i^*\right)\prec_w\sum_{i=1}^{n}s_j\left(A_i^*A_i+B_i^*B_i\right),\ \ j=1,2,\cdots.$$

The following inequality is an application of inequality (2.6) together with Lemma 2.3.

Corollary 2.9. *Let A, B, $X\in B(H)$ be positive. Then*

$$s_j\left(A^{1/2}XA^{1/2}+B^{1/2}XB^{1/2}\right)\leq\|X\|\,s_j\left(\left(A+\left|B^{1/2}A^{1/2}\right|\right)\oplus\left(B+\left|A^{1/2}B^{1/2}\right|\right)\right),$$

for $j=1,2,\cdots$. In particular,

$$s_j\left(A+B\right)\leq s_j\left(\left(A+\left|B^{1/2}A^{1/2}\right|\right)\oplus\left(B+\left|A^{1/2}B^{1/2}\right|\right)\right).$$

Proof. Let $A_1=B_1=A^{1/2}$, $A_2=B_2=B^{1/2}$, and $X_1=X_2=X$ in inequality (2.7). Then for $j=1,2,..$, we get

$$\begin{aligned}2s_j\left(A^{1/2}XA^{1/2}+B^{1/2}XB^{1/2}\right) &\leq \|X\|\ \ s_j^2\left(\left[\begin{array}{cc}A^{1/2} & B^{1/2}\\ A^{1/2} & B^{1/2}\end{array}\right]\right)\\ &= \|X\|\ \ s_j\left(\left[\begin{array}{cc}2A & 2A^{1/2}B^{1/2}\\ 2B^{1/2}A^{1/2} & 2B\end{array}\right]\right),\end{aligned}$$

so

$$s_j\left(A^{1/2}XA^{1/2}+B^{1/2}XB^{1/2}\right)\leq\|X\|\ \ s_j\left(\left[\begin{array}{cc}A & 0\\ 0 & B\end{array}\right]+\left[\begin{array}{cc}0 & A^{1/2}B^{1/2}\\ B^{1/2}A^{1/2} & 0\end{array}\right]\right).$$

But Lemma 2.3 implies that

$$\left[\begin{array}{cc}0 & A^{1/2}B^{1/2}\\ B^{1/2}A^{1/2} & 0\end{array}\right]\leq\left|\left[\begin{array}{cc}0 & A^{1/2}B^{1/2}\\ B^{1/2}A^{1/2} & 0\end{array}\right]\right|=\left[\begin{array}{cc}\left|B^{1/2}A^{1/2}\right| & 0\\ 0 & \left|A^{1/2}B^{1/2}\right|\end{array}\right].$$

Thus,

$$s_j\left(A^{1/2}XA^{1/2}+B^{1/2}XB^{1/2}\right)\leq$$

$$\|X\|\ s_j\left(\left[\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right]+\left[\begin{array}{cc} \left|B^{1/2}A^{1/2}\right| & 0 \\ 0 & \left|A^{1/2}B^{1/2}\right| \end{array}\right]\right),$$

which gives the desired inequality. $\square$

The following inequality is an application of inequality (2.6) and contains a generalization of inequality (1.6)

Corollary 2.10. *Let A, B, $X \in B(H)$ such that X is positive. Then*

$$s_j\left(AXA^* - BXB^*\right) \leq \|X\|\, s_j\left(A^*A \oplus B^*B\right), \quad j = 1, 2, \cdots. \tag{2.8}$$

In particular, if A, B are positive, then

$$s_j\left(A - B\right) \leq s_j\left(A \oplus B\right).$$

Proof. by letting $A_1 = B_1 = A$, $A_2 = -B_2 = B$, and $X_1 = X_2 = X$ in inequality (2.6), we get

$$2s_j\left(AXA^* - BXB^*\right) \leq \|X\|\, s_j\left(\left[\begin{array}{cc} 2A^*A & 0 \\ 0 & 2B^*B \end{array}\right]\right).$$

The particular case follows from inequality (2.8) by replacing A, B, and X by $A^{1/2}$,$B^{1/2}$, and I, respectively. $\square$

Remark 2.11. An equivalent form of inequality (2.8) can be stated as follows: Let A, B, and X be positive operators in $B\left(H\right)$. Then

$$s_j\left(A^{1/2}XA^{1/2} - B^{1/2}XB^{1/2}\right) \leq \|X\|\, s_j\left(A \oplus B\right), \quad j = 1, 2, \cdots.$$

Note that inequality (2.8) implies (2.2), to see this let $C = \left[\begin{array}{c} A \\ B \end{array}\right]$, $D = \left[\begin{array}{c} A \\ -B \end{array}\right]$. Then

$$\begin{aligned} 2s_j\left(\left[\begin{array}{cc} BXA^* & 0 \\ 0 & AXB^* \end{array}\right]\right) &= 2s_j\left(\left[\begin{array}{cc} 0 & AXB^* \\ BXA^* & 0 \end{array}\right]\right) \\ &= s_j\left(CXC^* - DXD^*\right) \\ &\leq \|X\|\, s_j\left(C^*C \oplus D^*D\right) \\ &= \|X\|\, s_j\left(\left[\begin{array}{cc} C^*C & 0 \\ 0 & D^*D \end{array}\right]\right) \\ &= \|X\|\, s_j\left(\left[\begin{array}{cc} A^*A + B^*B & 0 \\ 0 & A^*A + B^*B \end{array}\right]\right). \end{aligned}$$

Thus, we have $2s_j\left(AXB^*\right) \leq \|X\|\, s_j\left(A^*A + B^*B\right)$, for $j = 1, 2, \cdots$.

Recall that $\mathrm{ReA} = \frac{\mathrm{A}+\mathrm{A}^*}{2}$, and $\mathrm{ImA} = \frac{\mathrm{A}-\mathrm{A}^*}{2}$, so we end this section by the following corollary.

Corollary 2.12. *Let A, B, $X \in B(H)$, such that X is positive. Then*

$$s_j\left(\mathrm{Re}\left(\mathrm{AXA}\right)\right) \leq \|X\|\, s_j\left(\left(\mathrm{ReA}\right)^2 \oplus \left(\mathrm{ImA}\right)^2\right), \quad j = 1, 2, \cdots. \tag{2.9}$$

In particular,

$$s_j\left(\mathrm{Re}\left(\mathrm{A}^2\right)\right) \leq s_j\left(\left(\mathrm{ReA}\right)^2 \oplus \left(\mathrm{ImA}\right)^2\right), \quad j = 1, 2, \cdots.$$

Proof. For $j = 1, 2, \cdots$, we have

$$
\begin{aligned}
s_j\left((\mathrm{ReA})^2 \oplus (\mathrm{ImA})^2\right) &= s_j\left(\begin{bmatrix} (\mathrm{ReA})^2 & 0 \\ 0 & (\mathrm{ImA})^2 \end{bmatrix}\right) \\
&= s_j^2\left(\begin{bmatrix} \mathrm{ReA} & 0 \\ 0 & \mathrm{ImA} \end{bmatrix}\right) = \frac{1}{4}s_j^2\left(\begin{bmatrix} A+A^* & 0 \\ 0 & A-A^* \end{bmatrix}\right).
\end{aligned}
$$

But $\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}$ is unitarily equivalent to $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$, so

$$
s_j\left((\mathrm{ReA})^2 \oplus (\mathrm{ImA})^2\right) = \frac{1}{4}s_j^2\left(\begin{bmatrix} A & A^* \\ A^* & A \end{bmatrix}\right).
$$

Now inequality (2.7) implies that

$$
\begin{aligned}
s_j\left(\mathrm{Re}\left(\mathrm{AXA}\right)\right) &= \frac{1}{2}s_j\left(AXA + A^*XA^*\right) \\
&\leq \frac{1}{4}\|X\|\, s_j^2\left(\begin{bmatrix} A & A^* \\ A^* & A \end{bmatrix}\right) \\
&= \|X\|\, s_j\left((\mathrm{ReA})^2 \oplus (\mathrm{ImA})^2\right),
\end{aligned}
$$

as required. For the particular case, set $X = I$ in inequality (2.9) to get the result. □

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