# EXTENSION OF THE REFINED JENSEN'S OPERATOR INEQUALITY WITH CONDITION ON SPECTRA

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ABSTRACT. We give an extension of the refined Jensen's operator inequality for $n$−tuples of self-adjoint operators, unital $n$−tuples of positive linear mappings and real valued continuous convex functions with conditions on the spectra of the operators. We also study the order among quasi-arithmetic means under similar conditions.

## 1. INTRODUCTION

We recall some notations and definitions. Let $\mathcal{B}(H)$ be the $C^*$-algebra of all bounded linear operators on a Hilbert space $H$ and $1_H$ stands for the identity operator. We define bounds of a self-adjoint operator $A \in \mathcal{B}(H)$ by

$$m_A = \inf_{\|x\|=1} \langle Ax, x \rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x \rangle$$

for $x \in H$. If $\mathsf{Sp}(A)$ denotes the spectrum of $A$, then $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.

For an operator $A \in \mathcal{B}(H)$ we define operators $|A|$, $A^+$, $A^-$ by

$$|A| = (A^*A)^{1/2}, \qquad A^+ = (|A| + A)/2, \qquad A^- = (|A| - A)/2.$$

Obviously, if $A$ is self-adjoint, then $|A| = (A^2)^{1/2}$ and $A^+, A^- \geq 0$ (called positive and negative parts of $A = A^+ - A^-$).

B. Mond and J. Pečarić in [9] proved Jensen's operator inequality

$$f\left(\sum_{i=1}^{n} w_i \Phi_i(A_i)\right) \leq \sum_{i=1}^{n} w_i \Phi_i\left(f(A_i)\right), \tag{1.1}$$

for operator convex functions $f$ defined on an interval $I$, where $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, $i = 1, \ldots, n$, are unital positive linear mappings, $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in $I$ and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum_{i=1}^{n} w_i = 1$.

F. Hansen, J. Pečarić and I. Perić gave in [3] a generalization of (1.1) for a unital field of positive linear mappings. The following discrete version of their inequality holds

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) \leq \sum_{i=1}^{n} \Phi_i\left(f(A_i)\right), \tag{1.2}$$

for operator convex functions $f$ defined on an interval $I$, where $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, $i = 1, \ldots, n$, is a unital field of positive linear mappings (i.e. $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$), $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in $I$.

Recently, J. Mićić, Z. Pavić and J. Pečarić proved in [5, Theorem 1] that (1.2) stands without operator convexity of $f : I \rightarrow \mathbb{R}$ if a condition on spectra

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n$$

holds, where $m_i$ and $M_i$, $m_i \leq M_i$ are bounds of $A_i$, $i = 1, \ldots, n$; and $m_A$ and $M_A$, $m_A \leq M_A$, are bounds of $A = \sum_{i=1}^{n} \Phi_i(A_i)$ (provided that the interval $I$ contains all $m_i, M_i$).

Next, they considered in [6, Theorem 2.1] the case when $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ is valid for several $i \in \{1, \ldots, n\}$, but not for all $i = 1, \ldots, n$ and obtain an extension of (1.2) as follows.

*Theorem A.* Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds $m_i$ and $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \rightarrow B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let $m = \min\{m_1, \ldots, m_{n_1}\}$ and $M = \max\{M_1, \ldots, M_{n_1}\}$. If

$$(m, M) \cap [m_i, M_i] = \emptyset \qquad \text{for} \quad i = n_1 + 1, \ldots, n,$$

and one of two equalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

is valid, then

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)), \tag{1.3}$$

holds for every continuous convex function $f : I \rightarrow \mathbb{R}$ provided that the interval $I$ contains all $m_i, M_i$, $i = 1, \ldots, n,$.

If $f : I \rightarrow \mathbb{R}$ is concave, then the reverse inequality is valid in (1.3).

Very recently, J. Mićić, J. Pečarić and J. Perić gave in [7, Theorem 3] the following refinement of (1.2) with condition on spectra, i.e. a refinement of [5, Theorem 3] (see also [5, Corollary 5]).

*Theorem B.* Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds $m_i$ and $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Let

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n, \qquad \text{and} \qquad m < M,$$

where $m_A$ and $M_A$, $m_A \leq M_A$, are the bounds of the operator $A = \sum_{i=1}^n \Phi_i(A_i)$ and

$$m = \max\left\{M_i \colon M_i \leq m_A, i \in \{1, \ldots, n\}\right\}, \quad M = \min\left\{m_i \colon m_i \geq M_A, i \in \{1, \ldots, n\}\right\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval $I$ contains all $m_i, M_i$, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) - \delta_f \widetilde{A} \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \quad (1.4)$$

$$\text{(resp.} \quad f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \geq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) + \delta_f \widetilde{A} \geq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \text{)}$$

holds, where

$$\delta_f \equiv \delta_f(\bar{m}, \bar{M}) = f(\bar{m}) + f(\bar{M}) - 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right)$$

$$\text{(resp.} \quad \delta_f \equiv \delta_f(\bar{m}, \bar{M}) = 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M}) \text{)},$$

$$\widetilde{A} \equiv \widetilde{A}_A(\bar{m}, \bar{M}) = \tfrac{1}{2}1_K - \tfrac{1}{\bar{M}-\bar{m}}\left|A - \tfrac{\bar{m}+\bar{M}}{2}1_K\right|$$

and $\quad \bar{m} \in [m, m_A], \bar{M} \in [M_A, M], \bar{m} < \bar{M}, \quad$ are arbitrary numbers.

There is an extensive literature devoted to Jensens inequality concerning different refinements and extensive results, see, for example [1, 2, 4], [10]–[14].

In this paper we study an extension of Jensen's inequality given in Theorem B and a refinement of Theorem A. As an application of this result to the quasi-arithmetic mean with a weight, we give an extension of results given in [7] and a refinement of ones given in [6].

## 2. MAIN RESULTS

To obtain our main result we need a result [7, Lemma 2] given in the following lemma.

*Lemma C.* Let $A$ be a self-adjoint operator $A \in B(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$, for some scalars $m < M$. Then

$$f(A) \leq \frac{M1_H - A}{M - m}f(m) + \frac{A - m1_H}{M - m}f(M) - \delta_f \widetilde{A} \quad (2.1)$$

$$\text{(resp.} \quad f(A) \geq \frac{M1_H - A}{M - m}f(m) + \frac{A - m1_H}{M - m}f(M) + \delta_f \widetilde{A} \text{)}$$

holds for every continuous convex (resp. concave) function $f : [m, M] \to \mathbb{R}$, where

$$\delta_f = f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right) \quad (\text{resp. } \delta_f = 2f\left(\tfrac{m+M}{2}\right) - f(m) - f(M)),$$
$$\text{and} \quad \widetilde{A} = \tfrac{1}{2}1_H - \tfrac{1}{M-m}\left|A - \tfrac{m+M}{2}1_H\right|.$$

We shall give the proof for the convenience of the reader.

*Proof of Lemma C.* We prove only the convex case.

In [8, Theorem 1, p. 717] is prove that

$$\begin{aligned} &\min\{p_1, p_2\}\left[f(x) + f(y) - 2f\left(\tfrac{x+y}{2}\right)\right] \\ &\leq \; p_1 f(x) + p_2 f(y) - f(p_1 x + p_2 y) \end{aligned} \tag{2.2}$$

holds for every convex function $f$ on an interval $I$ and $x, y \in I$, $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$.

Putting $x = m, y = M$ in (2.2) it follows that

$$\begin{aligned} f\left(p_1 m + p_2 M\right) \;\leq\; & p_1 f(m) + p_2 f(M) \\ & - \min\{p_1, p_2\}\left(f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right)\right) \end{aligned} \tag{2.3}$$

holds for every $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. For any $t \in [m, M]$ we can write

$$f(t) = f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right).$$

Then by using (2.3) for $p_1 = \frac{M-t}{M-m}$ and $p_2 = \frac{t-m}{M-m}$ we get

$$\begin{aligned} f(t) \;\leq\; & \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\ & - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right), \end{aligned} \tag{2.4}$$

since

$$\min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|.$$

Finally we use the continuous functional calculus for a self-adjoint operator $A$: $f, g \in \mathcal{C}(I), Sp(A) \subseteq I$ and $f \geq g$ implies $f(A) \geq g(A)$; and $h(t) = |t|$ implies $h(A) = |A|$. Then by using (2.4) we obtain the desired inequality (2.1). $\square$

In the following theorem we give an extension of Jensen's inequality given in Theorem B and a refinement of Theorem A.

**Theorem 2.1.** *Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds $m_i$ and $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha \, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta \, 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ and*

$$\begin{aligned} m &= \begin{cases} m_L, & \text{if } \{M_i \colon M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\} = \varnothing, \\ \max\{M_i \colon M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise}, \end{cases} \\ M &= \begin{cases} M_R, & \text{if } \{m_i \colon m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\} = \varnothing, \\ \min\{m_i \colon m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise}. \end{cases} \end{aligned}$$

*If*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad for \quad i = n_1 + 1, \ldots, n, \qquad m < M,$$

*and one of two equalities*

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

*is valid, then*

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \;\leq\; \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A} \leq \sum_{i=1}^{n} \Phi_i(f(A_i))$$

$$\leq \; \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)), \text{ (2.5)}$$

*holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval $I$ contains all $m_i, M_i$, $i = 1, \ldots, n$, where*

$$\delta_f \equiv \delta_f(\bar{m}, \bar{M}) = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m} + \bar{M}}{2}\right)$$

$$\widetilde{A} \equiv \widetilde{A}_{A,\Phi,n_1,\alpha}(\bar{m}, \bar{M}) = \frac{1}{2} 1_K - \frac{1}{\alpha(\bar{M} - \bar{m})} \sum_{i=1}^{n_1} \Phi_i\left(\left|A_i - \frac{\bar{m} + \bar{M}}{2} 1_H\right|\right)$$

$$(2.6)$$

*and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.*

*If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in* (2.5).

*Proof.* We prove only the convex case.

Let us denote

$$A = \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i), \qquad B = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i), \qquad C = \sum_{i=1}^{n} \Phi_i(A_i).$$

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

Since $f$ is convex on $[\bar{m}, \bar{M}]$ and $\mathsf{Sp}(A_i) \subseteq [m_i, M_i] \subseteq [\bar{m}, \bar{M}]$ for $i = 1, \ldots, n_1$, it follows from Lemma C that

$$f(A_i) \leq \frac{\bar{M} 1_H - A_i}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{A_i - \bar{m} 1_H}{\bar{M} - \bar{m}} f(\bar{M}) - \delta_f \widetilde{A}_i, \qquad i = 1, \ldots, n_1$$

holds, where $\delta_f = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$ and $\widetilde{A}_i = \frac{1}{2} 1_H - \frac{1}{\bar{M}-\bar{m}} \left| A_i - \frac{\bar{m}+\bar{M}}{2} 1_H \right|$. Applying a positive linear mapping $\Phi_i$ and summing, we obtain

$$\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \;\leq\; \frac{\bar{M}\alpha 1_K - \sum_{i=1}^{n_1} \Phi_i(A_i)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^{n_1} \Phi_i(A_i) - \bar{m}\alpha 1_K}{\bar{M}-\bar{m}} f(\bar{M})$$
$$- \delta_f\left(\frac{\alpha}{2} 1_K - \frac{1}{\bar{M}-\bar{m}} \sum_{i=1}^{n_1} \Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2} 1_H\right|\right)\right),$$

since $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha 1_K$. It follows that

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{\bar{M} 1_K - A}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{A - \bar{m} 1_K}{\bar{M} - \bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \qquad (2.7)$$

where $\widetilde{A} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$.

In addition, since $f$ is convex on all $[m_i, M_i]$ and $(\bar{m}, \bar{M}) \cap [m_i, M_i] = \emptyset$ for $i = n_1 + 1, \ldots, n$, then

$$f(A_i) \geq \frac{\bar{M}1_H - A_i}{\bar{M} - \bar{m}}f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M} - \bar{m}}f(\bar{M}), \qquad i = n_1 + 1, \ldots, n.$$

It follows

$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A} \geq \frac{\bar{M}1_K - B}{\bar{M} - \bar{m}}f(\bar{m}) + \frac{B - \bar{m}1_K}{\bar{M} - \bar{m}}f(\bar{M}) - \delta_f\widetilde{A}. \quad (2.8)$$

Combining $(2.7)$ and $(2.8)$ and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A}. \qquad (2.9)$$

Next, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))$$

$$= \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \frac{\beta}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \quad \text{(by } \alpha + \beta = 1)$$

$$\leq \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \beta\delta_f\widetilde{A} \qquad \text{(by } (2.9))$$

$$\leq \frac{\alpha}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \alpha\delta_f\widetilde{A} + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \beta\delta_f\widetilde{A} \qquad \text{(by } (2.9))$$

$$= \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A} \quad \text{(by } \alpha + \beta = 1),$$

which gives the following double inequality

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \leq \sum_{i=1}^{n}\Phi_i(f(A_i)) - \beta\delta_f\widetilde{A} \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A}.$$

Adding $\beta\delta_f\widetilde{A}$ in the above inequalities, we get

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \beta\delta_f\widetilde{A} \leq \sum_{i=1}^{n}\Phi_i(f(A_i)) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \alpha\delta_f\widetilde{A}. \quad (2.10)$$

Now, we remark that $\delta_f \geq 0$ and $\widetilde{A} \geq 0$. (Indeed, since $f$ is convex, then $f\left((\bar{m} + \bar{M})/2\right) \leq (f(\bar{m}) + f(\bar{M}))/2$, which implies that $\delta_f \geq 0$. Also, since

$$\mathsf{Sp}(A_i) \subseteq [\bar{m}, \bar{M}] \quad \Rightarrow \quad \left|A_i - \frac{\bar{M} + \bar{m}}{2}1_H\right| \leq \frac{\bar{M} - \bar{m}}{2}1_H, \qquad \text{for } i = 1, \ldots, n_1,$$

then

$$\sum_{i=1}^{n_1} \Phi_i \left( \left| A_i - \frac{\bar{M} + \bar{m}}{2} 1_H \right| \right) \le \frac{\bar{M} - \bar{m}}{2} \alpha 1_K,$$

which gives

$$0 \le \frac{1}{2} 1_K - \frac{1}{\alpha(\bar{M} - \bar{m})} \sum_{i=1}^{n_1} \Phi_i \left( \left| A_i - \frac{\bar{M} + \bar{m}}{2} 1_H \right| \right) = \widetilde{A}. \qquad )$$

Consequently, the following inequalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A},$$

$$\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)),$$

hold, which with (2.10) proves the desired series inequalities (2.5).

$$\square$$

**Example 2.2.** We observe the matrix case of Theorem 2.1 for $f(t) = t^4$, which is the convex function but not operator convex, $n = 4$, $n_1 = 2$ and the bounds of matrices as in Figure 1.
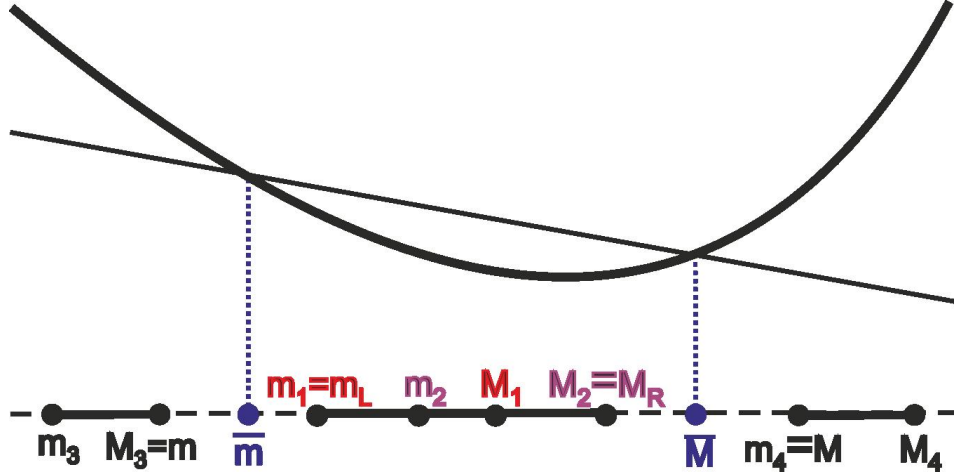


FIGURE 1. An example a convex function and the bounds of four operators

We show an example such that

$$\frac{1}{\alpha} \left( \Phi_1(A_1^4) + \Phi_2(A_2^4) \right) < \frac{1}{\alpha} \left( \Phi_1(A_1^4) + \Phi_2(A_2^4) \right) + \beta \delta_f \widetilde{A}$$
$$< \Phi_1(A_1^4) + \Phi_2(A_2^4) + \Phi_3(A_3^4) + \Phi_4(A_4^4) \qquad (2.11)$$
$$< \frac{1}{\beta} \left( \Phi_3(A_3^4) + \Phi_4(A_4^4) \right) - \alpha \delta_f \widetilde{A} < \frac{1}{\beta} \left( \Phi_3(A_3^4) + \Phi_4(A_4^4) \right)$$

holds, where $\delta_f = \bar{M}^4 + \bar{m}^4 - (\bar{M} + \bar{m})^4/8$ and

$$\widetilde{A} = \frac{1}{2} I_2 - \frac{1}{\alpha(\bar{M} - \bar{m})} \left( \Phi_1 \left( \left| A_1 - \frac{\bar{M} + \bar{m}}{2} I_h \right| \right) + \Phi_2 \left( \left| A_2 - \frac{\bar{M} + \bar{m}}{2} I_3 \right| \right) \right).$$

We define mappings $\Phi_i : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_i((a_{jk})_{1 \le j,k \le 3}) = \frac{1}{4}(a_{jk})_{1 \le j,k \le 2}$, $i = 1, \ldots, 4$. Then $\sum_{i=1}^{4} \Phi_i(I_3) = I_2$ and $\alpha = \beta = \frac{1}{2}$.
Let

$$A_1 = 2 \begin{pmatrix} 2 & 9/8 & 1 \\ 9/8 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \qquad A_2 = 3 \begin{pmatrix} 2 & 9/8 & 0 \\ 9/8 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$A_3 = -3 \begin{pmatrix} 4 & 1/2 & 1 \\ 1/2 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \qquad A_4 = 12 \begin{pmatrix} 5/3 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $m_1 = 1.28607$, $M_1 = 7.70771$, $m_2 = 0.53777$, $M_2 = 5.46221$, $m_3 = -14.15050$, $M_3 = -4.71071$, $m_4 = 12.91724$, $M_4 = 36.$, so $m_L = m_2$, $M_R = M_1$, $m = M_3$ and $M = m_4$ (rounded to five decimal places). Also,

$$\frac{1}{\alpha}\left(\Phi_1(A_1) + \Phi_2(A_2)\right) = \frac{1}{\beta}\left(\Phi_3(A_3) + \Phi_4(A_4)\right) = \begin{pmatrix} 4 & 9/4 \\ 9/4 & 3 \end{pmatrix},$$

and

$$A_f \equiv \frac{1}{\alpha}\left(\Phi_1(A_1^4) + \Phi_2(A_2^4)\right) = \begin{pmatrix} 989.00391 & 663.46875 \\ 663.46875 & 526.12891 \end{pmatrix},$$

$$C_f \equiv \Phi_1(A_1^4) + \Phi_2(A_2^4) + \Phi_3(A_3^4) + \Phi_4(A_4^4) = \begin{pmatrix} 68093.14258 & 48477.98437 \\ 48477.98437 & 51335.39258 \end{pmatrix},$$

$$B_f \equiv \frac{1}{\beta}\left(\Phi_3(A_3^4) + \Phi_4(A_4^4)\right) = \begin{pmatrix} 135197.28125 & 96292.5 \\ 96292.5 & 102144.65625 \end{pmatrix}.$$

Then

$$A_f < C_f < B_f \tag{2.12}$$

holds (which is consistent with (1.3)).

We will choose three pairs of numbers $(\bar{m}, \bar{M})$, $\bar{m} \in [-4.71071, 0.53777]$, $\bar{M} \in [7.70771, 12.91724]$ as follows:

**i)** $\bar{m} = m_L = 0.53777$, $\bar{M} = M_R = 7.70771$, then

$$\widetilde{\Delta}_1 = \beta\delta_f\widetilde{A} = 0.5 \cdot 2951.69249 \cdot \begin{pmatrix} 0.15678 & 0.09030 \\ 0.09030 & 0.15943 \end{pmatrix} = \begin{pmatrix} 231.38908 & 133.26139 \\ 133.26139 & 235.29515 \end{pmatrix},$$

**ii)** $\bar{m} = m = -4.71071$, $\bar{M} = M = 12.91724$, then

$$\widetilde{\Delta}_2 = \beta\delta_f\widetilde{A} = 0.5 \cdot 27766.07963 \cdot \begin{pmatrix} 0.36022 & 0.03573 \\ 0.03573 & 0.36155 \end{pmatrix} = \begin{pmatrix} 5000.89860 & 496.04498 \\ 496.04498 & 5019.50711 \end{pmatrix},$$

**iii)** $\bar{m} = -1$, $\bar{M} = 10$, then

$$\widetilde{\Delta}_3 = \beta\delta_f\widetilde{A} = 0.5 \cdot 9180.875 \cdot \begin{pmatrix} 0.28203 & 0.08975 \\ 0.08975 & 0.27557 \end{pmatrix} = \begin{pmatrix} 1294.66 & 411.999 \\ 411.999 & 1265. \end{pmatrix}.$$

New, we obtain the following improvement of (2.12) (see (2.11)):

**i)** $\qquad A_f < A_f + \widetilde{\Delta}_1 = \begin{pmatrix} 1220.39299 & 796.73014 \\ 796.73014 & 761.42406 \end{pmatrix}$

$$< C_f < \begin{pmatrix} 134965.89217 & 96159.23861 \\ 96159.23861 & 101909.36110 \end{pmatrix} = B_f - \widetilde{\Delta}_1 < B_f,$$

**ii)** $\quad A_f < A_f + \widetilde{\Delta}_2 = \begin{pmatrix} 5989.90251 & 1159.51373 \\ 1159.51373 & 5545.63601 \end{pmatrix}$

$$< C_f < \begin{pmatrix} 130196.38265 & 95796.45502 \\ 95796.45502 & 97125.14914 \end{pmatrix} = B_f - \widetilde{\Delta}_2 < B_f,$$

**iii)** $\quad A_f < A_f + \widetilde{\Delta}_3 = \begin{pmatrix} 2283.66362 & 1075.46746 \\ 1075.46746 & 1791.12874 \end{pmatrix}$

$$< C_f < \begin{pmatrix} 133902.62153 & 95880.50129 \\ 95880.50129 & 100879.65641 \end{pmatrix} = B_f - \widetilde{\Delta}_3 < B_f.$$

Using Theorem 2.1 we get the following result.

**Corollary 2.3.** *Let the assumptions of Theorem 2.1 hold. Then*

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \gamma_1 \delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \qquad (2.13)$$

*and*

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \gamma_2 \delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \qquad (2.14)$$

*holds for every $\gamma_1, \gamma_2$ in the close interval joining $\alpha$ and $\beta$, where $\delta_f$ and $\widetilde{A}$ are defined by (2.6).*

*Proof.* Adding $\alpha \delta_f \widetilde{A}$ in (2.5) and noticing $\delta_f \widetilde{A} \ge 0$, we obtain

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \alpha \delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)).$$

Taking into account the above inequality and the left hand side of (2.5) we obtain (2.13).

Similarly, subtracting $\beta \delta_f \widetilde{A}$ in (2.5) we obtain (2.14). $\qquad \square$

*Remark* 2.4. *Let the assumptions of Theorem 2.1 be valid.*
1)   We observe that the following inequality

$$f\left( \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) \right) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A}_\beta \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)),$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval $I$ contains all $m_i, M_i, i = 1, \ldots, n$, where $\delta_f$ is defined by (2.6),

$$\widetilde{A}_\beta \equiv \widetilde{A}_{\beta,A,\Phi,n_1}(\bar{m}, \bar{M}) = \frac{1}{2} 1_K - \frac{1}{\bar{M} - \bar{m}} \left| \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i A_i - \frac{\bar{m} + \bar{M}}{2} 1_K \right|$$

and $\quad \bar{m} \in [m, m_L], \bar{M} \in [M_R, M], \bar{m} < \bar{M}$, are arbitrary numbers.

Indeed, by the assumptions of Theorem 2.1 we have

$$m_L \alpha 1_H \leq \sum_{i=1}^{n_1} \Phi_i(A_i) \leq M_R \alpha 1_H \quad \text{and} \quad \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

which implies

$$m_L 1_H \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) \leq M_R 1_H.$$

Also $(m_L, M_R) \cap [m_i, M_i] = \varnothing$ for $i = n_1 + 1, \ldots, n$ and $\sum_{i=n_1+1}^{n} \frac{1}{\beta} \Phi_i(1_H) = 1_K$ hold. So we can apply Theorem B on operators $A_{n_1+1}, \ldots, A_n$ and mappings $\frac{1}{\beta} \Phi_i$. We obtain the desired inequality.

2) We denote by $m_C$ and $M_C$ the bounds of $C = \sum_{i=1}^{n} \Phi_i(A_i)$. If $(m_C, M_C) \cap [m_i, M_i] = \varnothing$, $i = 1, \ldots, n_1$, then series inequality (2.5) can be extended from the left side if we use refined Jensen's operator inequality (1.4)

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) = f\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)\right) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) - \delta_f \widetilde{A}_\alpha$$

$$\leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A} \leq \sum_{i=1}^{n} \Phi_i(f(A_i))$$

$$\leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)),$$

where $\delta_f$ and $\widetilde{A}$ are defined by (2.6),

$$\widetilde{A}_\alpha \equiv \widetilde{A}_{\alpha, A, \Phi, n_1}(\bar{m}, \bar{M}) = \frac{1}{2} 1_K - \frac{1}{\bar{M} - \bar{m}} \left| \frac{1}{\alpha} \sum_{i=n_1+1}^{n} \Phi_i A_i - \frac{\bar{m} + \bar{M}}{2} 1_K \right|$$

*Remark* 2.5. We obtain the equivalent inequalities to the ones in Theorem 2.1 in the case when $\sum_{i=1}^{n} \Phi_i(1_H) = \gamma 1_K$, for some positive scalar $\gamma$. If $\alpha + \beta = \gamma$ and one of two equalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) = \frac{1}{\gamma} \sum_{i=1}^{n} \Phi_i(A_i)$$

is valid, then

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \frac{\beta}{\gamma} \delta_f \widetilde{A} \leq \frac{1}{\gamma} \sum_{i=1}^{n} \Phi_i(f(A_i))$$

$$\leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \frac{\alpha}{\gamma} \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)),$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval $I$ contains all $m_i, M_i$, $i = 1, \ldots, n$, where $\delta_f$ and $\widetilde{A}$ are defined by (2.6).

With respect to Remark 2.5, we obtain the following obvious corollary of Theorem 2.1 with the convex combination of operators $A_i$, $i = 1, \ldots, n$.

**Corollary 2.6.** *Let* $(A_1, \ldots, A_n)$ *be an* $n-$*tuple of self-adjoint operators* $A_i \in B(H)$ *with the bounds* $m_i$ *and* $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. *Let* $(p_1, \ldots, p_n)$ *be an* $n-$*tuple of non-negative numbers such that* $0 < \sum_{i=1}^{n_1} p_i = \mathbf{p_{n_1}} < \mathbf{p_n} = \sum_{i=1}^{n} p_i$, *where* $1 \leq n_1 < n$. *Let*
$m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ *and*

$$m = \begin{cases} m_L, & \text{if } \{M_i \colon M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\} = \emptyset, \\ \max\{M_i \colon M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise,} \end{cases}$$
$$M = \begin{cases} M_R, & \text{if } \{m_i \colon m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\} = \emptyset, \\ \min\{m_i \colon m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise.} \end{cases}$$

*If*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad for \quad i = n_1 + 1, \ldots, n, \qquad m < M,$$

*and one of two equalities*

$$\frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i A_i = \frac{1}{\mathbf{p_n}} \sum_{i=1}^{n} p_i A_i = \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i A_i$$

*is valid, then*

$$\frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i f(A_i) \leq \frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i f(A_i) + \left(1 - \frac{\mathbf{p_{n_1}}}{\mathbf{p_n}}\right) \delta_f \widetilde{A} \leq \frac{1}{\mathbf{p_n}} \sum_{i=1}^{n} p_i f(A_i)$$
$$\leq \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i f(A_i) - \frac{\mathbf{p_{n_1}}}{\mathbf{p_n}} \delta_f \widetilde{A} \leq \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i f(A_i),$$

$$(2.15)$$

*holds for every continuous convex function* $f : I \to \mathbb{R}$ *provided that the interval* $I$ *contains all* $m_i, M_i$, $i = 1, \ldots, n$, *where where* $\delta_f$ *is defined by* (2.6),

$$\widetilde{A} \equiv \widetilde{A}_{A,p,n_1}(\bar{m}, \bar{M}) = \frac{1}{2} 1_H - \frac{1}{\mathbf{p_{n_1}}(\bar{M} - \bar{m})} \sum_{i=1}^{n_1} p_i \left( \left| A_i - \frac{\bar{m} + \bar{M}}{2} 1_H \right| \right)$$

*and* $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

*If* $f : I \to \mathbb{R}$ *is concave, then the reverse inequality is valid in* (2.15).

As a special case of Corollary 2.6 we obtain an extension of [7, Corollary 6].

**Corollary 2.7.** *Let* $(A_1, \ldots, A_n)$ *be an* $n-$*tuple of self-adjoint operators* $A_i \in B(H)$ *with the bounds* $m_i$ *and* $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. *Let* $(p_1, \ldots, p_n)$ *be an* $n-$*tuple of non-negative numbers such that* $\sum_{i=1}^{n} p_i = 1$. *Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad for\ i = 1, \ldots, n, \qquad and \qquad m < M,$$

*where* $m_A$ *and* $M_A$, $m_A \leq M_A$, *are the bounds of* $A = \sum_{i=1}^{n} p_i A_i$ *and*

$$m = \max\{M_i \leq m_A, i \in \{1, \ldots, n\}\}, \ M = \min\{m_i \geq M_A, i \in \{1, \ldots, n\}\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex function provided that the interval $I$ contains all $m_i, M_i$, then

$$f(\sum_{i=1}^n p_i A_i) \leq f(\sum_{i=1}^n p_i A_i) + \frac{1}{2}\delta_f \tilde{\tilde{A}} \leq \frac{1}{2}f(\sum_{i=1}^n p_i A_i) + \frac{1}{2}\sum_{i=1}^n p_i f(A_i)$$
$$\leq \sum_{i=1}^n p_i f(A_i) - \frac{1}{2}\delta_f \tilde{\tilde{A}} \leq \sum_{i=1}^n p_i f(A_i), \qquad (2.16)$$

holds, where $\delta_f$ is defined by (2.6), $\tilde{\tilde{A}} = \frac{1}{2}1_H - \frac{1}{M-\bar{m}}\left|\sum_{i=1}^n p_i A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$ and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (2.16).

*Proof.* We prove only the convex case.

We define $(n+1)$−tuple of operators $(B_1, \ldots, B_{n+1})$, $B_i \in B(H)$, by $B_1 = A = \sum_{i=1}^n p_i A_i$ and $B_i = A_{i-1}$, $i = 2, \ldots, n+1$. Then $m_{B_1} = m_A$, $M_{B_1} = M_A$ are the bounds of $B_1$ and $m_{B_i} = m_{i-1}$, $M_{B_i} = M_{i-1}$ are the ones of $B_i$, $i = 2, \ldots, n+1$. Also, we define $(n+1)$−tuple of non-negative numbers $(q_1, \ldots, q_{n+1})$ by $q_1 = 1$ and $q_i = p_{i-1}$, $i = 2, \ldots, n+1$. We have that $\sum_{i=1}^{n+1} q_i = 2$ and

$$(m_{B_1}, M_{B_1}) \cap [m_{B_i}, M_{B_i}] = \emptyset, \quad \text{for } i = 2, \ldots, n+1 \quad \text{and} \quad m < M \qquad (2.17)$$

holds. Since

$$\sum_{i=1}^{n+1} q_i B_i = B_1 + \sum_{i=2}^{n+1} q_i B_i = \sum_{i=1}^n p_i A_i + \sum_{i=1}^n p_i A_i = 2B_1,$$

then

$$q_1 B_1 = \frac{1}{2}\sum_{i=1}^{n+1} q_i B_i = \sum_{i=2}^{n+1} q_i B_i. \qquad (2.18)$$

Taking into account (2.17) and (2.18), we can apply Corollary 2.6 for $n_1 = 1$ and $B_i$, $q_i$ as above, and we get

$$q_1 f(B_1) \leq q_1 f(B_1) + \frac{1}{2}\delta_f \widetilde{B} \leq \frac{1}{2}\sum_{i=1}^{n+1} q_i f(B_i) \leq \sum_{i=2}^{n+1} q_i f(B_i) - \frac{1}{2}\delta_f \widetilde{B} \leq \sum_{i=2}^{n+1} q_i f(B_i),$$

where $\widetilde{B} = \frac{1}{2}1_H - \frac{1}{M-\bar{m}}\left|B_1 - \frac{\bar{m}+\bar{M}}{2}1_H\right|$, which gives the desired inequality (2.16). $\square$

## 3. Quasi-arithmetic means

In this section we study an application of Theorem 2.1 to the quasi-arithmetic mean with weight.

For a subset $\{A_{n_1}, \ldots, A_{n_2}\}$ of $\{A_1, \ldots, A_n\}$, we denote the quasi-arithmetic mean by

$$\mathcal{M}_\varphi(\gamma, \mathbf{A}, \mathbf{\Phi}, n_1, n_2) = \varphi^{-1}\left(\frac{1}{\gamma}\sum_{i=n_1}^{n_2} \Phi_i\left(\varphi(A_i)\right)\right), \qquad (3.1)$$

where $(A_{n_1}, \ldots, A_{n_2})$ are self-adjoint operators in $\mathcal{B}(H)$ with the spectra in $I$, $(\Phi_{n_1}, \ldots, \Phi_{n_2})$ are positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\sum_{i=n_1}^{n_2} \Phi_i(1_H) = \gamma 1_K$, and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

Under the same conditions, for convenience we introduce the following denotations

$$
\begin{aligned}
\delta_{\varphi,\psi}(m, M) &= \psi(m) + \psi(M) - 2\psi \circ \varphi^{-1}\left(\frac{\varphi(m)+\varphi(M)}{2}\right), \\
\widetilde{A}_{\varphi,n_1,\gamma}(m, M) &= \tfrac{1}{2}1_K - \frac{1}{\gamma(M-m)} \sum_{i=1}^{n_1} \Phi_i\left(\left|\varphi(A_i) - \frac{\varphi(M)+\varphi(m)}{2}1_H\right|\right),
\end{aligned} \tag{3.2}
$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $m, M \in I$, $m < M$. Of course, we include implicitly that $\widetilde{A}_{\varphi,n_1,\gamma}(m, M) \equiv \widetilde{A}_{\varphi,A,\Phi,n_1,\gamma}(m, M)$.

The following theorem is an extension of [7, Theorem 7] and a refinement of [6, Theorem 3.1].

**Theorem 3.1.** *Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds $m_i$ and $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $\varphi, \psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval $I$ which contains all $m_i, M_i$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let one of two equalities*

$$
\mathcal{M}_\varphi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n) \tag{3.3}
$$

*be valid and let*

$$
(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad for \quad i = n_1 + 1, \ldots, n, \qquad m < M,
$$

*where $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$,*

$$
\begin{aligned}
m &= \begin{cases} m_L, & \text{if } \{M_i : M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\} = \emptyset, \\ \max\{M_i : M_i \leq m_L, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise,} \end{cases} \\
M &= \begin{cases} M_R, & \text{if } \{m_i : m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\} = \emptyset, \\ \min\{m_i : m_i \geq M_R, i \in \{n_1 + 1, \ldots, n\}\}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

(i) *If $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone, then*

$$
\mathcal{M}_\psi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \psi^{-1}\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(\psi(A_i)) + \beta \delta_{\varphi,\psi} \widetilde{A}_{\varphi,n_1,\alpha}\right)
$$

$$
\leq \mathcal{M}_\psi(1, \mathbf{A}, \mathbf{\Phi}, 1, n) \leq \psi^{-1}\left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(\psi(A_i)) - \alpha \delta_{\varphi,\psi} \widetilde{A}_{\varphi,n_1,\alpha}\right)
$$

$$
\leq \mathcal{M}_\psi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n) \tag{3.4}
$$

*holds, where $\delta_{\varphi,\psi} \geq 0$ and $\widetilde{A}_{\varphi,n_1,\alpha} \geq 0$.*

(i') *If $\psi \circ \varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone, then the reverse inequality is valid in (3.4), where $\delta_{\varphi,\psi} \geq 0$ and $\widetilde{A}_{\varphi,n_1,\alpha} \geq 0$.*

(ii) *If $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone, then (3.4) holds, where $\delta_{\varphi,\psi} \leq 0$ and $\widetilde{A}_{\varphi,n_1,\alpha} \geq 0$.*

(ii′)   *If $\psi \circ \varphi^{-1}$ is concave and $\psi^{-1}$ is operator monotone, then the reverse inequality is valid in* (3.4)*, where $\delta_{\varphi,\psi} \le 0$ and $\widetilde{A}_{\varphi,n_1,\alpha} \ge 0$.*

*In all the above cases, we assume that $\delta_{\varphi,\psi} \equiv \delta_{\varphi,\psi}(\bar{m}, \bar{M})$, $\widetilde{A}_{\varphi,n_1,\alpha} \equiv \widetilde{A}_{\varphi,n_1,\alpha}(\bar{m}, \bar{M})$ are defined by* (3.2) *and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.*

*Proof.* We only prove the case (i). Suppose that $\varphi$ is a strictly increasing function. Then

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \qquad \text{for} \quad i = n_1 + 1, \ldots, n$$

implies

$$(\varphi(m_L), \varphi(M_R)) \cap [\varphi(m_i), \varphi(M_i)] = \emptyset \quad \text{for } i = n_1 + 1, \ldots, n. \tag{3.5}$$

Also, by using (3.3), we have

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(\varphi(A_i)\right) = \sum_{i=1}^{n} \Phi_i\left(\varphi(A_i)\right) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i\left(\varphi(A_i)\right).$$

Taking into account (3.5) and the above double equality, we obtain by Theorem 2.1

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f\left(\varphi(A_i)\right)) \le \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f\left(\varphi(A_i)\right)) + \beta\delta_f\widetilde{A}_{\varphi,n_1,\alpha} \le \sum_{i=1}^{n} \Phi_i(f\left(\varphi(A_i)\right))$$

$$\le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f\left(\varphi(A_i)\right)) - \alpha\delta_f\widetilde{A}_{\varphi,n_1,\alpha} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f\left(\varphi(A_i)\right)),$$

$$\tag{3.6}$$

for every continuous convex function $f : J \to \mathbb{R}$ on an interval $J$ which contains all $[\varphi(m_i), \varphi(M_i)] = \varphi([m_i, M_i])$, $i = 1, \ldots, n$, where $\delta_f = f(\varphi(m)) + f(\varphi(M)) - 2f\left(\frac{\varphi(m)+\varphi(M)}{2}\right)$.

Also, if $\varphi$ is strictly decreasing, then we check that (3.6) holds for convex $f : J \to \mathbb{R}$ on $J$ which contains all $[\varphi(M_i), \varphi(m_i)] = \varphi([m_i, M_i])$.

Putting $f = \psi \circ \varphi^{-1}$ in (3.6), we obtain

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(\psi(A_i)\right) \le \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(\psi(A_i)\right) + \beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha} \le \sum_{i=1}^{n} \Phi_i\left(\psi(A_i)\right)$$

$$\le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i\left(\psi(A_i)\right) - \alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i\left(\psi(A_i)\right).$$

Applying an operator monotone function $\psi^{-1}$ on the above double inequality, we obtain the desired inequality (3.4).                                                    □

We now give some particular results of interest that can be derived from Theorem 3.1, which are an extension of [7, Corollary 8, Corollary 10] and a refinement of [6, Corollary 3.3].

**Corollary 3.2.** *Let* $(A_1, \ldots, A_n)$ *and* $(\Phi_1, \ldots, \Phi_n)$, $m_i$, $M_i$, $m$, $M$, $m_L$, $M_R$, $\alpha$ *and* $\beta$ *be as in Theorem 3.1. Let $I$ be an interval which contains all $m_i$, $M_i$ and*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad for \quad i = n_1 + 1, \ldots, n, \qquad m < M.$$

***I)*** *If one of two equalities*

$$\mathcal{M}_\varphi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n)$$

*is valid, then*

$$
\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) + \beta \delta_{\varphi^{-1}} \widetilde{A}_{\varphi, n_1, \alpha} \leq \sum_{i=1}^{n} \Phi_i(A_i)
$$
$$
\leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) - \alpha \delta_{\varphi^{-1}} \widetilde{A}_{\varphi, n_1, \alpha} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i \Phi_i(A_i).
$$

(3.7)

*holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ such that $\varphi^{-1}$ is convex on $I$, where $\delta_{\varphi^{-1}} = \bar{m} + \bar{M} - 2 \, \varphi^{-1} \left( \frac{\varphi(\bar{m}) + \varphi(\bar{M})}{2} \right) \geq 0$, $\widetilde{A}_{\varphi, n_1, \alpha} = \frac{1}{2} 1_K -$
$\frac{1}{\alpha(\bar{M}-\bar{m})} \sum_{i=1}^{n_1} \Phi_i \left( \left| \varphi(A_i) - \frac{\varphi(\bar{M}) + \varphi(\bar{m})}{2} 1_H \right| \right)$ and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.*

*But, if $\varphi^{-1}$ is concave, then the reverse inequality is valid in (3.7) for $\delta_{\varphi^{-1}} \leq 0$.*

***II)*** *If one of two equalities*

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

*is valid, then*

$$
\mathcal{M}_\varphi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \varphi^{-1} \left( \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i \left( \varphi(A_i) \right) + \beta \delta_\varphi \widetilde{A}_{n_1} \right) \leq \mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n)
$$
$$
\leq \varphi^{-1} \left( \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i \left( \varphi(A_i) \right) - \alpha \delta_\varphi \widetilde{A}_{n_1} \right) \leq \mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n)
$$

(3.8)

*holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ such that one of the following conditions*

   (i) *$\varphi$ is convex and $\varphi^{-1}$ is operator monotone,*
   (i') *$\varphi$ is concave and $-\varphi^{-1}$ is operator monotone,*

*is satisfied, where $\delta_\varphi = \varphi(\bar{m}) + \varphi(\bar{M}) - 2\varphi \left( \frac{\bar{m}+\bar{M}}{2} \right)$, $\widetilde{A}_{n_1} = \frac{1}{2} 1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}$
$\times \sum_{i=1}^{n_1} \Phi_i \left( \left| A_i - \frac{\bar{m}+\bar{M}}{2} 1_H \right| \right)$ and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.*

*But, if one of the following conditions*

   (ii) *$\varphi$ is concave and $\varphi^{-1}$ is operator monotone,*
   (ii') *$\varphi$ is convex and $-\varphi^{-1}$ is operator monotone,*

*is satisfied, then the reverse inequality is valid in (3.8).*

*Proof.* The inequalities (3.7) follows from Theorem 3.1 by replacing $\psi$ with the identity function, while the inequalities (3.8) follows by replacing $\varphi$ with the identity function and $\psi$ with $\varphi$. □

*Remark* 3.3. Let the assumptions of Theorem 3.1 be valid.

1) We observe that if one of the following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone,
(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

is satisfied, then the following obvious inequality (see Remark 2.4.1))

$$\mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n) \leq \psi^{-1}\left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(\psi(A_i)) - \delta_\varphi \widetilde{A}_\beta\right)$$
$$\leq \mathcal{M}_\psi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n),$$

holds, $\delta_\varphi = \varphi(\bar{m}) + \varphi(\bar{M}) - 2\varphi\left(\frac{\bar{m}+\bar{M}}{2}\right)$, $\widetilde{A}_\beta = \frac{1}{2} 1_K - \frac{1}{\bar{M}-\bar{m}} \left| \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i A_i - \frac{\bar{m}+\bar{M}}{2} 1_K \right|$ and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

2) We denote by $m_\varphi$ and $M_\varphi$ the bounds of $\mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n)$. If $(m_\varphi, M_\varphi) \cap [m_i, M_i] = \emptyset$, $i = 1, \ldots, n_1$, and one of two following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone
(ii) $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone

is satisfied, then the double inequality (3.4) can be extended from the left side as follows

$$\mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \psi^{-1}\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) - \delta_{\varphi,\psi} \widetilde{A}_\alpha\right)$$
$$\leq \mathcal{M}_\psi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \psi^{-1}\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(\psi(A_i)) + \beta \delta_{\varphi,\psi} \widetilde{A}_{\varphi,n_1,\alpha}\right)$$
$$\leq \mathcal{M}_\psi(1, \mathbf{A}, \mathbf{\Phi}, 1, n) \leq \psi^{-1}\left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(\psi(A_i)) - \alpha \delta_{\varphi,\psi} \widetilde{A}_{\varphi,n_1,\alpha}\right)$$
$$\leq \mathcal{M}_\psi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n),$$

where $\delta_{\varphi,\psi}$ and $\widetilde{A}_{\varphi,n_1,\alpha}$ are defined by (3.2),

$$\widetilde{A}_\alpha = \frac{1}{2} 1_K - \frac{1}{\bar{M}-\bar{m}} \left| \frac{1}{\alpha} \sum_{i=n_1+1}^{n} \Phi_i A_i - \frac{\bar{m}+\bar{M}}{2} 1_K \right|.$$

As a special case of the quasi-arithmetic mean (3.1) we can study the weighted power mean as follows. For a subset $\{A_{p_1}, \ldots, A_{p_2}\}$ of $\{A_1, \ldots, A_n\}$ we denote

this mean by

$$
M^{[r]}(\gamma, \mathbf{A}, \mathbf{\Phi}, p_1, p_2) = \begin{cases} \left( \dfrac{1}{\gamma} \displaystyle\sum_{i=p_1}^{p_2} \Phi_i\left(A_i^r\right) \right)^{1/r}, & r \in \mathbb{R}\setminus\{0\}, \\[6mm] \exp\left( \dfrac{1}{\gamma} \displaystyle\sum_{i=p_1}^{p_2} \Phi_i\left(\ln\left(A_i\right)\right) \right), & r = 0, \end{cases}
$$

where $(A_{p_1}, \ldots, A_{p_2})$ are strictly positive operators, $(\Phi_{p_1}, \ldots, \Phi_{p_2})$ are positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\sum_{i=p_1}^{p_2} \Phi_i(1_H) = \gamma\, 1_K$.

Under the same conditions, for convenience we introduce denotations as a special case of (3.2) as follows

$$
\delta_{r,s}(m, M) = \begin{cases} m^s + M^s - 2\left(\frac{m^r+M^r}{2}\right)^{s/r}, & r \neq 0, \\[2mm] m^s + M^s - 2\left(mM\right)^{s/2}, & r = 0, \end{cases}
$$

$$
\widetilde{A}_r(m, M) = \begin{cases} \frac{1}{2}1_K - \frac{1}{|M^r-m^r|}\left| \sum_{i=1}^n \Phi_i(A_i^r) - \frac{M^r+m^r}{2}1_K \right|, & r \neq 0, \\[2mm] \frac{1}{2}1_K - \left| \ln\left(\frac{M}{m}\right) \right|^{-1} \left| \sum_{i=1}^n \Phi_i(\ln A_i) - \ln\sqrt{Mm}\,1_K \right|, & r = 0, \end{cases}
$$
(3.9)

where $m, M \in \mathbb{R}$, $0 < m < M$ and $r, s \in \mathbb{R}$, $r \leq s$. Of course, we include implicitly that $\widetilde{A}_r(m, M) \equiv \widetilde{A}_{r,A}(m, M)$, where $A = \sum_{i=1}^n \Phi_i(A_i^r)$ for $r \neq 0$ and $A = \sum_{i=1}^n \Phi_i(\ln A_i)$ for $r = 0$.

We obtain the following corollary by applying Theorem 3.1 to the above mean. This is an extension of [7, Corollary 13] and a refinement of [6, Corollary 3.4].

**Corollary 3.4.** *Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds $m_i$ and $M_i$, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^n \Phi_i(1_H) = \beta\, 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let*

$$
(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \text{for} \quad i = n_1+1, \ldots, n, \qquad m < M,
$$

*where $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ and*

$$
m = \begin{cases} m_L, & \text{if } \{M_i \colon M_i \leq m_L, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \max\{M_i \colon M_i \leq m_L, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise,} \end{cases}
$$
$$
M = \begin{cases} M_R, & \text{if } \{m_i \colon m_i \geq M_R, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \min\{m_i \colon m_i \geq M_R, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise.} \end{cases}
$$

(i) *If either $r \leq s$, $s \geq 1$ or $r \leq s \leq -1$ and also one of two equalities*

$$
\mathcal{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)
$$

*is valid, then*

$$
\mathcal{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \left( \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(A_i^s\right) + \beta\delta_{r,s}\widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \leq \mathcal{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n)
$$

$$
\leq \left( \frac{1}{\beta} \sum_{i=n_1+1}^n \Phi_i\left(A_i^s\right) - \alpha\delta_{r,s}\widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \leq \mathcal{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)
$$

holds, where $\delta_{r,s} \geq 0$ and $\widetilde{A}_{s,n_1,\alpha} \geq 0$.

In this case, we assume that $\delta_{r,s} \equiv \delta_{r,s}(\bar{m}, \bar{M})$, $\widetilde{A}_{s,n_1,\alpha} \equiv \widetilde{A}_{s,n_1,\alpha}(\bar{m}, \bar{M})$ are defined by (3.9) and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

(ii) If either $r \leq s$, $r \leq -1$ or $1 \leq r \leq s$ and also one of two equalities

$$\mathcal{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n)$$

is valid, then

$$\mathcal{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \geq \left( \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i \left( A_i^r \right) + \beta \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \geq \mathcal{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n)$$

$$\geq \left( \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i \left( A_i^r \right) - \alpha \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \geq \mathcal{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1 + 1, n)$$

holds, where $\delta_{s,r} \leq 0$ and $\widetilde{A}_{s,n_1,\alpha} \geq 0$.

In this case, we assume that $\delta_{s,r} \equiv \delta_{s,r}(\bar{m}, \bar{M})$, $\widetilde{A}_{r,n_1,\alpha} \equiv \widetilde{A}_{r,n_1,\alpha}(\bar{m}, \bar{M})$ are defined by (3.9) and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

*Proof.* In the case (i) we put $\psi(t) = t^s$ and $\varphi(t) = t^r$ if $r \neq 0$ or $\varphi(t) = \ln t$ if $r \neq 0$ in Theorem 3.1. In the case (ii) we put $\psi(t) = t^r$ and $\varphi(t) = t^s$ if $s \neq 0$ or $\varphi(t) = \ln t$ if $s \neq 0$. We omit the details. $\square$

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