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APPLICATION OF MULTIDIMENSIONAL HARDY OPERATOR AND ITS CONNECTION WITH A CERTAIN NONLINEAR DIFFERENTIAL EQUATION IN WEIGHTED VARIABLE LEBESGUE SPACES

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ABSTRACT. In this paper a two weight criterion for multidimensional geometric mean operator in variable exponent Lebesgue space is proved. Also, we found a criterion on weight functions expressing one-dimensional Hardy inequality via a certain nonlinear differential equation. In particular, considered nonlinear differential equation is nonlinear integro-differential equation.

1. Introduction and preliminaries

It is well known that the variable exponent Lebesgue space appeared in the literature for the first time already in [18]. Further development of this theory was connected with the theory of modular function spaces. The first systematic study of modular spaces is presented in [17]. In the appendix, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of more general spaces he considers. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [16]). The next step in the investigation of variable exponent spaces was given in [19] and in [14]. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [7]). Recently in [1] was investigated converse theorems of trigonometric approximation in variable exponent Lebesgue spaces with some Muckenhoupt weights.

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Inequalities are one of the most important instruments in many branches of mathematics such as functional analysis, theory of differential and integral equations, interpolation theory, harmonic analysis, probability theory etc. They are also useful in mechanics, physics and other sciences. It is well known that the classical two weight inequality for the geometric mean operator is closely connected to the one-dimensional Hardy inequality (see [9]). Analogously, the Pólya-Knopp type inequalities with multidimensional geometric mean operator is connected with multidimensional Hardy type operator. In the paper [8] the connection of the Hardy inequality with a nonlinear differential equation having a solution with certain special properties was considered. Therefore, the consideration of this problems in variable exponent Lebesgue space is actual.

Let R^n be the n-dimensional Euclidean space of points $x = (x_1, \cdots, x_n)$ and Ω be a Lebesgue measurable subset in R^n and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. Suppose that p is a Lebesgue measurable function on Ω such that $0 < \underline{p} \leq p(x) < \infty$, $\underline{p} = ess \inf_{x\in\Omega} p(x)$ and ω is a weight function on Ω, i.e. ω is a non-negative, almost everywhere (a.e.) positive function on Ω. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$. It is well known that $|B(0,1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$, where $B(0,1) = \{x \in R^n;\ |x| < 1\}$. Further, in this paper all sets and functions are supposed to be Lebesgue measurable. By $AC(0,\infty)$ we denote the set of absolutely continuous functions on $(0,\infty)$. For the sake of simplicity, the letter C always denotes a positive constant which may change from one step to the next.

Definition 1.1. By $L_{p(x),\omega}(\Omega)$ we denote the set of measurable functions f on Ω such that for some $\lambda_0 > 0$

$$\int_{\Omega} \left(\frac{|f(x)|\,\omega(x)}{\lambda_0}\right)^{p(x)} dx < \infty.$$

Note that the expression

$$\|\omega f\|_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(\cdot),\,\omega}(\Omega)} = \inf\left\{\lambda > 0 :\ \int_{\Omega} \left|\frac{f(x)\,\omega(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

defines a quasi-Banach spaces. In particular, for $1 \leq p(x) < \infty$ the space $L_{p(x),\omega}(\Omega)$ is a Banach function space (see [7]) with respect to the expression $\|f\|_{L_{p(x),\,\omega}(\Omega)}$.

For $\omega = 1$ the space $L_{p(x),\omega}(\Omega)$ coincides with the variable Lebesgue space $L_{p(x)}(\Omega)$.

We reduce two examples which characterize the norm of this space.

Example 1.2. Let $p(x) = \begin{cases} 2 & for \quad x \in \Omega \\ 3 & for \quad x \in R^n \setminus \Omega \end{cases}$ and $f \in L_2(R^n) \bigcap L_3(R^n)$.

We calculate $\|f\|_{L_{p(x)}(R^n)}$. By the definition we have

$$\|f\|_{L_{p(x)}(R^n)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{f(x)}{\lambda}\right|^2 dx + \int\limits_{R^n\setminus\Omega} \left|\frac{f(x)}{\lambda}\right|^3 dx \le 1\right\} =$$

$$= \inf\left\{\lambda > 0 : \frac{a_1}{\lambda^2} + \frac{a_2}{\lambda^3} \le 1\right\} = \inf\left\{\lambda > 0 : \lambda^3 - a_1\lambda - a_2 \ge 0\right\},$$

where $a_1 = \int\limits_{\Omega} f^2(x)\,dx$ and $a_2 = \int\limits_{R^n\setminus\Omega} |f(x)|^3\,dx$. Now we solve the inequality $\lambda^3 - a_1\lambda - a_2 \ge 0$. We consider three different cases.

Case 1. Let $\frac{a_2^2}{4} - \frac{a_1^3}{27} > 0$. Then the cubic equation $\lambda^3 - a_1\lambda - a_2 = 0$ has one real root and two complex conjugate roots. Namely, $\lambda_1 = \sqrt[3]{\frac{a_2}{2} + \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}} + \sqrt[3]{\frac{a_2}{2} - \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}}$, $\lambda_2 = -\frac{\lambda_1}{2} + i\sqrt{3}\,\frac{\sqrt[3]{\frac{a_2}{2} + \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}} - \sqrt[3]{\frac{a_2}{2} - \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}}}{2}$ and $\lambda_3 = \overline{\lambda}_2$. It is obvious that $\lambda^3 - a_1\lambda - a_2 = (\lambda - \lambda_1)\left(\lambda^2 + \lambda_1\lambda + |\lambda_2|^2\right)$ and $\lambda^2 + \lambda_1\lambda + |\lambda_2|^2 > 0$ for all $\lambda \in (-\infty, +\infty)$. Therefore the inequality $\lambda^3 - a_1\lambda - a_2 \ge 0$ holds if and only if $\lambda \ge \lambda_1$ and

$$\|f\|_{L_{p(x)}(R^n)} = \sqrt[3]{\frac{a_2}{2} + \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}} + \sqrt[3]{\frac{a_2}{2} - \sqrt{\frac{a_2^2}{4} - \frac{a_1^3}{27}}}.$$

Case 2. Let $\frac{a_2^2}{4} - \frac{a_1^3}{27} = 0$. Then $\lambda^3 - a_1\lambda - a_2 = \left(\lambda - 2\sqrt[3]{\frac{a_2}{2}}\right)\left(\lambda + \sqrt[3]{\frac{a_2}{2}}\right)^2$ and the inequality $\lambda^3 - a_1\lambda - a_2 \ge 0$ holds if and only if $\lambda \ge 2\sqrt[3]{\frac{a_2}{2}}$ and $\|f\|_{L_{p(x)}(R^n)} = 2\sqrt[3]{\frac{a_2}{2}}$.

Case 3. Let $\frac{a_2^2}{4} - \frac{a_1^3}{27} < 0$. Then the equation $\lambda^3 - a_1\lambda - a_2 = 0$ has three distinct real roots. We denote by α_1, α_2 and α_3 the roots of this equation. By Viète's formulas one root of this equation is positive and two roots are negative. Let $\alpha_1 > 0$. Then $\lambda^3 - a_1\lambda - a_2 = (\lambda - \alpha_1)\left(\lambda^2 + \alpha_1\lambda + \alpha_1^2 - a_1\right) = 0$, $\alpha_2 = \frac{-\alpha_1 + \sqrt{4a_1 - 3\alpha_1^2}}{2}$, $\alpha_3 = \frac{-\alpha_1 - \sqrt{4a_1 - 3\alpha_1^2}}{2}$ and $\sqrt{a_1} < \alpha_1 < \frac{2}{\sqrt{3}}\sqrt{a_1}$. It is obvious that $\alpha_3 < \alpha_2 < \alpha_1$. Therefore the inequality $\lambda^3 - a_1\lambda - a_2 \ge 0$ holds if and only if $\lambda \in [\alpha_3, \alpha_2] \bigcup [\alpha_1, \infty)$ and by the definition of the norm we have $\lambda \ge \alpha_1$. Thus, $\|f\|_{L_{p(x)}(R^n)} = \alpha_1$.

Example 1.3. Let $n = 1$, $x \in [1, \infty)$, $p(x) = x$ and $f(x) = 1$.

We calculate $\|1\|_{L_{p(x)}([1,\infty))}$. We have

$$\|1\|_{L_{p(x)}([1,\infty))} = \inf\left\{\lambda > 0 : \int_1^\infty \frac{1}{\lambda^x}\, dx \le 1\right\}.$$

It is obvious that $\int_1^\infty \frac{1}{\lambda^x}\, dx = \frac{1}{\lambda \ln \lambda}$ if $\lambda > 1$ and

$$\inf\left\{\lambda > 0 : \int_1^\infty \frac{1}{\lambda^x}\, dx \le 1\right\} = \inf\left\{\lambda > 1 : \frac{1}{\lambda \ln \lambda} \le 1\right\} = \inf\left\{\lambda > 1 : \lambda^\lambda \ge e\right\}.$$

Thus, $\|1\|_{L_{p(x)}([1,\infty))} = 1,7712\cdots$

In [2] the following theorem is proved.

Theorem 1.4. *Let $1 \le \underline{p} \le p(x) \le q(y) \le \overline{q} < \infty$ for all $x \in \Omega_1 \subset R^n$ and $y \in \Omega_2 \subset R^m$. If $p \in C\left(\overline{\Omega_1}\right)$, then the inequality*

$$\left\| \|f\|_{L_{p(\cdot)}(\Omega_1)} \right\|_{L_{q(\cdot)}(\Omega_2)} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q} - \underline{p}}{\overline{q}}\right)^{\frac{2}{p}} \left\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \right\|_{L_{p(\cdot)}(\Omega_1)}$$

is valid, where $\underline{q} = ess \inf_{\Omega_2} q(x)$, $\overline{q} = ess \sup_{\Omega_2} q(x)$ and $C\left(\Omega_1\right)$ is the space of continuous functions in Ω_1 and $f : \Omega_1 \times \Omega_2 \to R$ is any measurable function such that

$$\|\|f\|_{q,\Omega_2}\|_{p,\Omega_1} = \inf\left\{\mu > 0 : \int_{\Omega_1} \left(\frac{\|f(x,\cdot)\|_{q(\cdot),\Omega_2}}{\mu}\right)^{p(x)} dx \le 1\right\} < \infty.$$

Let $Hf(x) = \int_{|y|<|x|} f(y)\, dy$, where $f \ge 0$ and $B(0,|x|) = \{y \in R^n;\ |y| < |x|\}$.
Now we formulate the criteria on boundedness of multidimensional Hardy type operator in weighted variable Lebesgue spaces. In [3] the following theorem is proved.

Theorem 1.5. *Let $q(\cdot)$ be a measurable function on R^n, $1 < p \le q(x) \le \overline{q} < \infty$ and $p' = \frac{p}{p-1}$. Suppose that v and w are weights on R^n. Then the inequality*

$$\|Hf\|_{L_{q(\cdot),\, w}(R^n)} \le C\ \|f\|_{L_{p,\, v}(R^n)} \tag{1.1}$$

holds, for every $f \ge 0$ if and only if there exists $\alpha \in (0,1)$ such that

$$A(\alpha,p,q) = \sup_{t>0} \left(\int_{|y|<t} v^{-p'}(y)\, dy\right)^{\frac{\alpha}{p'}} \left\| \left(\int_{|y|<|\cdot|} v^{-p'}(y)\, dy\right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),\, w}(|x|>t)} < \infty. \tag{1.2}$$

Moreover, if $C > 0$ is the best possible constant in (1.1) *then*

$$\sup_{0<\alpha<1} \frac{p'\, A(\alpha, p, q)}{(1-\alpha)\,\left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)}\right]^{1/p}} \le C \le$$

$$\le \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \inf_{0<\alpha<1} \frac{A(\alpha,p,q)}{(1-\alpha)^{1/p'}}.$$

Remark 1.6. Note that Theorem 1.5 in the case $n = 1$, $q(x) = q = const$ for $x \in (0, \infty)$ and $\alpha = \frac{s-1}{p-1}$ $(1 < s < p)$ was proved in [20] and in multidimensional Hardy type operators it was proved in [3]. Two-weighted criterion for one-dimensional Hardy operator in weighted variable $L_{p(x),\,w}([0,1])$ spaces was proved in [13]. Note that Theorem 1.5 in the case $n = 1$, $p(x) = p = const$ $q(x) = q = const$ for $x \in (0, \infty)$ was proved in [4], [15] and etc.

2. Main results

We consider the multidimensional geometric mean operator defined as

$$Gf(x) = \exp\left(\frac{1}{|B(0,\,|x|)|}\int\limits_{B(0,|x|)} \ln\, f(y)\, dy\right),$$

where $f > 0$ and $|B(0,\,|x|)| = |B(0,1)|\,|x|^n$. It is obvious that $G\,(f_1 \cdot f_2)\,(x) = Gf_1(x) \cdot Gf_2(x)$.

Now we formulate a two-weight criterion on boundedness of multidimensional geometric mean operator in variable Lebesgue spaces.

Theorem 2.1. *Let $q(\cdot)$ be a measurable function on R^n and $0 < p \le q(x) \le \overline{q} < \infty$. Suppose that v and w are weights on R^n. Then the inequality*

$$\|Gf\|_{L_{q(\cdot),\,w}(R^n)} \le C\ \|f\|_{L_{p,\,v}(R^n)} \tag{2.1}$$

holds, for every $f > 0$ if and only if there exists $s \in (1, p)$ such that

$$D(s,p,q)$$

$$= \sup_{t>0} |B(0,t)|^{\frac{s-1}{p}} \left\| \frac{w(\cdot)}{|B(0,|\cdot|)|^{\frac{s}{p}}} \exp\left(\frac{1}{|B(0,|\cdot|)|}\int\limits_{B(0,|\cdot|)} \ln\frac{1}{v(y)}\, dy\right)\right\|_{L_{q(\cdot)}(|x|>t)} < \infty. \tag{2.2}$$

Moreover, if $C > 0$ is the best possible constant in (2.1), *then*

$$\sup_{s>1} \frac{e^{\frac{s}{p}}}{\left(e^s + \frac{1}{s-1}\right)^{1/p}} D(s,p,q) \le C \le \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \inf_{s>1} e^{\frac{s-1}{p}}\, D(s,p,q).$$

Proof. Let $\alpha = \dfrac{s-1}{p-1}$, where $1 < s < p$. We replace f with f^β, v with v^β, w with $\dfrac{w^\beta(x)}{|B(0,|x|)|}$, $0 < \beta < p$, and p with $\dfrac{p}{\beta}$ and $q(x)$ with $\dfrac{q(x)}{\beta}$ in (1.1), (1.2). We find that for $1 < s < \dfrac{p}{\beta}$

$$\left\| \frac{w^\beta}{|B(0,|\cdot|)|}\, H(f^\beta) \right\|_{L_{q(\cdot)/\beta}(R^n)} = \left\| \left(\frac{1}{|B(0,|\cdot|)|} \int\limits_{B(0,|\cdot|)} f^\beta(y)\, dy \right)^{1/\beta} \right\|^\beta_{L_{q(\cdot),\,w}(R^n)}$$

$$\leq C_\beta \left(\int\limits_{R^n} [f(y)v(y)]^p\, dy \right)^{\beta/p}.$$

Then the inequality

$$\left\| \left(\frac{1}{|B(0,|\cdot|)|} \int\limits_{B(0,|\cdot|)} f^\beta(y)\, dy \right)^{1/\beta} \right\|_{L_{q(\cdot),\,w}(R^n)} \leq C_\beta^{1/\beta} \left(\int\limits_{R^n} [f(y)v(y)]^p\, dy \right)^{1/p} \tag{2.3}$$

holds if and only if

$$A\left(\frac{s-1}{p-1}, \frac{p}{\beta}, \frac{q}{\beta} \right)$$

$$= \left[\sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-\frac{\beta p}{p-\beta}}\, dy \right)^{\frac{s-1}{p}} \left\| \left(\frac{1}{|B(0,|\cdot|)|^{\frac{p}{p-\beta s}}} \int\limits_{|y|<|\cdot|} [v(y)]^{-\frac{\beta p}{p-\beta}}\, dy \right)^{\frac{p-\beta s}{\beta p}} \right\|_{L_{q(\cdot),\,w}(|x|>t)} \right]^\beta$$

$$= B^\beta(s,p,q,\beta) < \infty$$

and

$$\sup_{1<s<\frac{p}{\beta}} \left[\frac{\left(\frac{p}{p-s\beta}\right)^{\frac{p}{\beta}}}{\left(\frac{p}{p-s\beta}\right)^{\frac{p}{\beta}} + \frac{1}{s-1}} \right]^{\beta/p} B^\beta(s,p,q,\beta) \leq C_\beta$$

$$\leq \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}} \right)^{\frac{2\beta}{p}} \inf_{1<s<\frac{p}{\beta}} \left(\frac{p-\beta}{p-s\beta} \right)^{\frac{p-\beta}{p}} B^\beta(s,p,q,\beta), \tag{2.4}$$

where $\underline{q}$ is replaced by $\dfrac{\underline{q}}{\beta}$ and $\overline{q}$ is replaced by $\dfrac{\overline{q}}{\beta}$. By L'Hospital rule, we get

$$\lim_{\beta\to+0} \left(\frac{1}{|B(0,|x|)|^{\frac{p}{p-\beta s}}} \int\limits_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}}\, dy \right)^{\frac{p-\beta s}{\beta p}}$$

$$= \lim_{\beta\to+0} \exp\left[\frac{p\,\ln\frac{1}{|B(0,|x|)|} + (p-\beta s)\,\ln\left(\int\limits_{|y|<|x|}[v(y)]^{-\frac{\beta p}{p-\beta}}\,dy\right)}{p\,\beta}\right]$$

$$= \lim_{\beta\to+0} \exp\left[-\frac{s}{p}\,\ln\left(\int\limits_{|y|<|x|}[v(y)]^{-\frac{\beta p}{p-\beta}}\,dy\right) + \frac{(p-\beta s)\left(\frac{p}{p-\beta}\right)^2 \int\limits_{|y|<|x|}[v(y)]^{-\frac{\beta p}{p-\beta}}\ln\frac{1}{v(y)}\,dy}{p\int\limits_{|y|<|x|}[v(y)]^{-\frac{\beta p}{p-\beta}}\,dy}\right]$$

$$= \exp\left[\frac{s}{p}\,\ln\frac{1}{|B(0,|x|)|} + \frac{\int\limits_{|y|<|x|}\ln\frac{1}{v(y)}\,dy}{|B(0,|x|)|}\right] = \frac{1}{|B(0,|x|)|^{\frac{s}{p}}}\,\exp\left(\frac{1}{|B(0,|x|)|}\int\limits_{B(0,|x|)}\ln\frac{1}{v(y)}\,dy\right).$$

Therefore

$$\lim_{\beta\to+0} B\,(s,p,q,\beta)$$

$$= \sup_{t>0}|B(0,t)|^{\frac{s-1}{p}}\left\|\frac{w(\cdot)}{|B(0,|\cdot|)|^{\frac{s}{p}}}\,\exp\left(\frac{1}{|B(0,|\cdot|)|}\int\limits_{B(0,|\cdot|)}\ln\frac{1}{v(y)}\,dy\right)\right\|_{L_{q(\cdot)}(|x|>t)}$$

$$= D(s,p,q) < \infty$$

and

$$\sup_{s>1}\frac{e^{\frac{s}{p}}}{\left(e^s+\frac{1}{s-1}\right)^{1/p}}D(s,p,q) \le \lim_{\beta\to+0} C_\beta^{1/\beta} \le \left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\inf_{s>1} e^{\frac{s-1}{p}}\,D(s,p,q). \quad (2.5)$$

Further, we have

$$\lim_{\beta\to+0}\left(\frac{1}{|B(0,|x|)|}\int\limits_{B(0,|x|)}f^\beta(y)\,dy\right)^{1/\beta} = \exp\left(\frac{1}{|B(0,\,|x|)|}\int\limits_{B(0,|x|)}\ln\,f(y)\,dy\right).$$

Formula (2.4) implies $\lim\limits_{\beta\to+0} C_\beta = 1$, and according to (2.2) and (2.5) $\lim\limits_{\beta\to+0} C_\beta^{1/\beta}$ $= C < \infty$. Therefore the inequality (2.3) is valid. Moreover, from (2.3) for $\beta\to+0$ we obtain

$$\|Gf\|_{L_{q(\cdot),\,w}(R^n)} \le C\ \|f\|_{L_{p,\,v}(R^n)}$$

and by (2.5)

$$\sup_{s>1}\frac{e^{\frac{s}{p}}}{\left(e^s+\frac{1}{s-1}\right)^{1/p}}\,D(s,p,q) \le C \le \left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\inf_{s>1} e^{\frac{s-1}{p}}\,D(s,p,q).$$

This completes the proof of Theorem 2.1. $\square$

Remark 2.2. Let $q(x) = q = const$ and $n = 1$. Note that the simplest case of (2.1) with $v = w = 1$ and $p = q = 1$ was considered in [9] and in [12]. Later on, this inequality was generalized in various ways by many authors in [5], [6], [10], [11], [20] and etc.

Corollary 2.3. *Let $q(x) = q = const$, $0 < p \le q < \infty$ and let f be a positive function on R^n. Then*

$$\left(\int\limits_{R^n} [Gf(x)]^q \, |x|^{\gamma\, q} \, dx\right)^{1/q} \le C \left(\int\limits_{R^n} f^p(x) \, |x|^{\beta\, p} \, dx\right)^{1/p} \tag{2.6}$$

holds with a finite constant C if and only if

$$\gamma + \frac{n}{q} = \frac{\beta}{n} + \frac{n}{p}$$

and the best constant C has the following condition:

$$\sqrt[q]{\frac{p}{nq}} \, e^{\frac{\beta}{n^2}} \, |B(0,\,1)|^{\frac{1}{q}-\frac{1}{p}} \sup_{s>1} \frac{e^{\frac{s}{p}} \, (s-1)^{\frac{1}{p}-\frac{1}{q}}}{[(s-1)e^s+1]^{1/p}} \le C \le \frac{|B(0,\,1)|^{\frac{1}{q}-\frac{1}{p}} \; e^{\frac{\beta}{n^2}+\frac{1}{q}}}{\sqrt[q]{n}}.$$

Remark 2.4. Note that if $p = q$, then the inequality (2.6) is sharp with the constant $C = \dfrac{e^{\frac{\beta}{n^2}+\frac{1}{p}}}{\sqrt[p]{n}}$.

Corollary 2.5. *Let $x \in R^n$, $0 < p \le q(x) < \infty$, $q(x) = \begin{cases} 1 & for \quad |x| < 1 \\ 2 & for \quad |x| \ge 1, \end{cases}$ and let f be a positive function on R^n. Suppose that $v(x) = 1$ and $w(x) = |x|^\beta$. Then*

$$\|Gf\|_{L_{q(\cdot),\,|\cdot|^\beta}(R^n)} \le C \left(\int\limits_{R^n} f^p(x) \, dx\right)^{1/p}$$

holds with a finite constant C if and only if

$$n\left(\frac{s}{p} - 1\right) \le \beta \le n\left(\frac{1}{p} - \frac{1}{2}\right), \quad s \in \left(1, 1 + \frac{p}{2}\right]$$

and the best constant C has the following condition:

$$\sup_{1<s\le 1+\frac{p}{2}} \frac{e^{\frac{s}{p}}}{\left(e^s + \frac{1}{s-1}\right)^{1/p}} D'(s,p,q) \le C \le \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \inf_{1<s\le 1+\frac{p}{2}} e^{\frac{s-1}{p}} D'(s,p,q),$$

where $D'(s,p,q) = |B(0,1)|^{-\frac{1}{p}} \sup\limits_{t>0} t^{\frac{n(s-1)}{p}} \left\| \, | \cdot |^{\beta - \frac{ns}{p}} \right\|_{L_{q(\cdot)}(|\cdot|>t)} < \infty.$

Now we consider an application in the theory of nonlinear ordinary differential equation. Let $L(t,\omega,y) = \left\|\omega \, y^{1/p'}\right\|_{L_{q(x)}(x>t)}$, where $t \in (0,\infty)$ and ω is a weight function defined on $(0,\infty)$.

Lemma 2.6. *Let $1 < p \le q(x) \le \overline{q} < \infty$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$. Let the equation*

$$L\left(t,\omega_2,y\right) - \lambda\omega_1(t)\left(y'(t)\right)^{1/p'} = 0, \quad (\lambda > 0) \tag{2.7}$$

have a solution y such that

$$y(t) > 0, \;\; y'(t) > 0, \;\; y \in AC(0,\,\infty). \tag{2.8}$$

Then the weighted norm inequality

$$\|u\|_{L_{q(\cdot),\omega_2}(0,\infty)} \le \lambda \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \|u'\|_{L_{p,\omega_1}(0,\infty)}$$

holds, where $u \in AC(0,\,\infty)$ and $u(0) = \lim\limits_{t\to+0} u(t) = 0$.

Proof. Applying the Hölder inequality we have

$$u(x) = \int_0^x u'(t)\,dt = \int_0^x u'(t)(y'(t))^{-\frac{1}{p'}}\left(y'(t)\right)^{\frac{1}{p'}}dt$$

$$\le \left(\int_0^x y'(t)\,dt\right)^{\frac{1}{p'}} \left\|u'\,\left(y'\right)^{-\frac{1}{p'}}\right\|_{L_p(0,\,x)} \le (y(x))^{\frac{1}{p'}} \left\|u'\,\left(y'\right)^{-\frac{1}{p'}}\right\|_{L_p(0,\,x)}.$$

Thus

$$\|u\|_{L_{q(\cdot),\,\omega_2}(0,\infty)} \le \left\|\omega_2(\cdot)\,(y(\cdot))^{\frac{1}{p'}}\;\|u'\,\left(y'\right)^{-\frac{1}{p'}}\|_{L_p(0,\,\cdot)}\right\|_{L_{q(\cdot)}(0,\infty)}$$

$$= \left\|\left\|\omega_2(\cdot)(y(\cdot))^{\frac{1}{p'}}\,u'\,\left(y'\right)^{-\frac{1}{p'}}\,\chi_{(0,\,\cdot)}\right\|_{L_p(0,\infty)}\right\|_{L_{q(\cdot)}(0,\infty)}.$$

By using Theorem 1.4 we have

$$\left\|\left\|\omega_2(\cdot)(y(\cdot))^{\frac{1}{p'}}\,u'\,\left(y'\right)^{-\frac{1}{p'}}\,\chi_{(0,\,\cdot)}\right\|_{L_p(0,\infty)}\right\|_{L_{q(\cdot)}(0,\infty)}$$

$$\le \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \left\|\left\|\omega_2(\cdot)(y(\cdot))^{\frac{1}{p'}}\,u'\,\left(y'\right)^{-\frac{1}{p'}}\,\chi_{(0,\,\cdot)}\right\|_{L_{q(\cdot)}(0,\infty)}\right\|_{L_p(0,\infty)}$$

$$= \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \left\|\left\|\omega_2\,y^{\frac{1}{p'}}\right\|_{L_{q(\cdot)}(t,\infty)}\,u'\,\left(y'\right)^{-\frac{1}{p'}}\right\|_{L_p(0,\infty)}$$

$$= \lambda\left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \left\|\omega_1\,\left(y'\right)^{\frac{1}{p'}}\,u'\,\left(y'\right)^{-\frac{1}{p'}}\right\|_{L_p(0,\infty)}$$

$$= \lambda\left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \|\omega_1\,u'\|_{L_p(0,\infty)} = \lambda\left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \|u'\|_{L_{p,\,\omega_1}(0,\infty)}.$$

This proves the Lemma 2.6. □

Put

$$K = p' \inf \sup_{x>0} \frac{1}{f(x) - x - p' \int\limits_0^t \frac{\omega_1'(s)\, f(s)}{\omega_1(s)}\, ds} \int\limits_0^x \frac{(f(t))^{1/p'+1} P\,(t, \omega_2, y, f)}{\omega_1(t)\,(y(t))^{1/p'}}\, dt, \quad (2.9)$$

where $P\,(t, \omega_2, y, f) \geq 0$ for all $t > 0$, $y(t)$ is a fixed positive solution of equation (2.7) and the infimum is taken over the class of measurable functions such that

$$f(x) > x + p' \int\limits_0^t \frac{\omega_1'(s)\, f(s)}{\omega_1(s)}\, ds \quad \text{for all} \quad x > 0.$$

The following lemma gives the relation between the number K and the problem (2.7), (2.8).

Lemma 2.7. *Let $\lambda > 0$ be the number from Lemma 2.6 and let K be given by (2.9). Suppose that ω_1 and ω_2 are weight functions defined on $(0, \infty)$ and the derivative $\omega_1'(t)$ exists for all $t \in (0, \infty)$ and $\omega_1(t) \geq \omega_1(0) > 0$.*

Then the following statements are equivalent:

(i) if the problem (2.7), (2.8) has a solution with a locally absolutely continuous first derivative, then $\lambda \geq K$;

(ii) if $K < +\infty$, then the problem (2.7), (2.8) has a solution for every $\lambda > K$.

Proof. Assume that (i) holds. Let $y_0(x)$ be a solution of (2.7), (2.8). Let us take $w = \dfrac{y_0}{y_0'}$. Assume that $P\,(t, \omega_2, y_0, w) = -\dfrac{d}{dt} L\,(t, \omega_2, y_0) = P(t)$. It is obvious that $P(t) \geq 0$ for all $t > 0$. Then by virtue of (2.7) w is a positive solution of the equation

$$w'(t) = \frac{p'\, \omega_1'(t)\, w(t)}{\omega_1(t)} + \frac{p'(w(t))^{1/p'+1} P(t)}{\lambda\, \omega_1(t)\, \left(y_0(t)\right)^{1/p'}} + 1. \quad (2.10)$$

Hence (2.10) implies

$$w(t) \geq \int\limits_0^t w'(s)\, ds = p' \int\limits_0^t \frac{\omega_1'(s)\, w(s)}{\omega_1(s)}\, ds + \frac{p'}{\lambda} \int\limits_0^t \frac{(w(s))^{1/p'+1}\, P(s)}{\omega_1(s)\, \left(y_0(s)\right)^{1/p'}}\, ds + t. \quad (2.11)$$

From (2.11) implies that

$$\lambda \geq \frac{p'}{w(t) - t - p' \int\limits_0^t \frac{\omega_1'(s)\, w(s)}{\omega_1(s)}\, ds} \int\limits_0^t \frac{(w(s))^{1/p'+1}\, P(s)}{\omega_1(s)\, \left(y_0(s)\right)^{1/p'}}\, ds. \quad (2.12)$$

From (2.11), (2.12) and (2.9) it follows that $\lambda \geq K$ and the proof of $(i) \Rightarrow (ii)$ is complete.

Assume now (ii) holds. Let us fix $\lambda > K$. By the definition of K there exists a measurable function $f(x)$ such that

$$f(x) \geq x + p' \int_0^x \frac{\omega_1'(t)\, f(t)}{\omega_1(t)}\, dt + \frac{p'}{\lambda} \int_0^x \frac{(f(t))^{1/p'+1} P(t)}{\omega_1(t)\ (y(t))^{1/p'}}\, dt. \tag{2.13}$$

Let us define a sequence $w_n(x)$ by setting

$$w_0(x) = f(x),$$

$$w_n(x) = x + p' \int_0^t \frac{\omega_1'(s)\, w_{n-1}(s)}{\omega_1(s)}\, ds + \frac{p'}{\lambda} \int_0^x \frac{(w_{n-1}(t))^{1/p'+1}\, P_{n-1}(t)}{\omega_1(t)\ (y(t))^{1/p'}}\, dt, \tag{2.14}$$

where $P_0(t) = P(t)$ and $P_n(t) \geq 0$ for all $n \in \mathbb{N}$. From (2.13) it follows that $w_0(x) \geq w_1(x)$. We put $w_{n-1}(x) \geq w_n(x)$ and let $P_n(t)$ be decreasing sequences with respect to n on $(0, \infty)$, where $n \in \mathbb{N}$. Then

$$\int_0^x \frac{(w_{n-1}(t))^{1/p'+1}\, P_{n-1}(t)}{\omega_1(t)\ (y(t))^{1/p'}}\, dt \geq \int_0^x \frac{(w_n(t))^{1/p'+1}\, P_n(t)}{\omega_1(t)\ (y(t))^{1/p'}}\, dt$$

and

$$w_n(x) - w_{n+1}(x) \geq p' \int_0^x \frac{\omega_1'(s)\ (w_{n-1}(s) - w_n(s))}{\omega_1(s)}\, ds$$

$$\geq \inf_{s\in(0,\infty)} (w_{n-1}(s) - w_n(s)) \int_0^x \frac{\omega_1'(s)}{\omega_1(s)}\, ds = \inf_{t\in(0,\infty)} (w_{n-1}(t) - w_n(t)) \ln \frac{\omega_1(x)}{\omega_1(0)} \geq 0.$$

Since $w_n(x) \geq 0$, the sequence (2.14) converges. We denote its limit by $w(x)$. By the Levi monotone convergence theorem it follows that w is a nonnegative solution of the equation

$$w(x) = x + p' \int_0^x \frac{\omega_1'(t)\, w(t)}{\omega_1(t)}\, dt + \frac{p'}{\lambda} \int_0^x \frac{(w(t))^{1/p'+1} P(t)}{\omega_1(t)\ (y(t))^{1/p'}}\, dt,$$

where $P(t) = \lim\limits_{n\to\infty} P_n(t)$. Hence, w is absolutely continuous and satisfies the equation

$$w'(x) = 1 + \frac{p'\, \omega_1'(x)\, w(x)}{\omega_1(x)} + \frac{p'}{\lambda}\, \frac{(w(x))^{1/p'+1} P(x)}{\omega_1(x)\ (y(x))^{1/p'}}.$$

Therefore the function

$$y_0(x) = e^{\int_a^x \frac{dt}{w(t)}} \quad (a \ \text{ be fixed in } (0, \infty))$$

satisfies the problem (2.7), (2.8).

This completes the proof of Lemma 2.7. $\square$

Thus, we have the following

Theorem 2.8. *Let $1 < p \leq q(x) \leq \overline{q} < \infty$ and $K < +\infty$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$ and the derivative $\omega_1'(t)$ exists for all $t \in (0,\infty)$ and $\omega_1(t) \geq \omega_1(0) > 0$. Then the following statements are equivalent:*

a) there is a positive solution of the equation

$$L\left(t,\omega_2,y\right) - \lambda\omega_1(t)\left(y'(t)\right)^{1/p'} = 0,$$

$$y(t) > 0,\ \ y'(t) > 0,\ \ y \in AC(0,\,\infty),$$

where $\lambda > 0$;

b) the weighted norm inequality

$$\|u\|_{L_{q(\cdot),\omega_2}(0,\infty)} \leq C_0\|u'\|_{L_{p,\omega_1}(0,\infty)}$$

holds, where $u \in AC(0,\,\infty)$, $u(0) = \lim\limits_{t\to+0} u(t) = 0$ and $C_0 > 0$ is independent of u.

Remark 2.9. Note that for $q(x) = q = const$ and $1 < p \leq q < \infty$ Theorem 2.8 was proved in [8] and the proof of Theorem 2.8 is based on the paper [8].

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