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PROPERTIES OF SOME DYNAMIC INTEGRAL EQUATION ON TIME SCALES

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ABSTRACT. The main objective of the paper is to study the existence, uniqueness and other properties of solution of certain dynamic integral equation on time scales. The tools employed are based on the application of the Banach fixed point theorem and certain integral inequality with explicit estimates on time scales.

1. Introduction

In 1988 Hilger [7] initiated the study of time scales which unifies the continuous and discrete analysis. An excellent account on time scale can be found in two recent books [5, 6] by Bohner and Peterson. Since then many authors have studied Properties of solutions of integral equations on time scales [2, 8, 10, 11]. In [3] the authors have studied the integral equation in which the functions involved under the integral sign contains the derivative of a unknown functions using Pervo's fixed point theorem, and successive approximation. In [1] continuous dependence of solution is studied. In this paper we consider the general nonlinear dynamic integral equations on time scales of the type

$$x(t) = f(t) + \int_{\alpha}^{t} g\left(t, \tau, x\left(\tau\right), x^{\Delta}\left(\tau\right)\right) \Delta\tau + \int_{\alpha}^{\beta} h\left(t, \tau, x\left(\tau\right), x^{\Delta}\left(\tau\right)\right) \Delta\tau, \tag{1.1}$$

f, g, h are given functions and x is unknown function to be found, $f : I_{\mathbb{T}} \to \mathbb{R}^n$, $g, h : I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous function, t is from a time scale $\mathbb{T}$, which is nonempty closed subset of $\mathbb{R}$, the set of real numbers $\tau \leq t$ and $I_{\mathbb{T}} = I \cap \mathbb{T}$,

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$I = [t_0, \infty)$ the given subset of $\mathbb{R}$, $\mathbb{R}^n$ the real n dimensional Euclidean space with appropriate norm defined by $|.|$. The integral sign represents the delta integral.

2. Preliminaries

In what follows $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes arbitrary time scale, jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ denoted by

$$\sigma(t) = \inf\{s \in T : s > t\}, \quad \rho(t) = \sup\{s \in T : s > t\}.$$

The jump operators classify the points of time scale $\mathbb{T}$ as left dense, left scattered, right dense and right scattered according to whether $\rho(t) = t$ or $\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$ respectively for $t \in \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum M define $\mathbb{T}^k = \mathbb{T} - m$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - M$, otherwise $\mathbb{T}^k = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t is denoted by $f^{\Delta}(t)$ is the number (provided if exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t such that

$$\left| f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}$. In this case we define delta integral of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Excellent information about introduction to time scales can be found in [5, 6].

3. Main Results

By a solution of equation (1.1) means a rd-continuous function $x(t)$ for $t \in \mathbb{T}$ which is rd-continuous delta differentiable with respect to t and satisfies the equation (1.1). For every rd-continuous delta differentiable function $x(t)$ together with its rd-continuous delta derivative $x^{\Delta}(t)$ for $t \in I_{\mathbb{T}}$. We denote by

$$|x(t)|_1 = |x(t)| + \left| x^{\Delta}(t) \right|.$$

For $t \in I_{\mathbb{T}}$ the notation $a(t) = O(b(t))$ for $|t \to \alpha|$ means that there exists a constant $q > 0$ such that $\left| \frac{a(t)}{b(t)} \right| \leq q$ on some right hand neighborhood of the point t_0, we denote by G the space of all rd-continuous function $x(t)$ whose delta derivative $x^{\Delta}(t)$ exists, which fulfill the condition

$$|x(t)|_1 = O(e_\lambda(t, t_0)), \tag{3.1}$$

where λ is positive constant. In the space G we define the norm

$$|x|_G = \sup\left\{ |x(t)|_1 e_\lambda(t, t_0) : t \in I_{\mathbb{T}} \right\}. \tag{3.2}$$

It is easy to see that G with norm defined in (3.2) is a Banach space. The condition (3.2) implies that there is a constant $N \geq 0$ such that

$$|x(t)|_1 \leq N e_\lambda(t, t_0), \tag{3.3}$$

Using this fact in (3.2) we observe that

$$|x|_G \leq N.$$

We need following inequality which is important in proving our results.

Lemma 3.1. *Let $u(t) \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$, $h(t,s), k(t,s) \in C_{rd}(I_{\mathbb{T}}^2, \mathbb{R}_+)$ be nondecreasing in $t \in \mathbb{T}$ for each $\tau \in I_{\mathbb{T}}$ and*

$$u(t) \leq c + \int_{\alpha}^{t} h(t,\tau)\, u(\tau)\, \Delta\tau + \int_{\alpha}^{\beta} K(t,\tau) u(\tau)\, \Delta\tau, \tag{3.4}$$

for $t \in I_{\mathbb{T}}$ where $c \geq 0$ is a constant. If

$$d(t) = \int_{\alpha}^{\beta} k(t,\tau) e_k(\tau,\alpha)\, \Delta\tau < 1, \tag{3.5}$$

for $t \in I_{\mathbb{T}}$, then

$$u(t) \leq \frac{c}{1-dt} e_h(t,\alpha). \tag{3.6}$$

Proof. Fix any X, $\alpha \leq X \leq \beta$, then for $\alpha \leq t \leq X$ we have

$$u(t) \leq c + \int_{\alpha}^{t} h(X,\tau) u(\tau) + \int_{\alpha}^{\beta} k(X,\tau)\, u(\tau)\, \Delta\tau, \tag{3.7}$$

Define a function $p(t,X)$, $\alpha \leq t \leq X$ on the right hand side of (3.4). Then $u(t) \leq p(t,X)$, $\alpha \leq t \leq X$

$$p(\alpha, X) = c + \int_{\alpha}^{\beta} k(X,\tau) u(\tau)\, \Delta\tau, \tag{3.8}$$

and

$$p^{\Delta t}(t,X) = h(X,t)\, u(t) \leq h(X,t)\, p(t,X), \tag{3.9}$$

for $\alpha \leq X$. By setting $t = \tau$ in (3.6) and integrating it with respect to τ from α to X we get

$$p(X, \sigma(X)) \leq p(\alpha, X)\, e_h(X,\alpha), \tag{3.10}$$

Since X is arbitrary from (3.7) and (3.5) with X replaced by t and $u(t) \leq p(t,\sigma(t))$ we have

$$u(t) \leq p(\alpha,t)\, e_h(t,\alpha), \tag{3.11}$$

where

$$p(\alpha,t) = c + \int_{\alpha}^{\beta} k(t,\tau) u(\tau)\, \Delta\tau, \tag{3.12}$$

Using (3.8) on the right hand side of (3.9) and condition (3.2). It is easy to see that

$$p(\alpha,t) = \frac{c}{1-d(t)}, \tag{3.13}$$

Using (3.10) in (3.8) we get the desired inequality (3.3). □

In what follows, we assume that the functions $g, g^{\Delta}, h, h^{\Delta} : I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous.

Theorem 3.2. *Assume that*

(i) The function g, h and its delta derivative with respect to t are rd-continuous and satisfy the conditions

$$|g(t,\tau,u,v) - g(t,\tau,\bar{u},\bar{v})| \le p_1(t,\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{3.14}$$

$$\left|g^{\Delta}(t,\tau,u,v) - g^{\Delta}(t,\tau,\bar{u},\bar{v})\right| \le p_2(t,\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{3.15}$$

$$|h(t,\tau,u,v) - h(t,\tau,\bar{u},\bar{v})| \le q_1(t,\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{3.16}$$

$$\left|h^{\Delta}(t,\tau,u,v) - h^{\Delta}(t,\tau,\bar{u},\bar{v})\right| \le q_2(t,\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{3.17}$$

where $p_i(t,\tau)$, $q_i(t,\tau) \in C_{rd}\left(I_{\mathbb{T}}^2, \mathbb{R}_+\right)$, $(i = 1, 2)$,

(ii)for λ as in

(a) there exists a nonnegative constant μ such that $\mu \le 1$ and

$$p_1(\sigma(t),t)\, e_{\lambda}(t,\alpha) + \int_{\alpha}^{t} p(t,\tau) e_{\lambda}(\tau,\alpha)\,\Delta\tau + \int_{\alpha}^{\beta} q(t,\tau) e_{\lambda}(\tau,\alpha)\,\Delta\tau \le \alpha e_{\lambda}(t,\alpha), \tag{3.18}$$

for $t \in \mathbb{T}$, where $p(t,\tau) = p_1(t,\tau) + p_2(t,\tau)$, $q(t,\tau) = q_1(t,\tau) + q_2(t,\tau)$,

(b) there exists nonnegative constant ξ such that

$$\begin{aligned}|f(t)| + \left|f^{\Delta}(t)\right| + |g(\sigma(t),t,0)| + \int_{\alpha}^{t}\left[|g(t,\tau,0,0)| + \left|g^{\Delta}(t,\tau,0,0)\right|\right]\Delta\tau \\ + \int_{\alpha}^{\beta}\left[|h(t,\tau,0,0)| + \left|h^{\Delta}(t,\tau,0,0)\right|\right]\Delta\tau \le \xi e_{\lambda}(t,\alpha),\end{aligned} \tag{3.19}$$

where f, g, h are as in equation (1.1).

Then equation (1.1) has a unique solution $x(t)$ in G on $I_{\mathbb{T}}$.

Proof. Let $x : I_{\mathbb{T}} \to R^n$ be rd-continuous and define the operator S by

$$(Sx)(t) = f(t) + \int_{\alpha}^{t} g\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\Delta\tau + \int_{\alpha}^{\beta} h\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\Delta\tau. \tag{3.20}$$

Differentiating both sides of (3.20) with respect to t we have

$$\begin{aligned}(Sx)^{\Delta}(t) &= f(t) + g\left(\sigma(t),t,x(t),x^{\Delta}(t)\right) \\ &\quad + \int_{\alpha}^{t} g^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\Delta\tau + \int_{\alpha}^{\beta} h^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\Delta\tau.\end{aligned} \tag{3.21}$$

Now we show that Sx maps G onto itself, Sx and Sx^{Δ} are rd-continuous on $I_{\mathbb{T}}$. We first verify that (3.2) is satisfied. From (3.20) and (3.21), we have

$$\begin{aligned}|(Sx)(t)|_1 &= |(Sx)(t)| + \left|(Sx)^{\Delta}(t)\right| \\ &\le |f(t)| + \left|f^{\Delta}(t)\right| + \left|g\left(\sigma(t),t,x(t),x^{\Delta}(t)\right) - g(\sigma(t),t,0,0)\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|g\left(\sigma\left(t\right),t,0,0\right)\right|+\int\limits_{\alpha}^{t}\left|g\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(s\right)\right)-g(t,\tau,0,0)\right|\Delta s\\
&+\int\limits_{\alpha}^{t}\left|g\left(t,\tau,0,0\right)\right|\Delta s+\int\limits_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-g^{\Delta}\left(t,\tau,0,0\right)\right|\Delta s\\
&+\int\limits_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,0,0\right)\right|\Delta\tau+\int\limits_{\alpha}^{\beta}\left|h\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h(t,\tau,0,0)\right|\Delta\tau\\
&+\int\limits_{\alpha}^{\beta}\left|h\left(t,\tau,0,0\right)\right|\Delta\tau+\int\limits_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h^{\Delta}(t,\tau,0,0)\right|\Delta\tau\\
&+\int\limits_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,0,0\right)\right|\Delta\tau\\
&\leq\xi e_{\lambda}\left(t,\alpha\right)+p_{1}\left(\sigma\left(t\right),t\right)\left|x\left(t\right)\right|_{1}+\int\limits_{\alpha}^{t}p\left(t,\tau\right)\left|x\left(\tau\right)\right|_{1}\Delta\tau\\
&+\int\limits_{\alpha}^{\beta}q\left(t,\tau\right)\left|x\left(\tau\right)\right|_{1}\Delta\tau\\
&\leq\xi e_{\lambda}\left(t,\alpha\right)+\left|x\right|_{G}\left\{p_{1}\left(\sigma\left(t\right),t\right)e_{\lambda}\left(t,\alpha\right)+\int\limits_{\alpha}^{t}p\left(t,\tau\right)e_{\lambda}\left(t,\alpha\right)\Delta\tau\right.\\
&\left.+\int\limits_{\alpha}^{\beta}q\left(t,\tau\right)e_{\lambda}\left(\tau,\alpha\right)\Delta\tau\right\}\\
&\leq\beta e_{\lambda}\left(t,\alpha\right)+\left|x\right|_{G}\mu e_{\lambda}\left(t,\alpha\right)\\
&\leq\left[\beta+N\mu\right]e_{\lambda}\left(t,\alpha\right).
\end{aligned}
\tag{3.22}
$$

From (3.22) it follows that $S_x \in G$. This proves that S maps G onto itself. Now we verify that S is a contraction map.

$$
\begin{aligned}
\left|\left(Sx\right)\left(t\right)-\left(Sy\right)\left(t\right)\right|_{1}&=\left|\left(Sx\right)\left(t\right)-\left(Sy\right)\left(t\right)\right|+\left|\left(Sx\right)^{\Delta}\left(t\right)-\left(Sy\right)^{\Delta}\left(t\right)\right|\\
&\leq\left|g\left(\sigma\left(t\right),t,x\left(t\right),x\left(t\right)\right)-g\left(\sigma\left(t\right),t,y\left(t\right),y\left(t\right)\right)\right|\\
&+\int\limits_{\alpha}^{\beta}\left|g\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-g\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau
\end{aligned}
$$

$$+\int_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-g^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$\leq p_1\left(\sigma\left(t\right),t\right)\left|x\left(t\right)-y\left(t\right)\right|_1+\int_{\alpha}^{t}p\left(t,\tau\right)\left|x\left(\tau\right)-y\left(\tau\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}q\left(t,\tau\right)\left|x\left(\tau\right)-y\left(\tau\right)\right|_1\Delta\tau$$

$$\leq\left|x-y\right|_G\Bigg\{p_1\left(\sigma\left(t\right),t\right)e_{\lambda}\left(t,\alpha\right)+\int_{\alpha}^{t}p\left(t,\tau\right)e_{\lambda}\left(\tau,\alpha\right)\Delta\tau$$

$$+\int_{\alpha}^{\beta}q\left(t,\tau\right)e_{\lambda}\left(\tau,\alpha\right)\Delta\tau\Bigg\}$$

$$\leq\left|x-y\right|_G\mu e_{\lambda}\left(t,\alpha\right). \tag{3.23}$$

From (3.23) we obtain

$$\left|Sx-Sy\right|_G\leq\mu\left|x-y\right|_G.$$

Since $\alpha<1$, it follows from Banach Fixed Theorem that S has a unique fixed point in G. The fixed point of S is however solution of equation (1.1). □

Remark 3.3. We note that Bielecki [4] first used the norm defined in (3.2) for proving the global existence and uniqueness of solution of differential equation. It is now frequently used to obtain global existence and uniqueness results for wide class of equations.

The following theorem deals with the uniqueness of solution of equation (1.1).

Theorem 3.4. *Assume that function g,h in equation (1.1) and their delta derivative with respect to t satisfy the conditions (3.15)-(3.17). Further assume that the functions $h_i(t,\tau),q_i(t,\tau),(i=1,2)$ in (3.15)-(3.17) are nondecreasing in $t\in I_{\mathbb{T}}$ and each $\tau\in I_{\mathbb{T}}$*

$$p_1\left(\sigma\left(t\right),t\right)\leq m, \tag{3.24}$$

for $t\in I_{\mathbb{T}}$ where $m\geq 0$ is a constant such that $m<1$

$$r\left(t\right)=\int_{\alpha}^{\beta}\frac{1}{1-m}q\left(t,\tau\right)e_{\frac{1}{1-\alpha}p}\left(\tau,\alpha\right)\Delta\tau<1, \tag{3.25}$$

where

$$p(t,s) = p_1(t,s) + p_2(t,s), \quad q(t,s) = q_1(t,s) + q_2(t,s),$$

Then the equation (1.1) has at most one solution on $I_{\mathbb{T}}$.

Proof. Let $x(t)$ and $y(t)$ be two solutions of (1.1) and

$$w(t) = |x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right|.$$

Then by hypothesis we have

$$\begin{aligned} w(t) \leq & \int_{\alpha}^{t} \left|g\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - g\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \\ & + \int_{\alpha}^{\beta} \left|h\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - h\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \\ & + \left|g\left(\sigma(t),t,x(t),x^{\Delta}(t)\right) - g\left(\sigma(t),t,y(t),y^{\Delta}(t)\right)\right| \\ & + \int_{\alpha}^{\beta} \left|g^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - g^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \\ & + \int_{\alpha}^{\beta} \left|h^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - h^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \\ & \leq \int_{\alpha}^{\tau} p_1(t,\tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau \\ & + \int_{\alpha}^{\beta} q_1(t,\tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau \\ & + p_1(\sigma(t),t)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right] \\ & + \int_{\alpha}^{\tau} p_2(t,\tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau \\ & + \int_{\alpha}^{\beta} q_2(t,\tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau. \end{aligned} \tag{3.26}$$

Using (3.24) in (3.26) we observe that

$$w(t) \leq \frac{1}{1-m}\int_{\alpha}^{t} p(t,\tau)w(\tau)\,\Delta\tau + \frac{1}{1-m}\int_{\alpha}^{\beta} q(t,\tau)w(\tau)\,\Delta\tau. \tag{3.27}$$

Now a suitable application of Lemma 3.1 to (3.27) yields

$$|x(t) - y(t)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right| \leq 0,$$

and hence $x(t) = y(t)$, which proves the uniqueness of solution of equation (1.1) on $I_{\mathbb{T}}$. □

4. Bounds on Solutions

In this section we obtain estimates on the solutions of equation (1.1) under some suitable conditions on the functions involved and their derivatives.

The following theorem concerning an estimate on the solution of equation (1.1) holds.

Theorem 4.1. *Assume that function f, g, h in equation (1.1) and their delta derivative with respect to t satisfy the conditions*

$$|f(t)| + \left|f^{\Delta}(t)\right| \le \bar{c}, \tag{4.1}$$

$$|g(t,\tau,u,v)| \le m_1(t,\tau)[|u|+|v|], \tag{4.2}$$

$$\left|g^{\Delta}(t,\tau,u,v)\right| \le m_2(t,\tau)[|u|+|v|], \tag{4.3}$$

$$|h(t,\tau,u,v)| \le n_1(t,\tau)[|u|+|v|], \tag{4.4}$$

$$\left|h^{\Delta}(t,\tau,u,v)\right| \le n_2(t,\tau)[|u|+|v|], \tag{4.5}$$

where $\bar{c} \ge 0$ is a constant and for $i = 1, 2$, $m_i(t,\tau), n_i(t,\tau) \in C_{rd}(I_{\mathbb{T}}^2, \mathbb{R}_+)$, and they are nondecreasing for $t \in I_{\mathbb{T}}$ for each $\tau \in I_{\mathbb{T}}$ further assume that

$$m_1(\sigma(t), t) \le \overline{d}, \tag{4.6}$$

$$\bar{r}(t) = \int_{\alpha}^{\beta} \frac{1}{1-\overline{d}} n(t,\tau)\, e_{\frac{1}{1-\overline{d}}m}(\tau,\alpha)\,\Delta\tau < 1, \tag{4.7}$$

for $t \in I_{\mathbb{T}}$ where $\overline{d} \ge 0$ is a constant such that $\overline{d} < 1$ and

$$m(t,\tau) = m_1(t,\tau) + m_2(t,\tau), \quad n(t,\tau) = n_1(t,\tau) + n_2(t,\tau),$$

If $x(t)$, $t \in I_{\mathbb{T}}$ is any solution of equation (1.1), then

$$|x(t)| + \left|x^{\Delta}(t)\right| \le \left(\frac{\bar{c}}{1-\overline{d}}\right)\left(\frac{1}{1-\bar{r}(t)}\right) e_m(t,\alpha), \tag{4.8}$$

for $t \in I_{\mathbb{T}}$.

Proof. Let $u(t) = |x(t)| + \left|x^{\Delta}(t)\right|$ for $t \in I_{\mathbb{T}}$. Using the fact that $x(t)$ is a solution of equation (1.1) and the hypotheses we have

$$\begin{aligned} u(t) \le{}& |f(t)| + \left|f^{\Delta}(t)\right| + \int_{\alpha}^{t} \left|g\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau \\ &+ \int_{\alpha}^{t} \left|h\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau + \left|g\left(\sigma(t),t,x(t),x^{\Delta}(t)\right)\right| \\ &+ \int_{\alpha}^{\beta} \left|g^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau + \int_{\alpha}^{\beta} \left|h^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau \end{aligned}$$

$$\leq \bar{c} + \int_{\alpha}^{t} m_1(t,\tau)\, u(\tau)\Delta\tau + \int_{\alpha}^{\beta} n_1(t,\tau)\, u(\tau)\,\Delta\tau$$

$$+ m_1(\sigma(t),t)\, u(t) + \int_{\alpha}^{t} m_2(t,\tau) u(\tau)\,\Delta\tau + \int_{\alpha}^{\beta} n_2(t,\tau)\, u(\tau)\,\Delta\tau. \tag{4.9}$$

Using (4.6) in (4.9) we have

$$u(t) \leq \frac{\bar{c}}{1-\bar{d}} + \frac{1}{1-\bar{d}} \int_{\alpha}^{t} m(t,\tau)\, u(\tau)\Delta\tau + \frac{1}{1-\bar{d}} \int_{\alpha}^{\beta} n(t,\tau) u(\tau)\,\Delta\tau. \tag{4.10}$$

□

Now an application of 3.1 to (4.10) yields (4.8).

Remark 4.2. We note that estimate obtained in (4.10) yields not only the bound on the solution of equation (1.1) but also the bound on its delta derivative. If the estimate on the right hand side in (4.8) is bounded, then the solution of equation (1.1) and its delta derivative is bounded on $I_{\mathbb{T}}$.

Now we obtain estimate on the solution equation (1.1) assuming that the functions g, h and their delta derivatives with respect to t satisfy the Lipschitz type conditions.

Theorem 4.3. *Assume that the hypotheses of Theorem 3.4 hold. Suppose that*

$$\int_{\alpha}^{t} \left|g\left(t,\tau,f(\tau),f^{\Delta}(\tau)\right)\right| \Delta\tau + \int_{\alpha}^{\beta} \left|h\left(t,\tau,f(\tau),f^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+ \left|g\left(\sigma(t),t,f(t),f^{\Delta}(t)\right)\right| + \int_{\alpha}^{\beta} \left|g^{\Delta}\left(t,\tau,f(\tau),f^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+ \int_{\alpha}^{\beta} \left|h^{\Delta}\left(t,\tau,f(\tau),f^{\Delta}(\tau)\right)\right|\Delta\tau \leq D, \tag{4.11}$$

for $t \in I_{\mathbb{T}}$, where $D \geq 0$ is a constant. If $x(t)$, $t \in I_{\mathbb{T}}$ is any solution of equation (1.1) then

$$|x(t) - f(t)| + \left|x^{\Delta}(t) - f^{\Delta}(t)\right| \leq \left(\frac{D}{1-d}\right)\left(\frac{1}{1-r(t)}\right) e_p(t,\alpha). \tag{4.12}$$

Proof. Let $u(t) = |x(t) - f(t)| + \left|x^{\Delta}(t) - f^{\Delta}(t)\right|$ for $t \in I_{\mathbb{T}}$. Using the fact that $x(t)$ is a solution of equation (1.1) and the hypotheses we have

$$u(t) \leq \int_{\alpha}^{t} \left|g\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - g\left(t,\tau,f(\tau),f^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{t}\left|g\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\left|g\left(\sigma(t),t,f(t),x^{\Delta}\left(t\right)\right)-g\left(\sigma(t),t,f(t),f^{\Delta}\left(t\right)\right)+g\left(\sigma(t),t,f(t),f^{\Delta}\left(t\right)\right)\right|$$

$$\int_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-g^{\Delta}\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|$$

$$+\int_{\alpha}^{t}g^{\Delta}\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h^{\Delta}\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{t}\left|h^{\Delta}\left(t,\tau,f\left(\tau\right),f^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$\leq D+\int_{\alpha}^{t}p_{1}\left(t,\tau\right)u\left(\tau\right)\Delta\tau+\int_{\alpha}^{\beta}q_{1}\left(t,\tau\right)u\left(\tau\right)\Delta\tau$$

$$+p_{1}\left(\sigma(t),t\right)u(t)+\int_{\alpha}^{t}p_{2}\left(t,\tau\right)u\left(\tau\right)\Delta\tau+\int_{\alpha}^{\beta}q_{2}\left(t,\tau\right)u\left(\tau\right)\Delta\tau. \tag{4.13}$$

Using (3.24) into (4.13) we have

$$u\left(t\right)\leq\frac{D}{1-m}+\frac{1}{1-m}\int_{\alpha}^{t}p\left(t,\tau\right)u\left(\tau\right)\Delta\tau+\frac{1}{1-m}\int_{\alpha}^{\beta}q\left(t,\tau\right)u\left(\tau\right)\Delta\tau. \tag{4.14}$$

Now an application of Lemma 3.1 to (4.14) yields (4.12). □

5. Continuous Dependence

In this section we study continuous dependence of solutions of (1.1).

Consider the equation (1.1) and following Dynamic integral equation on time scales

$$y(t) = F(t) + \int_{\alpha}^{t} G\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\Delta\tau + \int_{\alpha}^{\beta} H\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\Delta\tau, \quad (5.1)$$

for $t \in I_T$, where y, F, G, H are in R^n, we assume that $F \in C_{rd}\left(I_{\mathbb{T}}, \mathbb{R}^n\right)$, $G, H \in C_{rd}\left(I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right)$ and are rd-continuous and delta differentiable with respect to t on the respective domain of their definition.
The following theorem deals with the continuous dependence of solutions of equation (1.1) on the functions involved therein.

Theorem 5.1. *Assume that the hypotheses of Theorem 3.4 hold. Suppose that*

$$\begin{aligned}
&|f(t) - F(t)| + \left|f^{\Delta}(t) - f^{\Delta}(t)\right| \\
&+ \left|g\left(\sigma(t), t, y(t), y^{\Delta}(t)\right) - G\left(\sigma\left(t\right), t, y(t), y^{\Delta}(t)\right)\right| \\
&+ \int_{\alpha}^{t} \left|g\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right) - G\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau \\
&+ \int_{\alpha}^{t} \left|g^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right) - G^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau \\
&+ \int_{\alpha}^{\beta} \left|h\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right) - H\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau \\
&+ \int_{\alpha}^{t} \left|h^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right) - H^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau \le \varepsilon, \quad (5.2)
\end{aligned}$$

where f, g, h and F, G, H are the functions involved in equations (1.1) and (5.1), $y(t)$ is a solution of equation (5.1) and $\varepsilon > 0$ is an arbitrary small constant. Then the solution $x(t), t \in I_{\mathbb{T}}$ of equation (1.1) depends rd-continuous on the functions involved on the right hand side of equation (1.1).

Proof. Let $z\left(t\right) = \left|x\left(t\right) - y\left(t\right)\right| + \left|x^{\Delta}\left(t\right) - y^{\Delta}\left(t\right)\right|$ for $t \in I_{\mathbb{T}}$. Using the facts that $x(t)$ and $y(t)$ are solutions of equations (1.1) and (5.1) and the hypothesis we have

$$\begin{aligned}
z\left(t\right) \le\; &\left|f\left(t\right) - F\left(t\right)\right| + \left|f^{\Delta}\left(t\right) - F^{\Delta}\left(t\right)\right| \\
&+ \left|g\left(\sigma\left(t\right), t, x(t), x^{\Delta}\left(t\right)\right) - g\left(\sigma\left(t\right), t, y(t), y^{\Delta}\left(t\right)\right)\right| \\
&+ \left|g\left(\sigma\left(t\right), t, y(t), y^{\Delta}\left(t\right)\right) - G\left(\sigma\left(t\right), t, y(t), y^{\Delta}\left(t\right)\right)\right| \\
&+ \int_{\alpha}^{t} \left|g\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right) - g\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau
\end{aligned}$$

$$+\int_{\alpha}^{t}\left|g\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)-G\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-H\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)-H\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-g^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{t}\left|g^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)-G^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right)-h^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$+\int_{\alpha}^{\beta}\left|h^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)-H^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right|\Delta\tau$$

$$\leq\varepsilon+p_1\left(\sigma(t),t\right)z(t)+\int_{\alpha}^{t}p\left(t,\tau\right)z\left(\tau\right)\Delta\tau+\int_{\alpha}^{\beta}q\left(t,\tau\right)z\left(\tau\right)\Delta\tau. \quad (5.3)$$

Using (3.24) in (5.3) we have

$$z(t)\leq\frac{\varepsilon}{1-m}+\frac{1}{1-m}\int_{\alpha}^{t}p\left(t,\tau\right)z\left(\tau\right)\Delta\tau+\frac{1}{1-m}\int_{\alpha}^{\beta}q\left(t,\tau\right)z\left(\tau\right)\Delta\tau. \quad (5.4)$$

Now an application of Lemma 3.1 to (5.4) yields

$$\left|x(t)-y(t)\right|+\left|x^{\Delta}\left(t\right)-y^{\Delta}\left(t\right)\right|\leq\left(\frac{\varepsilon}{1-m}\right)\left(\frac{1}{1-r(t)}\right)e_p\left(t,\alpha\right), \quad (5.5)$$

for $t\in I_{\mathbb{T}}$. From (5.4) it follows that the solution of equation (1.1) depend rd-continuously on the function involved on the right hand side of equation (1.1). □

Next we consider following dynamic integral equations on time scales,

$$z\left(t\right)=f(t)+\int_{\alpha}^{t}g\left(t,\tau,z\left(\tau\right),z^{\Delta}\left(\tau\right),\phi\right)\Delta\tau+\int_{\alpha}^{\beta}h\left(t,\tau,z\left(\tau\right),z^{\Delta}\left(\tau\right),\phi\right)\Delta\tau, \quad (5.6)$$

and

$$z(t) = f(t) + \int_{\alpha}^{t} g\left(t, \tau, z(\tau), z^{\Delta}(\tau), \phi_0\right)\Delta\tau + \int_{\alpha}^{\beta} h\left(t, \tau, z(\tau), z^{\Delta}(\tau), \phi_0\right)\Delta\tau, \tag{5.7}$$

for $t \in I_{\mathbb{T}}$, where z, f, g, h are in $\mathbb{R}^n$ and ϕ, ϕ_0 are real parameters. We assume that $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n), g, h \in C_{rd}(I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and are rd-continuous and delta differentiable with respect to t.

Now we present theorem which deals with the continuous dependence of solutions of equations (5.6) and (5.7) on parameters.

Theorem 5.2. *Assume that the functions g, h in equations (5.6) and (5.7) and their delta derivatives with respect to t satisfy the conditions*

$$|g(t, \tau, u, v, \phi) - g(t, \tau, \bar{u}, \bar{v}, \phi)| \le k_1(t, \tau)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{5.8}$$

$$|g(t, \tau, u, v, \phi) - g(t, \tau, u, v, \phi_0)| \le \delta_1(t, \tau)|\phi - \phi_0|, \tag{5.9}$$

$$|h(t, \tau, u, v, \phi) - h(t, \tau, \bar{u}, \bar{v}, \phi)| \le r_1(t, \tau)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{5.10}$$

$$|h(t, \tau, u, v, \phi) - h(t, \tau, u, v, \phi_0)| \le \gamma_1(t, \tau)|\phi - \phi_0|, \tag{5.11}$$

$$\left|g^{\Delta}(t, \tau, u, v, \phi) - g^{\Delta}(t, \tau, \bar{u}, \bar{v}, \phi_0)\right| \le k_2(t, \tau)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{5.12}$$

$$\left|g^{\Delta}(t, \tau, u, v, \phi) - g^{\Delta}(t, \tau, u, v, \phi_0)\right| \le \delta_2(t, s)|\phi - \phi_0| \tag{5.13}$$

$$\left|h^{\Delta}(t, \tau, u, v, \phi) - h^{\Delta}(t, \tau, \bar{u}, \bar{v}, \phi)\right| \le r_2(t, \tau)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{5.14}$$

$$\left|h^{\Delta}(t, \tau, u, v, \phi) - h^{\Delta}(t, \tau, \bar{u}, \bar{v}, \varphi_0)\right| \le \gamma_2(t, \tau)|\phi - \phi_0|, \tag{5.15}$$

where $k_i(t,\tau), r_i(t,\tau) \in C_{rd}(I_{\mathbb{T}}^2, \mathbb{R}_+)$ $(i = 1, 2)$ are nondecreasing in $t \in I_{\mathbb{T}}$, for each $\tau \in I_{\mathbb{T}}$ and $\delta_i(t, \tau), \gamma_i(t, \tau)\, C_{rd}(I_{\mathbb{T}}^2, \mathbb{R}_+)$ $(i = 1, 2)$. Further suppose that

$$k_i(\sigma(t), t) \le \lambda, \tag{5.16}$$

$$w_0(t) = \int_{\alpha}^{\beta} \frac{1}{1-\lambda}\, \bar{r}(t, \tau)\, e_{\frac{1}{1-\lambda}\bar{k}}(\tau, \alpha)\,\Delta\tau < 1, \tag{5.17}$$

$$\delta_1(\sigma(t), t) + \int_{\alpha}^{t} [\delta_1(t, \tau) + \delta_2(t, \tau)]\Delta\tau + \int_{\alpha}^{\beta} [\gamma_1(t, \tau) + \gamma_2(t, \tau)]\Delta\tau \le M, \tag{5.18}$$

for $t \in I_{\mathbb{T}}$ where λ, M are nonnegative constants such that $\lambda < 1$ and

$$\bar{k}(t, \tau) = k_1(t, \tau) + k_2(t, \tau), \quad \bar{r}(t, \tau) = r_1(t, \tau) + r_2(t, \tau). \tag{5.19}$$

Let $z_1(t)$ and $z_2(t)$ be the solutions of equations (5.6) and (5.7) respectively. Then

$$|z_1(t) - z_2(t)| + \left|z_1^{\Delta}(t) - z_2^{\Delta}(t)\right| \le \left(\frac{|\phi - \phi_0|\, M}{1-\lambda}\right)\left(\frac{1}{1 - w_0(t)}\right) e_{\bar{k}}(t, \alpha), \tag{5.20}$$

for $t \in I_{\mathbb{T}}$.

Proof. Let $u(t) = |z_1(t) - z_2(t)| + \left|z_1^{\Delta}(t) - z_2^{\Delta}(t)\right|$ for $t \in I_{\mathbb{T}}$. As $z_1(t)$ and $z_2(t)$ are solutions of equation (5.6) and (5.7) and hypotheses we have

$$
\begin{aligned}
u(t) \leq & \int_{\alpha}^{t} \left|g\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - g\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right)\right|\Delta\tau \\
& + \int_{\alpha}^{t} \left|g\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right) - g\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi_0\right)\right|\Delta\tau \\
& + \int_{\alpha}^{\beta} \left|h\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - h\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right)\right|\Delta\tau \\
& + \int_{\alpha}^{\beta} \left|h\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - h\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi_0\right)\right|\Delta\tau \\
& + \left|g\left(\sigma(t),t,z_1(t),z_1^{\Delta}(t),\phi\right) - g\left(\sigma(t),t,z_2(t),z_2^{\Delta}(t),\phi\right)\right| \\
& + \left|g\left(\sigma(t),t,z_2(t),z_2^{\Delta}(t),\phi\right) - g\left(\sigma(t),t,z_2(t),z_2^{\Delta}(t),\phi_0\right)\right| \\
& + \int_{\alpha}^{t} \left|g^{\Delta}\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - g^{\Delta}\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right)\right|\Delta\tau \\
& + \int_{\alpha}^{t} \left|g^{\Delta}\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right) - g^{\Delta}\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi_0\right)\right|\Delta\tau \\
& + \int_{\alpha}^{\beta} \left|g^{\Delta}\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - g^{\Delta}\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi\right)\right|\Delta\tau \\
& + \int_{\alpha}^{\beta} \left|g^{\Delta}\left(t,\tau,z_1(\tau),z_1^{\Delta}(\tau),\phi\right) - g^{\Delta}\left(t,\tau,z_2(\tau),z_2^{\Delta}(\tau),\phi_0\right)\right|\Delta\tau \\
& \leq \int_{\alpha}^{t} k_1(t,\tau)u(\tau)\,\Delta\tau + \int_{\alpha}^{t} \delta_1(t,\tau)\,|\phi-\phi_0|\Delta\tau \\
& + \int_{\alpha}^{\beta} r_1(t,\tau)u(\tau)\,\Delta\tau + + \int_{\alpha}^{\beta} \gamma_1(t,\tau)\,|\phi-\phi_0|\,\Delta\tau \\
& + k_1(\sigma(t),t)\,u(\tau)\,\Delta\tau + \delta_1(\sigma(t),t)\,|\phi-\phi_0| \\
& + \int_{\alpha}^{\beta} k_2(t,\tau)\,u(\tau)\,\Delta\tau + \int_{\alpha}^{t} \delta_2(t,\tau)\,|\phi-\phi_0|\Delta\tau
\end{aligned}
$$

$$+\int_{\alpha}^{\beta} r_2(t,\tau)u(\tau)\,\Delta\tau + \int_{\alpha}^{\beta} \gamma_2(t,\tau)\,|\phi-\phi_0|\,\Delta\tau. \tag{5.21}$$

Using (5.16), (5.18) in (5.21) we have

$$u(t) \le \frac{|\phi-\phi_0|\,M}{1-\lambda} + \frac{1}{1-\lambda}\int_{\alpha}^{t} \bar{k}(t,\tau)\,u(\tau)\,\Delta\tau + \frac{1}{1-\lambda}\int_{\alpha}^{\beta} \bar{r}(t,\tau)\,u(\tau)\,\Delta\tau. \tag{5.22}$$

Now an application of Lemma 3.1 to (5.22) yields (5.20) which shows the dependency of (5.6) and (5.7) on parameters. □

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