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WEIGHTED COMPOSITION OPERATORS AND DYNAMICAL SYSTEMS ON WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS ON BANACH SPACES

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ABSTRACT. Let B_X and B_Y be the open unit balls of the Banach Spaces X and Y, respectively. Let V and W be two countable families of weights on B_X and B_Y, respectively. Let $HV(B_X)$ (or $HV_0(B_X)$) and $HW(B_Y)$ (or $HW_0(B_Y)$) be the weighted Fréchet spaces of holomorphic functions. In this paper, we investigate the holomorphic mappings $\phi : B_X \to B_Y$ and $\psi : B_X \to \mathbb{C}$ which characterize continuous weighted composition operators between the spaces $HV(B_X)$ (or $HV_0(B_X)$) and $HW(B_Y)$ (or $HW_0(B_Y)$). Also, we obtained a (linear) dynamical system induced by multiplication operators on these weighted spaces.

1. INTRODUCTION

Weighted composition operators have been appearing in a natural way on different spaces of analytic functions. For example, it is well known that isometries on most of the spaces of analytic functions are described as weighted composition operators. For details on isometries and weighted composition operators, we refer to the monographs of Fleming and Jamison(see [11, 12]). In recent years, the theory of weighted composition operators on different spaces of analytic functions is gaining more importance as it includes two nice classes of operators such as composition operators and multiplication operators. A detailed account of composition operators can be found in three monographs (see Cowen and MacCluer [10], Shapiro [21] and Singh and Manhas [22]).

In the last few years many authors are engaged in studying the behaviour of these operators between the weighted spaces of holomorphic functions $H_v(B)$

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whenever B is the unit disk of $\mathbb{C}$ or, more in general, an open subset of $\mathbb{C}^N$. We refer to [3, 4, 6, 7, 8, 13, 23] for more information on composition operators on these weighted spaces. Also, Contreras and Hernández-Díaz [9], Montes-Rodríguez [18] and Manhas [17] has made a study of weighted composition operators on weighted spaces of analytic functions. Recently, García, Maestre and Sevilla–Peris [15, 16] have explored a study of composition operators on the weighted spaces $H_v(B_X)$ and $HV(B_X)$ where B_X is the open unit ball of a Banach space X. In this paper, strongly inspired by the work of García, Maestre and Sevilla–Peris [15, 16], we study weighted composition operators between the weighted spaces $H_v(B_X)$ and $H_w(B_Y)$ and between the spaces $H_{v_0}(B_X)$ and $H_{w_0}(B_Y)$. Also, we characterize weighted composition operators between the weighted Fréchet spaces $HV(B_X)$ and $HW(B_Y)$ and between the spaces $HV_0(B_X)$ and $HW_0(B_Y)$, where V and W are two countable families of weights on B_X and B_Y, respectively.

2. Preliminaries

Let X be a complex Banach space and let U_X be a balanced open subset of X. By a weight we mean an upper semicontinuous function $v : U_X \to [0,\infty)$. A set of weights V is called a Nachbin family if for every $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ such that $\lambda v_1 \leq v$ and $\lambda v_2 \leq v$ on U_X. In what follows V denotes a Nachbin family of continuous weight functions such that for every $x \in U_X$, there exists $v \in V$ for which $v(x) > 0$. A subset $B \subseteq U_X$ is U_X -bounded if it is bounded and its distance to $X \backslash U_X$ is greater that zero. A function $f : U_X \to [0,\infty)$ is said to vanish at infinity outside U_X -bounded sets if for each $\epsilon > 0$, there exists a U_X -bounded set B such that $f(x) < \epsilon$, for every $x \in U_X \backslash B$. Let V and W be two Nachbin families of continuous weight functions on U_X. Then we say that $V \leq W$ if for every $v \in V$, there exists $w \in W$ such that $v \leq w$. Let $H(U_X)$ be the space of all holomorphic functions $f : U_X \to \mathbb{C}$.

Now, the weighted spaces of holomorphic functions associated with V are defined as follows:

$$HV(U_X) = \left\{ \begin{array}{c} f \in H(U_X) : \|f\|_v = \sup\{v(x)|f(x)| : x \in U_X\} < \infty, \\ \text{for every } v \in V \end{array} \right\},$$

and

$$HV_0(U_X) = \left\{ \begin{array}{c} f \in HV(U_X) : v|f| \text{ vanishes at infinity outside} \\ U_X - \text{bounded sets for every } v \in V \end{array} \right\}.$$

Both spaces are endowed with the weighted topology τ_V generated by the family $\{\|.\|_v : v \in V\}$ of seminorms. The family of closed absolutely convex neighbourhoods of the form

$$B_v = \{f \in HV(U_X)\,(resp. HV_0(U_X)) : \|f\|_v \leq 1\}$$

is a basis of these spaces. It is observed that if X is a Banach space of finite dimension the elements B in the definition of $HV_0(U_X)$ are considered to be compact, but any compact subset of an infinite-dimensional Banach space has empty interior, hence every $f : U_X \to [0,\infty)$ continuous on U_X and vanishing at infinity outside compact subsets of U_X is identically zero. By $H_b(U_X)$ we denote the subspace of $H(U_X)$ of those functions which are bounded on U_X -bounded subsets of U_X. If V is a countable family of continous weights, then $HV(U_X)$ and $HV_0(U_X)$ are weighted Fréchet spaces. If the family of continuous weights has a unique element $V = \{v\}$ such that $v(x) > 0$, for all $x \in U_X$, then $HV(U_X)$ and $HV_0(U_X)$ endowed with $\|.\|_v$ are Banach spaces and are denoted by $H_v(U_X)$ and $H_{v_0}(U_X)$, respectively. The space of bounded holomorphic functions is denoted by $H^\infty(U_X)$.

Following [14, Definition 1], we say that a family V of weigths defined on U_X satisfies Condition-I if for each U_X -bounded subset B of U_X, there exists $v \in V$ such that $\inf\{v(x) : x \in B\} > 0$. If V satisfies Condition-I, then $HV(U_X) \subseteq H_b(U_X)$ ([14, Proposition 2]). Also, if $V = \{v\}$ and $U_X = B_X$, the open unit ball of a Banach space X, then in this setting, a set $A \subseteq B_X$ is said to B_X- bounded if there exists $0 < \gamma < 1$ such that $A \subseteq \gamma B_X$. Also, a weight v satisfies Condition-I if $\inf\{v(x) : x \in \gamma B_X\} > 0$ for every $0 < \gamma < 1$. If X is finite dimensional, then all weights on B_X satisfy Condition-I. By B_v we denote the closed unit ball of $H_v(B_X)$. It is well-known that in $H_v(B_X)$ the τ_V (norm) topology is finer than the τ_0 (compact-open) topology ([19, Section 3]) and that B_v is τ_0- compact ([19, p. 349]).

A weight v is said to be radial if $v(\lambda x) = v(x)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and every $x \in B_X$ ([1, 3]). Given any weight v, following [2], we consider an associated growth condition $u : B_X \to (0, +\infty)$ defined by $u(x) = \dfrac{1}{v(x)}$.With this new function we can rewrite

$$B_v = \{f \in H_v(B_X) : |f| \le u\}.$$

From this, $\tilde{u} : B_X \to (0, +\infty)$ is defined by

$$\tilde{u}(x) = \sup\{|f(x)| : f \in B_v\}$$

and a new associated weight $\tilde{v} = \dfrac{1}{\tilde{u}}$. All these functions are defined by Bierstedt, Bonet and Taskinen for open subsets of $\mathbb{C}^N$ in [2]. In ([2], Proposition 1.2) the following relations between weights for open sets in $\mathbb{C}^N$ are proved. The same arguments work for the unit ball of a Banach space.

Proposition 2.1. *Let X be a Banach space and v a weight defined on B_X. Then the following hold:*

(i) $0 < v \le \tilde{v}$ *and $\tilde{v}$ is bounded and continuous i.e, $\tilde{v}$ is a weight.*

(ii) $\tilde{u}$ *(respectively $\tilde{v}$) is radial and decreasing or increasing whenever u (respectively v) is so.*

(iii) $\|f\|_v \le 1$ *if and only if* $\|f\|_{\tilde{v}} \le 1$.

(iv) For each $x \in B_X$, there exists $f_x \in B_v$ such that $f_x(x) = \dfrac{1}{\tilde{v}(x)}$.

Also, we use the property of the associated weight: $v(x) \le \tilde{v}(x)$, for every $x \in B_X$, the element δ_x, the point evaluation functional of $(H_v(B_X))^*$ defined by $\delta_x(f) = f(x)$, for every $f \in H_v(B_X)$ satisfies $\|\delta_x\|_v = \dfrac{1}{\tilde{v}(x)}$. A weight v is said to be essential if there exists $c > 0$ such that $v(x) \leq \tilde{v}(x) \leq cv(x)$, for all $x \in B_X$ [23]. For more details on weighted spaces of holomorphic functions defined on open subsets of $\mathbb{C}^N$, we refer to [1, 2]. Also, we refer to [14, 19] for more information on weighted spaces of holomorphic functions defined on the unit ball of a Banach space.

3. Characterizations of Weighted Composition Operators

Let X and Y be Banach spaces. Let U_X and U_Y be balanced open subsets of X and Y, respectively. Let $\phi : U_X \to U_Y$ and $\psi : U_X \to \mathbb{C}$ be holomorphic mappings. Then the weighted composition operator $W_{\phi,\psi} : (H(U_Y), \tau_0) \to (H(U_X), \tau_0)$, defined by $W_{\phi,\psi} f = \psi . f o \phi$, for every $f \in H(U_Y)$, is clearly linear and continuous with respect to the compact-open topology τ_0. Now, we shall discuss the continuity of the weighted composition operators between the weighted spaces $HV(U_X)$ and $HW(U_Y)$ and between the spaces $HV_0(U_X)$ and $HW_0(U_Y)$.

Proposition 3.1. *Let V and W be Nachbin families of weights on U_X and U_Y, respectively. Let $\phi : U_Y \to U_X$ and $\psi : U_Y \to \mathbb{C}$ be holomorphic mappings. Then $W_{\phi,\psi} : HV(U_X) \to HW(U_Y)$ is continuous if $W|\psi| \leq V o \phi$.*

Proof. To show that $W_{\phi,\psi}$ is continuous, it is enough to show that $W_{\phi,\psi}$ is continuous at the origin. For, let $w \in W$ and B_w be a neighbourhood of the origin in $HW(U_Y)$.Then by the given condition, there exists $v \in V$ such that $w|\psi| \leq v o \phi$. That is, $w(x)|\psi(x)| \leq v(\phi(x))$, for every $x \in U_Y$. We claim that $W_{\phi,\psi}(B_v) \subseteq B_w$. Let $f \in B_v$. Then $\|f\|_v \leq 1$ and

$$\begin{aligned}
\|W_{\phi,\psi} f\|_w &= \sup\{w(x)|\psi(x)||f(\phi(x))| : x \in U_Y\} \\
&\leq \sup\{v(\phi(x))|f(\phi(x))| : x \in U_Y\} \\
&\leq \sup\{v(y)|f(y)| : y \in U_X\} \\
&= \|f\|_v \leq 1.
\end{aligned}$$

This proves our claim and $W_{\phi,\psi}$ is continuous. □

Proposition 3.2. *Let V and W be Nachbin families of weights on U_X and U_Y, respectively such that V satisfies Condition-I. Let $\phi : U_Y \to U_X$ and $\psi : U_Y \to \mathbb{C}$ be holomorphic mappings. Then $W_{\phi,\psi} : HV_0(U_X) \to HW_0(U_Y)$ is continuous if*

(i) $W|\psi| \leq V o \phi$;

(ii) for every $w \in W, \epsilon > 0$ and U_X-bounded set A, the set $\phi^{-1}(A) \cap F(w|\psi|, \epsilon)$ is U_Y-bounded, where $F(w|\psi|, \epsilon) = \{y \in U_Y : w(y)|\psi(y)| \geq \epsilon\}$.

Proof. According to Proposotion 3.1, condition (i) implies that $W_{\phi,\psi} : HV(U_X) \to HW(U_Y)$ is continuous. To show that $W_{\phi,\psi} : HV_0(U_X) \to HW_0(U_Y)$ is continuous, it is enough to show that $W_{\phi,\psi}$ is an into map. Let $f \in HV_0(U_X)$.To show that $W_{\phi,\psi} f \in HW_0(U_Y)$, we need to show that for $w \in W$, the function $w.|\psi.f o \phi|$ vanishes at infinity outside U_Y-bounded sets. Let $w \in W$ and $\epsilon > 0$. Then consider the set $K = \{y \in U_Y : w(y)|\psi(y)||f(\phi(y))| \geq \epsilon\}$. We shall show that K is U_Y-bounded set. By Condition (i), there exist $v \in V$

such that $w(y)|\psi(y)| \leq v(\phi(y))$, for every $y \in U_Y$. Since $f \in HV_0(U_X)$, there exists a set $B \subseteq U_X$ which is U_X-bounded such that $v(x)|f(x)| < \epsilon$, for every $x \in U_X \backslash B$. Also, since V satisfies Condition-I, $f \in H_b(U_X)$. Let $\alpha = \sup\{|f(x)| : x \in B\}$. Clearly $\phi(K) \subseteq B$. Also, $K \subseteq F\left(w|\psi|, \dfrac{\epsilon}{\alpha}\right)$ and hence $K \subseteq \phi^{-1}(B) \cap F\left(w|\psi|, \dfrac{\epsilon}{\alpha}\right)$. By Condition (ii), the set $\phi^{-1}(B) \cap F\left(w|\psi|, \dfrac{\epsilon}{\alpha}\right)$ is U_Y-bounded and hence K being a subset of U_Y-bounded set is also U_Y-bounded. This shows that $w(y)|\psi(y)||f(\phi(y))| < \epsilon$, for every $y \in U_Y \backslash K$. This completes the proof. □

Corollary 3.3. *Let V and W be Nachbin families of weights on U_X and U_Y, respectively. Let $\phi : U_Y \to U_X$ be a holomorphic mapping.*

Then

(i) $W_{\phi,\psi} : HV(U_X) \to HW(U_Y)$ is continuous if $\psi \in H^\infty(U_Y)$ and $W \leq V o\phi$.

(ii) $W_{\phi,\psi} : HV_0(U_X) \to HW_0(U_Y)$ is continuous if $\psi \in HW_0(U_Y)$ and $W|\psi| \leq V o\phi$, where V satisfies condition-I.

Remark 3.4. If $U_X = U_Y = G$ is an open connected subset of $\mathbb{C}^{\mathbb{N}}\,(N \geq 1)$, then Proposition 3.1 and Proposition 3.2 reduce to Theorem 3.1 and Theorem 3.2 of [17].

Next, for given single weights v and w, our efforts are to obtain necessary and sufficient conditions for the weighted composition operator $W_{\phi,\psi}$ to be continuous on the weighted Banach spaces and then using these results we shall establish the characterization of the continuity of $W_{\phi,\psi}$ on the weighted Fréchet spaces.

Remark 3.5. Let H_1 and H_2 be two Banach spaces of holomorphic functions whose topologies are stronger than the pointwise convergence topology. Then by the closed graph theorem $W_{\phi,\psi} : H_1 \to H_2$ is continuous if it is well-defined.

Proposition 3.6. *Let $\phi : U_X \to U_Y$ be a holomorphic mapping such that $\phi(U_X)$ is U_Y-bounded set, and let $\psi \in H^\infty(U_X)$. Let v and w be continuous weights on U_X and U_Y, respectively such that w satisfies Condition-I and v is bounded. Then $W_{\phi,\psi} : H_w(U_Y) \to H_v(U_X)$ is continuous*

Proof. Let $f \in H_w(U_Y)$. Then $f \in H_b(U_Y)$ because w satisfies Condition-I and so $H_w(U_Y) \subseteq H_b(U_Y)$. Since $\phi(U_X)$ is U_Y-bounded set, there exists $m > 0$ such that $\sup\{|f(\phi(x))| : x \in U_X\} \leq m$. Also, since $\psi \in H^\infty(U_X)$, there exists $k > 0$ such that $\sup\{|\psi(x)| : x \in U_X\} \leq k$. Since v is bounded, there exists $\alpha > 0$ such that $\sup\{v(x) : x \in U_X\} \leq \alpha$. Now, consider,

$$\begin{aligned}\|W_{\phi,\psi} f\|_v &= \sup\{v(x)|\psi(x)||f(\phi(x))| : x \in U_X\} \\ &\leq \sup_{x\in U_x} v(x) . \sup_{x\in U_X} |\psi(x)| . \sup_{x\in U_X} |f(\phi(x))| \\ &\leq \alpha k m.\end{aligned}$$

This shows that $W_{\phi,\psi} f \in H_v(U_X)$. Thus $W_{\phi,\psi}$ is well defined and hence by Remark 3.5, $W_{\phi,\psi}$ is continuous. □

Theorem 3.7. *Let B_X and B_Y be the open unit balls of the Banach spaces X and Y, respectively. Let $\phi : B_X \to B_Y$ and $\psi : B_X \to \mathbb{C}$ be holomorphic mappings. Let w and v be continuous weights on B_X and B_Y, respectively. Then the following are equivalent:*

(i) $W_{\phi,\psi} : H_v(B_Y) \to H_w(B_X)$ *is continuous;*

(ii) $\sup\left\{\dfrac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\} < \infty$;

(iii) $\sup\left\{\dfrac{\tilde{w}(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\} < \infty$.

Moreover, the following holds

$$\|W_{\phi,\psi}\| = \sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\}.$$

If v is an essential weight, then $W_{\phi,\psi} : H_v(B_Y) \to H_w(B_X)$ is continuous if and only if $\sup\left\{\dfrac{w(x)|\psi(x)|}{v(\phi(x))} : x \in B_X\right\} < \infty$.

Proof. Since $w \leq \tilde{w}$, clearly $(iii) \Rightarrow (ii)$. Now, assume that (ii) holds. Let $M = \sup\left\{\dfrac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\}$ and let $f \in H_v(B_Y)$. Then we have

$$\begin{aligned}
\|W_{\phi,\psi}f\|_w &= \sup\{w(x)|\psi(x)f(\phi(x))| : x \in B_X\} \\
&= \sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))}\tilde{v}(\phi(x))|f(\phi(x))| : x \in B_X\right\} \\
&\leq \sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\}\sup\{\tilde{v}(\phi(x))|f(\phi(x))| : x \in B_X\} \\
&\leq M\|f\|_{\tilde{v}} = M\|f\|_v.
\end{aligned}$$

This proves that the operator $W_{\phi,\psi}$ is continuous and hence establishes Condition (i). Now suppose that $W_{\phi,\psi}$ is continuous. Assume that Condition (iii) does not hold. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_X$ such that $\dfrac{\tilde{w}(x_n)|\psi(x_n)|}{\tilde{v}(\phi(x_n))} > n$, for all n. For each $\phi(x_n)$, there exists $f_n \in B_v$ such that $|f_n(\phi(x_n))| > \dfrac{1}{2\tilde{v}(\phi(x_n))}$. But for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|W_{\phi,\psi}f_n\|_w = \|W_{\phi,\psi}f_n\|_{\tilde{w}} &\geq \tilde{w}(x_n)|\psi(x_n)||f_n(\phi(x_n))| \\
&> \frac{\tilde{w}(x_n)|\psi(x_n)|}{2\tilde{v}(\phi(x_n))} > \frac{n}{2}.
\end{aligned}$$

This contradicts the fact that $W_{\phi,\psi}(B_v)$ is bounded. Finally, we estimate $\|W_{\phi,\psi}\|$. If we put $M = \sup\left\{\dfrac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\}$, then we have already seen in the proof that

$$\|W_{\phi,\psi}f\|_w \leq M\|f\|_v, \text{ for every } f \in H_v(B_Y).$$

From this it clearly follows that

$$\|W_{\phi,\psi}\| \leq \sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\}.$$

Now, for each $x \in B_X$, the point evaluation functional $\delta_x \in (H_w(B_X))^*$. Also, since $(W_{\phi,\psi})^*(\delta_x) = \psi(x)\,\delta_{\phi(x)}$, for each $x \in B_X$, we have

$$\begin{aligned}
\|W_{\phi,\psi}\| = \|(W_{\phi,\psi})^*\| &\geq \frac{\|(W_{\phi,\psi})^*(\delta_x)\|_v}{\|\delta_x\|_w} \\
&= \frac{|\psi(x)|\,\|\delta_{\phi(x)}\|_v}{\|\delta_x\|_w} \\
&= \frac{|\psi(x)|\,\tilde{w}(x)}{\tilde{v}(\phi(x))}, \\
&\geq \frac{|\psi(x)|\,w(x)}{\tilde{v}(\phi(x))},
\end{aligned}$$

for every $x \in B_X$. This proves that

$$\|W_{\phi,\psi}\| = \sup\left\{\frac{|\psi(x)|\,w(x)}{\tilde{v}(\phi(x))} : x \in B_X\right\}.$$

With this the proof of the theorem is completed. □

Theorem 3.8. *Let B_X and B_Y be the open unit balls of X and Y, respectively. Let $\phi : B_X \to B_Y$ and $\psi : B_X \to \mathbb{C}$ be holomorphic mappings. Let w and v be continuous weights on B_X and B_Y, respectively, such that v is bounded and satisfies Condition-I. Then $W_{\phi,\psi} : H_v(B_Y) \to H_w(B_X)$ is continuous if and only if $\psi \in H_w(B_X)$ and*

$$\sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : \|\phi(x)\| > \gamma_0\right\} < \infty, \ \textit{for some } 0 < \gamma_0 < 1.$$

Proof. Suppose that $W_{\phi,\psi}$ is continuous. Since the constant function 1 belongs to $H_v(B_Y)$, we have $W_{\phi,\psi}1 = \psi \in H_w(B_X)$. Also, from Theorem 3.7, it follows that $\sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\} < \infty$. Clearly it implies that

$$\sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : \|\phi(x)\| > \gamma_0\right\} < \infty, \quad \text{for some } \ 0 < \gamma_0 < 1.$$

Conversely, suppose that given conditions hold. Let

$$M = \sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : \|\phi(x)\| > \gamma_0\right\} < \infty.$$

Since $\psi \in H_w(B_X)$, we have

$$k = \sup\{w(x)|\psi(x)| : x \in B_X\} < \infty. \text{ Let } f \in H_v(B_Y)$$

Then we show that

$$\|W_{\phi,\psi}f\|_w = \sup\{w(x)|\psi(x)f(\phi(x))| : x \in B_X\} < \infty.$$

Let $x \in B_X$ be such that $\|\phi(x)\| > \gamma_0$. Then

$$w(x)\,|\psi(x))f(\phi(x))| = \frac{w(x)\,|\psi(x)|}{\tilde{v}(\phi(x))}.\tilde{v}(\phi(x))\,|f(\phi(x))| \le M\,\|f\|_{\tilde{v}} = M\,\|f\|_v\,.$$

Let $x \in B_X$ be such that $\|\phi(x)\| \le \gamma_0$. Since v satisfies Condition-I, $f \in H_b(B_Y)$ and hence there exists $\alpha > 0$ such that $\sup\left\{|f(y)| : y \in \gamma_0 \bar{B}_Y\right\} \le \alpha$. Further, it implies that

$$\begin{aligned} w(x)\,|\psi(x))f(\phi(x))| &\le \sup\{w(x)\,|\psi(x)| : x \in B_X\} \sup\left\{|f(y)| : y \in \gamma_0 \bar{B}_Y\right\} \\ &\le k\alpha. \end{aligned}$$

Thus it follows that

$$\|W_{\phi,\psi}f\|_w = \sup\{w(x)\,|\psi(x)\,f(\phi(x))| : x \in B_X\} < \infty.$$

This shows that $W_{\phi,\psi}f \in H_w(B_X)$, for every $f \in H_v(B_Y)$. Hence by Remark 3.5, $W_{\phi,\psi}$ is continuous. □

Remark 3.9. (i) If $\psi(x) = 1$, for every $x \in B_X$, then it is already seen in [15, Example 2.4] that Condition-I for the weight v in Theorem 3.8 is necessary to prove that $W_{\phi,\psi}$ is a continuous operator if$\sup\left\{\frac{w(x)\,|\psi(x)|}{\tilde{v}(\phi(x))} : \|\phi(x)\| > \gamma_0\right\} < \infty$, for some $0 < \gamma_0 < 1$.

(ii) If $\psi(x) = 1$, for every $x \in B_X$, then Theorem 3.7 and Theorem 3.8 reduce to Proposition 2.3 of [15].

(iii) If $B_X = B_Y = D$, the open unit disk in the complex plane, then Theorem 3.7 reduces to Proposition 3.1 of [9] and the boundedness results given in Theorem 2.1 of [18].

(iv) If $\psi(x) = 1$, for every $x \in B_X$ and $B_X = B_Y = D$, then Theorem 3.7 reduces to Proposition 2.1 and Corollary 2.2 of [6]).

In the following Corollary 3.10, we record a special case of Theorem 3.7 which characterizes multiplication operators $M_\psi : H_v(B_X) \to H_w(B_X)$, which we defined as $M_\psi f = \psi.f$, for every $f \in H_v(B_X)$.

Corollary 3.10. *Let $\psi : B_X \to \mathbb{C}$ be a holomorphic mapping. Let w and v be continuous weights defined on B_X. Then the following are equivalent:*

(i) $M_\psi : H_v(B_X) \to H_w(B_X)$ *is continuous;(ii)* $\sup\left\{\frac{w(x)\,|\psi(x)|}{\tilde{v}(x)} : x \in B_X\right\} < \infty$;

(iii) $\sup\left\{\frac{\tilde{w}(x)\,|\psi(x)|}{\tilde{v}(x)} : x \in B_X\right\} < \infty$.

Moreover, the following holds$\|M_\psi\| = \sup\left\{\frac{w(x)\,|\psi(x)|}{\tilde{v}(x)} : x \in B_X\right\}$.

If v is an essential weight, then $M_\psi : H_v(B_X) \to H_w(B_X)$ is continuous if and only if $\sup\left\{\frac{w(x)\,|\psi(x)|}{v(x)} : x \in B_X\right\} < \infty$.

Remark 3.11. If $B_X = D$, the open unit disk in the complex plane and $v = w$ with v essential, then Corollary 3.10 reduces to Proposition 2.1 of [5].

Theorem 3.12. *Let B_X and B_Y be the open unit balls of the Banach spaces X and Y, respectively. Let $\phi : B_X \to B_Y$ and $\psi : B_X \to \mathbb{C}$ be holomorphic mappings. Let w and v be continuous weights on B_X and B_Y, respectively, such that v satisfies Condition-I and vanishes at infinity outside B_Y-bounded sets. Then the operator $W_{\phi,\psi} : H_{v_0}(B_Y) \to H_{w_0}(B_X)$ is continuous if and only if (i) $\psi \in H_{w_0}(B_X)$ and (ii)* $\sup\left\{\dfrac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\} < \infty.$

Proof. Suppose that given conditions are satisfied. Since $\psi \in H_{w_0}(B_X)$, $w|\psi|$ vanishes at infinity outside B_X-bounded sets. Let $\epsilon > 0$. Then there exists a B_X-bounded set S of B_X such that $w(x)|\psi(x)| < \epsilon$, for every $x \in B_X \backslash S$. Clearly the set $F(w|\psi|,\epsilon) = \{x \in B_X : w(x)|\psi(x)| \geq \epsilon\}$ is B_X-bounded and hence the condition (ii) of Proposition 3.2 is satisfied. Thus from Proposition 3.2, it follows that $W_{\phi,\psi}$ is continuous. Conversely, suppose that $W_{\phi,\psi}$ is continuous.Then using the same arguments of Theorem 3.7, it can be proved that $\sup\left\{\dfrac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_X\right\} < \infty$. Also, since the constant 1 function belongs to $H_{v_0}(B_Y)$, we have $W_{\phi,\psi}(1) = \psi \in H_{w_0}(B_X)$. This completes the proof. □

Remark 3.13. If $B_X = B_Y = D$, the open unit disk in $\mathbb{C}$, then Theorem 3.12 reduces to Proposition 3.2 of [9].

4. Weighted Composition Operators on Weighted Fréchet Spaces

Let V and W be two countable families of continuous bounded weights on B_X and B_Y, respectively. Then we consider the weighted composition operators $W_{\phi,\psi} : HV(B_X) \to HW(B_Y)$. García, Maestre and Sevilla–Peris [16, Proposition 10] proved a general result which allows them to give conditions on the continuity of the composition operators ([16, Proposition 11]), on these weighted Fréchet spaces.The same general result also permits us to give a characterization of the continuity of the weighted composition operators on these weighted Fréchet spaces. First we state this general result. Let (H,τ) and (G,τ') be Hausdorff locally convex spaces. For each n, let E_n and F_n be Banach spaces with closed unit balls B_n and C_n and norms $\|.\|_n$ and $|.|_n$. Suppose that $E_{n+1} \subseteq E_n \subseteq E_1 \subseteq H$, $B_{n+1} \subseteq B_n$ and $F_{n+1} \subseteq F_n \subseteq F_1 \subseteq G$, $C_{n+1} \subseteq C_n$, for every n. Suppose that for each n, both B_n and C_n are compact in (H,τ) and (G,τ'), respectively. Let E be the projective limit of $(E_n)_n$ and F be the projective limit of $(F_n)_n$. Let us assume that for every $n \in \mathbb{N}$ and all $x \in E_n$, there exists a sequence $\{y_k\}_k \subseteq E$ converging to x in (H,τ) such that $\|y_k\|_n \leq \|x\|_n$, for all k.

Theorem 4.1. *Let $T : (H,\tau) \to (G,\tau')$ be a continuous linear operator:*

(a) The following are equivalent:

(i) $TE \subseteq F$;

(ii) $T \in L(E,F)$;

(iii) For each m, there is n such that $TE_n \subseteq F_n$;

(iv) For each m, there is n such that $T : E_n \to F_n$ is well defined and continuous.

(b) The following are equivalent:

(i) $T : E \to F$ *is bounded;*

(ii) *There exists* n *such that for all* m, $TE_n \subseteq F_m$;

(iii) *There exists* n *such that for all* m, $T : E_n \to F_m$ *is well defined and continuous.*

(c) *The following are equivalent:*

(i) $T : E \to F$ *is compact (resp. weekly compact);*

(ii) *There exists* n *such that for all* m, $T : E_n \to F_m$ *is compact (resp. weekly compact).*

Let $(H, \tau) = (H(B_X), \tau_0)$ and $(G, \tau') = (H(B_Y), \tau_0)$. Then the weighted composition operator $W_{\phi,\psi} : (H(B_X), \tau_0) \to (H(B_Y), \tau_0)$ is linear and continuous, where $\phi : B_Y \to B_X$ and $\psi : B_Y \to \mathbb{C}$ are holomorphic mappings. Let $V = \{v_n\}_{n=1}^{\infty}$ and $W = \{w_n\}_{n=1}^{\infty}$ be two increasing families of continuous bounded weights satisfying Condition-I on B_X and B_Y, respectively. Let $E_n = H_{v_n}(B_X)$ $\left(\text{or } H_{(v_n)_0}(B_X)\right)$ and $F_n = H_{w_n}(B_Y)$ $\left(\text{or } H_{(w_n)_0}(B_Y)\right)$. Then each of them is a Banach space and they satisfy $H_{v_{n+1}}(B_X) \subseteq H_{v_n}(B_X) \subseteq H_{v_1}(B_X) \subseteq H(B_X)$, the closed unit ball $\bar{B}_{v_n}$ is τ_0-compact ([6, 19]) and $\bar{B}_{v_{n+1}} \subseteq \bar{B}_{v_n}$ for all n (the same is true for $H_{w_n}(B_Y)$, $H_{(w_n)_0}(B_Y)$ and $H_{(v_n)_0}(B_X)$). Let $E = HV(B_X)$ (or $HV_0(B_X)$) and $F = HW(B_Y)$ (or $HW_0(B_Y)$.

For $f \in H(B_X)$, consider its Taylor series expansion at zero, $f = \sum_{k=0}^{\infty} P_m f$. For each $k \in \mathbb{N}$, the k-th Cesàro mean ([14, Proposition 4] or [1, section 1]) is defined by

$$C_k f(x) = \frac{1}{k+1} \sum_{l=0}^{k} \left(\sum_{m=0}^{l} P_m f(x) \right) = \sum_{m=0}^{k} \left(1 - \frac{m}{k+1} \right) P_m f(x).$$

Since every weight is bounded on B_X, every polynomial belongs to $HV(B_X)$. In particular, for every $f \in H(B_X)$, the sequence $(C_k f)_k$ is in $HV(B_X)$. Also, $C_k f \to f$ in τ_0 (see [1, 14]). If v is a radial weight then for all $f \in H_v(B_X)$,

$$\sup\{v(x)\,|C_k f(x)| : x \in B_X\} \le \sup\{v(x)\,|f(x)| : x \in B_X\}$$

(see [1, Proposition 1.2 (b)] also [14]). Hence, if every $v \in V$ is radial, then the spaces and the weighted composition operator satisfy all the above conditions to apply Theorem 4.1 in a very similar way to that used by García, Maestre and Sevilla–Peris to obtain the following generalizations of [16, Proposition 11].

Theorem 4.2. *Let* $\phi : B_Y \to B_X$ *and* $\psi : B_Y \to \mathbb{C}$ *be holomorphic mappings. Let* $V = \{v_n\}_{n=1}^{\infty}$ *and* $W = \{w_n\}_{n=1}^{\infty}$ *be increasing countable families of continuous bounded weights satisfying Condition-I on* B_X *and* B_Y, *respectively such that each* v_n *is radial. Then the following are equivalent:*

(i) $W_{\phi,\psi} : HV(B_X) \to HW(B_Y)$ *is continuous;*

(ii) *For each* $w \in W$, *there exists* $v \in V$ *such that* $W_{\phi,\psi} : H_v(B_X) \to H_w(B_Y)$ *is continuous;*

(iii) *For each* $w \in W$, *there exists* $v \in V$ *such that*

$$\sup\left\{ \frac{w(x)\,|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_Y \right\} < \infty.$$

Proof. Follows from Theorem 4.1 and Theorem 3.7. $\square$

Corollary 4.3. *Let $\psi : B_X \to \mathbb{C}$ be a holomorphic mapping. Let $V = \{v_n\}_{n=1}^{\infty}$ and $W = \{w_n\}_{n=1}^{\infty}$ be increasing countable families of continuous bounded weights satisfying Condition-I on B_X such that each v_n is radial. Then the following are equivalent:*

(i) $M_\psi : HV(B_X) \to HW(B_X)$ *is continuous;*

(ii) For each $w \in W$, there exists $v \in V$ such that $M_\psi : H_v(B_X) \to H_w(B_Y)$ is continuous;

(iii) For each $w \in W$, there exists $v \in V$ such that

$$\sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(x)} : x \in B_X\right\} < \infty.$$

Proof. Follows from Theorem 4.1 and Theorem 3.8. □

Theorem 4.4. *Let $\phi : B_Y \to B_X$ and $\psi : B_Y \to \mathbb{C}$ be holomorphic mappings. Let $V = \{v_n\}_{n=1}^{\infty}$ and $W = \{w_n\}_{n=1}^{\infty}$ be increasing countable families of bounded continuous radial weights satisfying Condition-I on B_X and B_Y, respectively such that each v_n vanishes at infinity outside B_X-bounded sets. Then the following are equivalent:*

(i) $W_{\phi,\psi} : HV_0(B_X) \to HW_0(B_Y)$ *is continuous;*

(ii) For each $w \in W$, there exists $v \in V$ such that $W_{\phi,\psi} : H_{v_0}(B_X) \to H_{w_0}(B_Y)$ is continuous;

(iii) For each $w \in W$, there exists $v \in V$ such that

$$\sup\left\{\frac{w(x)|\psi(x)|}{\tilde{v}(\phi(x))} : x \in B_Y\right\} < \infty$$

and $\psi \in H_{w_0}(B_Y)$.

Proof. Follows from Theorem 4.1 and Theorem 3.12. □

Corollary 4.5. *Let U_X be a balanced open subset of a Banach space X. Then every $\psi \in H^\infty(U_X)$ induces a multiplication operator M_ψ on $HV(U_X)$.*

Proof. Let $f \in HV(U_X)$ and let $v \in V$. Then

$$\begin{aligned}\|M_\psi f\|_v &= \sup\{v(x)|\psi(x)||f(x)| : x \in B_X\}\\ &\le \|\psi\|_\infty \|f\|_v.\end{aligned}$$

This implies that M_ψ is a multiplication operator. □

5. Dynamical System Induced by Multiplication Operators

Let $g : U_X \to \mathbb{C}$ be a bounded analytic function. Then for each $t \in \mathbb{R}$, we define $\psi_t : U_X \to \mathbb{C}$ as $\psi_t(x) = e^{tg(x)}$, for every $x \in U_X$. Clearly each ψ_t is bounded analytic function on U_X. Thus according to Corollary 4.5, each ψ_t induces a multiplication operator M_{ψ_t} on $HV(U_X)$.

Theorem 5.1. *Let h_n be a sequence converging to h in $H^\infty(U_X)$ and let f_n be a sequence converging to f in $HV(U_X)$. Then the product of f_n and h_n converges to fh in $HV(U_X)$.*

Proof. Let $v \in V$. Then

$$\begin{aligned}
\|f_n h_n - fh\|_v &= \sup\{v(x)|f_n(x)h_n(x) - f(x)h(x)| : x \in U_X\} \\
&= \sup\left\{v(x)\left|\begin{array}{c} f_n(x)h_n(x) - f_n(x)h(x) + f_n(x)h(x) \\ -f(x)h(x) \end{array}\right| : x \in U_X\right\} \\
&\leq \sup\{v(x)|f_n(x)||h_n(x) - h(x)| : x \in U_X\} \\
&+ \sup\{v(x)|h(x)||f_n(x) - f(x)| : x \in U_X\} \\
&\leq \|f_n\|_v \|h_n - h\|_\infty + \|h\|_\infty \|f_n - f\|_v \to 0
\end{aligned}$$

as $\|h_n - h\|_\infty \to 0$ and $\|f_n - f\|_v \to 0$. □

Theorem 5.2. *Let V be a countable family of continuous weights on U_X. Let $\prod : \mathbb{R} \times HV(U_X) \to H(U_X)$ be defined as $\prod(t, f) = M_{\psi_t} f$, for every $t \in \mathbb{R}$ and $f \in HV(U_X)$. Then $\prod$ is a (linear) dynamical system on $HV(U_X)$.*

Proof. Since for every $t \in \mathbb{R}$, M_{ψ_t} is a multiplication operator on $HV(U_X)$, it follows that $\prod(t, f) \in HV(U_X)$, for every $t \in \mathbb{R}$ and $f \in HV(U_X)$. Thus $\prod$ is a function from $\mathbb{R} \times HV(U_X) \to HV(U_X)$. Now, it can be easily seen that $\prod$ is linear and $\prod(0, f) = f$, for every $f \in HV(U_X)$. Also, it is obvious that $\prod(t + s, f) = \prod(t, \prod(s, f))$,for every $t, s \in \mathbb{R}$ and $f \in HV(U_X)$. In order to show that $\prod$ is a dynamical system on $HV(U_X)$, it is enough to show that $\prod$ is jointly continuous. Let $\{(t_n, f_n)\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R} \times HV(U_X)$ such that $(t_n, f_n) \to (t, f)$ in $\mathbb{R} \times HV(U_X)$. We shall show that $\prod(t_n, f_n) \to \prod(t, f)$ in $HV(U_X)$. That is, we have to prove that $\psi_{t_n} f_n$ converges to $\psi_t f$ in $HV(U_X)$. But this follows from a more general fact that we have proved in Theorem 5.1. Hence $\prod$ is a linear dynamical system on $HV(U_X)$. □

Remark 5.3. (i) Let $\mathcal{F} = \{M_{\psi_t} : t \in \mathbb{R}\}$ be the family of multiplication operators defined above. Then the following observations are straightforward:

(a) $M_{\psi_{t+s}} f = M_{\psi_t}(M_{\psi_s} f)$, for every $t, s \in \mathbb{R}$ and $f \in HV(U_X)$.
(b) $M_{\psi_0} f = f$, for every $f \in HV(U_X)$.
(c) $\lim\limits_{t \to 0} M_{\psi_t} f = f$, for every $f \in HV(U_X)$.

Thus the family $\mathcal{F}$ is a C_0-group of multiplication operators on $HV(U_X)$.

(ii) Also, we show that the family $\mathcal{F}$ is locally equicontinuous in $B(HV(U_X))$. That is, for every fixed $s \in \mathbb{R}$, the subfamily $\mathcal{F}_s = \{M_{\psi_t} : -s \leq t \leq s\}$ is equicontinuous on $HV(U_X)$. Since the map $t \to M_{\psi_t}$ is continuous, where $B(HV(U_X))$ is the space of all continuous linear operators on $HV(U_X)$ with the strong operator topology , we conclude that for each $s \in \mathbb{R}$, the family $\mathcal{F}_s = \{M_{\psi_t} : -s \leq t \leq s\}$ is a bounded set in $B(HV(U_X))$. Also, for each $f \in HV(U_X)$, the set $\mathcal{F}_s(f) = \{M_{\psi_t} f : -s \leq t \leq s\}$ is bounded in $HV(U_X)$. Now, from a Corollary of the Banach–Steinhaus Theorem ([20, Theorem 2.6]), it follows that the family $\mathcal{F}$ is locally equicontinuous.

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