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ON RELATIONS AMONG SOLUTIONS OF THE HERMITIAN MATRIX EQUATION $AXA^* = B$ AND ITS THREE SMALL EQUATIONS

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Dedicated to Professor Tsuyoshi Ando for his significant contributions in matrix and operator theory

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ABSTRACT. Assume that the linear matrix equation $AXA^* = B = B^*$ has a Hermitian solution and is partitioned as $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X[\,A_1^*,\, A_2^*\,] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21}^* & B_{22} \end{bmatrix}$. We study in this paper relations among the Hermitian solutions of the equation and the three small-size matrix equations $A_1X_1A_1^* = B_{11}$, $A_1X_2A_2^* = B_{12}$ and $A_2X_3A_2^* = B_{22}$. In particular, we establish closed-form formulas for calculating the maximal and minimal ranks and inertias of $X - X_1 - X_2 - X_2^* - X_3$, and use the formulas to derive necessary and sufficient conditions for the Hermitian matrix equality $X = X_1 + X_2 + X_2^* + X_3$ to hold and Hermitian matrix inequalities $X > (\geqslant, <, \leqslant)\, X_1 + X_2 + X_2^* + X_3$ to hold in the Löwner partial ordering.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\mathrm{H}}^{m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively; the symbols A^T, A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the transpose, conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. $A > 0$ ($A \geqslant 0$) means that A is Hermitian positive definite (Hermitian positive

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semi-definite). Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ ($A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive definite (positive semi-definite). The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^{*} = AX, \quad \text{(iv) } (XA)^{*} = XA.$$

Further, denote $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$. The ranks of E_A and F_A are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. The inertia of A is defined to be the triplet $\mathrm{In}(A) = \{\, i_{+}(A),\, i_{-}(A),\, i_{0}(A)\,\}$, where $i_{+}(A)$, $i_{-}(A)$ and $i_{0}(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_{+}(A)$ and $i_{-}(A)$ are usually called partial inertia of A. For a matrix $A \in \mathbb{C}^{m}_{\mathrm{H}}$, both $r(A) = i_{+}(A) + i_{-}(A)$ and $i_{0}(A) = m - r(A)$ hold.

Consider the following well-known Hermitian linear matrix equation

$$AXA^{*} = B, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m}_{\mathrm{H}}$ are given, and $X \in \mathbb{C}^{n\times n}$ is an unknown matrix. Eq. (1.1) is one of the simplest linear matrix equations with symmetric pattern, which has attracted much attention of many authors since 1970s. If the known matrices in (1.1) are given in partitioned form

$$A_i \in \mathbb{C}^{m_i\times n}, \quad B_{ii} \in \mathbb{C}^{m_i}_{\mathrm{H}}, \quad B_{12} \in \mathbb{C}^{m_1\times m_2}, \quad m_1 + m_2 = m, \quad i = 1, 2,$$

we can rewrite (1.1) as

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X[\, A_1^{*},\, A_2^{*}\,] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21}^{*} & B_{22} \end{bmatrix}, \tag{1.2}$$

Comparing both sides of (1.2), we obtain the following triple equations

$$A_1XA_1^{*} = B_{11}, \quad A_1XA_2^{*} = B_{12}, \quad A_2XA_2^{*} = B_{22}. \tag{1.3}$$

In other words, (1.2) can be regarded as a combination of three small-size equations. The triple equations in (1.3) do not necessarily have a common Hermitian solution. In this case, we rewrite (1.3) as three independent matrix equations as follows

$$A_1X_1A_1^{*} = B_{11}, \quad A_1X_2A_2^{*} = B_{12}, \quad A_2X_3A_2^{*} = B_{22}. \tag{1.4}$$

It is obvious that solvability conditions and general (Hermitian) solutions of the triple matrix equations are not necessarily the same as those for (1.2), and therefore, it would be of interest to consider possible relations among the four equations in (1.2) and (1.4) in general cases. In this paper, we consider the following decomposition of a Hermitian solution X of (1.1) into the sum of solutions of the equations in (1.4):

$$X = X_1 + X_2 + X_2^{*} + X_3. \tag{1.5}$$

In addition, we consider the following four Hermitian matrix inequalities

$$X > \ (\geqslant, <, \leqslant)\ X_1 + X_2 + X_2^{*} + X_3 \tag{1.6}$$

in the Löwner partial ordering.

It has just been realized in matrix theory that expansion formulas for ranks and inertias of matrices are simple and useful methods for characterizing properties of matrices and their operations. In the past two decades, many types of closed-form formula for calculating (maximal and minimal) ranks and inertias of matrices were systematically established. These formulas bring deep insights into relations among matrices and their operations, and lead to many essential developments in elementary linear algebra and matrix theory. One of remarkable applications of matrix rank and inertias formulas, as described in Lemma 1.1 below, is to establish or characterize various complicated matrix equalities and inequalities. In this paper, we first establish a group of analytical formulas for calculating the maximal and minimal ranks and inertias of the matrix $X - X_1 - X_2 - X_2^* - X_3$, and then use them to characterize the equality in (1.5) and the inequalities in (1.6). This work is motivated by some recent results on additive decompositions of solutions of the matrix equation $AXB = C$ in [15] and of g-inverses of general matrices in [21], as well as additive decomposition of Hermitian solutions of the matrix equation $AXA^* = B$ and the open problems in [17].

We shall use the following results on ranks and inertias of matrices in the latter part of this paper.

Lemma 1.1 ([14])**.** *Let $\mathcal{S}$ be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_{\mathrm{H}}^{m}$. Then, the following hold.*

(a) *Under $m = n$, $\mathcal{S}$ has a nonsingular matrix if and only if* $\max_{X\in\mathcal{S}} r(X) = m$.

(b) *Under $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if* $\min_{X\in\mathcal{S}} r(X) = m$.

(c) $0 \in \mathcal{S}$ *if and only if* $\min_{X\in\mathcal{S}} r(X) = 0$.

(d) $\mathcal{S} = \{0\}$ *if and only if* $\max_{X\in\mathcal{S}} r(X) = 0$.

(e) *There exists a matrix $X > 0$ ($X < 0$) in $\mathcal{H}$ if and only if*

$$\max_{X\in\mathcal{H}} i_+(X) = m \quad \left(\max_{X\in\mathcal{H}} i_-(X) = m\right).$$

(f) *All $X \in \mathcal{H}$ satisfy $X > 0$ ($X < 0$) if and only if*

$$\min_{X\in\mathcal{H}} i_+(X) = m \quad \left(\min_{X\in\mathcal{H}} i_-(X) = m\right).$$

(g) *There exists a matrix $X \geqslant 0$ ($X \leqslant 0$) in $\mathcal{H}$ if and only if*

$$\min_{X\in\mathcal{H}} i_-(X) = 0 \quad \left(\min_{X\in\mathcal{H}} i_+(X) = 0\right).$$

(h) *All $X \in \mathcal{H}$ satisfy $X \geqslant 0$ ($X \leqslant 0$) if and only if*

$$\max_{X\in\mathcal{H}} i_-(X) = 0 \quad \left(\max_{X\in\mathcal{H}} i_+(X) = 0\right).$$

Lemma 1.2 ([12]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then, the following rank expansion formulas hold*

$$r[\,A,\, B\,] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.7}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.8}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.9}$$

The following rank expansion formulas follow directly from (1.7)–(1.9)

$$r\begin{bmatrix} A & BF_P \\ E_Q C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q), \tag{1.10}$$

$$r\begin{bmatrix} E_P A F_Q & E_P B \\ CF_Q & D \end{bmatrix} = r\begin{bmatrix} A & B & P \\ C & D & 0 \\ Q & 0 & 0 \end{bmatrix} - r(P) - r(Q), \tag{1.11}$$

$$r\begin{bmatrix} M & N \\ E_P A & E_P B \end{bmatrix} = r\begin{bmatrix} M & N & 0 \\ A & B & P \end{bmatrix} - r(P), \tag{1.12}$$

$$r\begin{bmatrix} M & AF_P \\ N & BF_P \end{bmatrix} = r\begin{bmatrix} M & A \\ N & B \\ 0 & P \end{bmatrix} - r(P). \tag{1.13}$$

The following results are well known.

Lemma 1.3. *Let $A \in \mathbb{C}^m_{\mathrm{H}}$, $B \in \mathbb{C}^n_{\mathrm{H}}$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then,*

$$i_{\pm}(PAP^*) = i_{\pm}(A), \tag{1.14}$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & if\ \lambda > 0 \\ i_{\mp}(A) & if\ \lambda < 0 \end{cases}, \tag{1.15}$$

$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \tag{1.16}$$

$$i_{+}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_{-}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \tag{1.17}$$

Lemma 1.4 ([14]). *Let $A \in \mathbb{C}^m_{\mathrm{H}}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{m\times n}$, and let*

$$U = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad V = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the following inertia expansion formulas hold

$$i_{\pm}(U) = r(B) + i_{\pm}(E_B A E_B), \tag{1.18}$$

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}. \tag{1.19}$$

In particular, the following hold.

(a) *If* $A \geqslant 0$, *then*

$$i_+(U) = r[\,A,\, B\,], \quad i_-(U) = r(B), \quad r(U) = r[\,A,\, B\,] + r(B). \tag{1.20}$$

(b) *If* $A \leqslant 0$, *then*

$$i_+(U) = r(B), \quad i_-(U) = r[\,A,\, B\,], \quad r(U) = r[\,A,\, B\,] + r(B). \tag{1.21}$$

(c) *If* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *then*

$$i_\pm(V) = i_\pm(A) + i_\pm(\,D - B^*A^\dagger B\,), \quad r(V) = r(A) + r(\,D - B^*A^\dagger B\,). \tag{1.22}$$

(d) *If* $\mathscr{R}(B) \cap \mathscr{R}(A) = \{0\}$ *and* $\mathscr{R}(B^*) \cap \mathscr{R}(D) = \{0\}$, *then*

$$i_\pm(V) = i_\pm(A) + i_\pm(D) + r(B), \quad r(V) = r(A) + 2r(B) + r(D). \tag{1.23}$$

The following inertia expansion formulas follow directly from (1.18) and (1.19)

$$i_\pm\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P), \tag{1.24}$$

$$i_\pm\begin{bmatrix} E_QAE_Q & E_QB \\ B^*E_Q & D \end{bmatrix} = i_\pm\begin{bmatrix} A & B & Q \\ B^* & D & 0 \\ Q^* & 0 & 0 \end{bmatrix} - r(Q). \tag{1.25}$$

The two-sided matrix equations $AXA^* = B$ and $AXB = C$ have many essential applications in mathematics and other fields, and were extensively studied in the literature; see, e.g., [1, 3, 5, 6, 7, 9, 11, 13, 14, 18, 19, 20]. Concerning the consistency and general solutions of $AXA^* = B$ and $AXB = C$, the following results are well known; see, e.g., [5, 7, 13, 14].

Lemma 1.5. *There exists an* $X \in \mathbb{C}^n_{\mathrm{H}}$ *such that* (1.1) *holds if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *or equivalently,* $AA^\dagger B = B$. *In this case, the general Hermitian solution of* (1.1) *can be written as*

$$X = A^\dagger B(A^\dagger)^* + F_AV + V^*F_A, \tag{1.26}$$

where $V \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Lemma 1.6. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, *and* $C \in \mathbb{C}^{m\times q}$ *be given. Then, the matrix equation*

$$AXB = C \tag{1.27}$$

has a solution for $X \in C^{n\times p}$ *if and only if* $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$, *or equivalently,* $AA^\dagger CB^\dagger B = C$. *In this case, the general solution of* (1.27) *can be written in the following parametric form*

$$X = A^\dagger CB^\dagger + F_AV_1 + V_2E_B, \tag{1.28}$$

where V_1, $V_2 \in \mathbb{C}^{n\times p}$ *are arbitrary.*

Lemma 1.7 ([10, 16]). *Let $A \in \mathbb{C}_{\rm H}^{m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then, closed-form formulas for calculating the global maximal and minimal rank and inertias of $A - BXC - (BXC)^*$ are given by*

$$\max_{X\in\mathbb{C}^{n\times p}} r[\, A - BXC - (BXC)^* \,] \tag{1.29}$$

$$= \min\left\{ r[\, A,\, B,\, C^* \,],\quad r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},\quad r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{1.30}$$

$$\min_{X\in\mathbb{C}^{n\times p}} r[\, A - BXC - (BXC)^* \,] \tag{1.31}$$

$$= 2r[\, A,\, B,\, C^* \,] + \max\{\, s_+ + s_-,\quad t_+ + t_-,\quad s_+ + t_-,\quad s_- + t_+ \,\}, \tag{1.32}$$

and

$$\max_{X\in\mathbb{C}^{n\times p}} i_{\pm}[\, A - BXC - (BXC)^* \,] = \min\left\{ i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},\ i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{1.33}$$

$$\min_{X\in\mathbb{C}^{n\times p}} i_{\pm}[\, A - BXC - (BXC)^* \,] = r[\, A,\, B,\, C^* \,] + \max\{\, s_{\pm},\ t_{\pm} \,\}, \tag{1.34}$$

where

$$s_{\pm} = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r\begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix},\quad t_{\pm} = i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}.$$

In particular, if $\mathscr{R}(B) \subseteq \mathscr{R}(C^*)$, then

$$\max_{X\in\mathbb{C}^{n\times p}} r[\, A - BXC - (BXC)^* \,] = \min\left\{ r[\, A,\, C^* \,],\quad r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \tag{1.35}$$

$$\min_{X\in\mathbb{C}^{n\times p}} r[\, A - BXC - (BXC)^* \,] = 2r[\, A,\, C^* \,] + r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.36}$$

$$\max_{X\in\mathbb{C}^{n\times p}} i_{\pm}[\, A - BXC - (BXC)^* \,] = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{1.37}$$

$$\min_{X\in\mathbb{C}^{n\times p}} i_{\pm}[\, A - BXC - (BXC)^* \,] = r[\, A,\, C^* \,] + i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.38}$$

and

$$\max_{X\in\mathbb{C}^{n\times m}} r[\, A - BX - (BX)^* \,] = \min\left\{ m,\quad r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \tag{1.39}$$

$$\min_{X\in\mathbb{C}^{n\times m}} r[\, A - BX - (BX)^* \,] = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B), \tag{1.40}$$

$$\max_{X\in\mathbb{C}^{n\times m}} i_{\pm}[\, A - BX - (BX)^* \,] = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{1.41}$$

$$\min_{X\in\mathbb{C}^{n\times m}} i_{\pm}[\, A - BX - (BX)^* \,] = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(B). \tag{1.42}$$

The matrices X that satisfy (1.30)–(1.42) (namely, the global maximizers and minimizers of the objective rank and inertia functions) are not necessarily unique and their expressions were also given in [10] by using certain simultaneous decomposition of the three given matrices.

2. Equalities for solutions of $AXA^* = B$ and the small-size equations

Theorem 2.1. *Assume that the matrix equation (1.1) has a Hermitian solution, and Let $\mathcal{S}$ be the set of all Hermitian solutions of (1.1), $\mathcal{T}$ be the set of all sums of $X_1 + X_2 + X_2^* + X_3$, where X_1 and X_3 are Hermitian solutions of the first and third matrix equations in (1.4), respectively, X_2 is solution of the second matrix equation in (1.4). Also let*

$$P = \begin{bmatrix} 0 & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix}. \tag{2.1}$$

Then, the following hold.

(a) $\mathcal{S} \cap \mathcal{T} \neq \emptyset$ *if and only if* $r\begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} = 2r(A)$.

(b) $\mathcal{S} \supseteq \mathcal{T}$ *if and only if* $A = 0$ *or* $r\begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} + 2r(A) = 2r(A_1) + 2r(A_2)$.

(c) $\mathcal{S} \subseteq \mathcal{T}$ *if and only if* $r(A) = r(A_1) + r(A_2)$ *or* $r\begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} = 2r(A)$.

Proof. It is easy to see from the definition of rank of matrix that for two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ consisting of matrices of the same size, the following assertions

$$\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset \Leftrightarrow \min_{A \in \mathcal{S}_1,\, B \in \mathcal{S}_2} r(\, A - B\,) = 0, \tag{2.2}$$

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow \max_{A \in \mathcal{S}_1} \min_{B \in \mathcal{S}_2} r(\, A - B\,) = 0 \tag{2.3}$$

hold. Hence, we see from (2.2) and (2.3) that (1.6) is equivalent to

$$\min_{X \in \mathcal{S},\, X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0; \tag{2.4}$$

while the set inclusion $\mathcal{S} \supseteq \mathcal{T}$ is equivalent to

$$\max_{X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}} \min_{X \in \mathcal{S}} r(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0; \tag{2.5}$$

and set inclusion $\mathcal{S} \subseteq \mathcal{T}$ is equivalent to

$$\max_{X \in \mathcal{S}} \min_{X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0. \tag{2.6}$$

By Lemmas 1.5 and 1.6, $X - X_1 - X_2 - X_2^* - X_3$ can be written as

$$\begin{aligned} & X - X_1 - X_2 - X_2^* - X_3 \\ & = G + F_A V + V^* F_A - F_{A_1} V_1 - V_1^* F_{A_1} - F_{A_1} U_1 - U_2 F_{A_2} - U_1^* F_{A_1} - F_{A_2} U_2^* \\ & \quad - F_{A_2} V_2 - V_2^* F_{A_2} \\ & = G + HW + W^* H^*, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} G = {} & A^\dagger B (A^\dagger)^* - (A_1)^\dagger B_{11} (A_1^\dagger)^* - (A_1)^\dagger B_{12} (A_2^\dagger)^* \\ & - (A_2)^\dagger B_{12}^* (A_1^\dagger)^* - (A_2)^\dagger B_{22} (A_2^\dagger)^*, \\ H = {} & [\, F_A,\ F_{A_1},\ F_{A_2},\ F_{A_1},\ F_{A_2}\,], \end{aligned}$$

and $W^* = [\, V^*,\ -V_1^*,\ -U_2,\ -U_1^*,\ -V_2^* \,]$ is arbitrary. Applying (1.40) to (2.7) gives

$$\min_{X \in \mathcal{S},\, X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3 \,)$$
$$= \min_{W} r(\, G + HW + W^*H^*) = r\begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} - 2r(H). \tag{2.8}$$

Applying (1.10) to the row block matrix in (2.8) and simplifying, we obtain

$$\begin{aligned} r(H) &= r[\, F_A,\ F_{A_1},\ F_{A_2},\ F_{A_1},\ F_{A_2} \,] = r[\, F_A,\ F_{A_1},\ F_{A_2} \,] \\ &= r\begin{bmatrix} I_n & I_n & I_n \\ A & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} - r(A) - r(A_1) - r(A_2) \\ &= n + r\begin{bmatrix} A & A \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} - r(A) - r(A_1) - r(A_2) \\ &= n + r(A) - r(A_1) - r(A_2), \qquad (2.9) \\ r[\, F_{A_1},\ F_{A_2} \,] &= r\begin{bmatrix} I_n & I_n \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} - r(A_1) - r(A_2) \\ &= n + r(A) - r(A_1) - r(A_2). \qquad (2.10) \end{aligned}$$

Comparing (2.9) and (2.10) gives $r[\, F_A,\ F_{A_1},\ F_{A_2} \,] = r[\, F_{A_1},\ F_{A_2} \,]$, that is

$$\mathscr{R}(\, F_A \,) \subseteq \mathscr{R}[\, F_{A_1},\ F_{A_2} \,]. \tag{2.11}$$

From (2.11) and the given assumptions $AA^{\dagger}B = B$, $A_1A_1^{\dagger}B_{11} = B_{11}$, $A_2A_2^{\dagger}B_{22} = B_{22}$, $A_1A_1^{\dagger}B_{12}(A_2^{\dagger})^*A_2^* = B_{12}$, $A_2A_2^{\dagger}B_{12}^*(A_1^{\dagger})^*A_1^* = B_{12}^*$, we obtain

$$r\begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} = r\begin{bmatrix} G & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 \\ F_{A_2} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} G & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A_1^* & 0 \\ I_n & 0 & 0 & 0 & A_2^* \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{bmatrix} - 2r(A_1) - 2r(A_2)$$

$$= r\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A_1^* & 0 \\ I_n & 0 & 0 & 0 & A_2^* \\ B_{11}(A_1^{\dagger})^* + A_1A_1^{\dagger}B_{12}(A_2^{\dagger})^* & A_1 & 0 & 0 & 0 \\ B_{22}((A_2)^{\dagger})^* + A_2A_2^{\dagger}B_{12}^*(A_1^{\dagger})^* & 0 & A_2 & 0 & 0 \end{bmatrix} - 2r(A_1) - 2r(A_2)$$

$$= r\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & -A_1^* & 0 \\ 0 & 0 & 0 & A_1^* & A_2^* \\ 0 & -A_1 & A_1 & -B_{11} & B_{12} \\ 0 & 0 & A_2 & B_{12}^* & -B_{22} \end{bmatrix} - 2r(A_1) - 2r(A_2)$$

$$= r\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & 0 & 0 \\ I_n & 0 & 0 & A^*\begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & A^* \\ 0 & \begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}A & A & \begin{bmatrix} -B_{11} & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix} \end{bmatrix} - 2r(A_1) - 2r(A_2)$$

$$= r\begin{bmatrix} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & A^*\begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & A^* \\ 0 & \begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}A & A & \begin{bmatrix} 0 & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix} \end{bmatrix} - 2r(A_1) - 2r(A_2)$$

$$= 2n + r\begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} - 2r(A_1) - 2r(A_2). \tag{2.12}$$

Substituting (2.9) and (2.12) into (2.8) yields

$$\min_{X\in\mathcal{S},\ X_1+X_2+X_2^*+X_3\in\mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3\,) = r\begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} - 2r(A). \tag{2.13}$$

Substituting (2.13) into (2.4) leads to (a).

Applying (1.42) to (2.7) gives

$$\begin{aligned}&\min_{X\in\mathcal{S}} r(\,X - X_1 - X_2 - X_2^* - X_3\,)\\&= \min_{V} r\left[A^\dagger B(A^\dagger)^* + F_A V + V^* F_A - X_1 - X_2 - X_2^* - X_3\right]\\&= r\begin{bmatrix} A^\dagger B(A^\dagger)^* - X_1 - X_2 - X_2^* - X_3 & F_A\\ F_A & 0\end{bmatrix} - 2r(F_A).\end{aligned}\tag{2.14}$$

Applying (1.37) to the block matrix in (2.14) and simplifying, we obtain

$$\begin{aligned}&\max_{X_1+X_2+X_2^*+X_3\in\mathcal{T}} r\begin{bmatrix} A^\dagger B(A^\dagger)^* - X_1 - X_2 - X_2^* - X_3 & F_A\\ F_A & 0\end{bmatrix}\\&= \max_{V_1,\,V_2,\,U_1,\,U_2} r\left(\begin{bmatrix} G & F_A\\ F_A & 0\end{bmatrix} - \begin{bmatrix} F_{A_1} & F_{A_2} & F_{A_1} & F_{A_2}\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_1\\ U_2^*\\ U_1\\ V_2\end{bmatrix}[\,I_n,\, 0\,]\right.\\&\qquad\left. - \begin{bmatrix} I_n\\ 0\end{bmatrix}[\,V_1^*,\, U_2,\, U_1^*,\, V_2^*\,]\begin{bmatrix} F_{A_1} & 0\\ F_{A_2} & 0\\ F_{A_1} & 0\\ F_{A_2} & 0\end{bmatrix}\right)\\&= \min\left\{ r\begin{bmatrix} G & F_A & I_n\\ F_A & 0 & 0\end{bmatrix},\ r\begin{bmatrix} G & H\\ H^* & 0\end{bmatrix}\right\}\\&= \min\left\{ 2n - r(A),\ 2n + r\begin{bmatrix} 0 & A^*\\ A & P\end{bmatrix} - 2r(A_1) - 2r(A_2)\right\}.\end{aligned}\tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\begin{aligned}&\max_{X_1+X_2+X_2^*+X_3\in\mathcal{T}}\ \min_{X\in\mathcal{S}} r(\,X - X_1 - X_2 - X_2^* - X_3\,)\\&= \min\left\{ r(A),\ \ r\begin{bmatrix} 0 & A^*\\ A & P\end{bmatrix} + 2r(A) - 2r(A_1) - 2r(A_2)\right\}.\end{aligned}\tag{2.16}$$

Substituting (2.16) into (2.5) leads to

$$A = 0 \ \text{ or } \ r\begin{bmatrix} 0 & A^*\\ A & P\end{bmatrix} + 2r(A) = 2r(A_1) + 2r(A_2).\tag{2.17}$$

Also note that if $A = 0$, then $B = 0$ in (1.1) under the given assumption. In such a case, the second rank equality in (2.17) holds as well under the given assumption. Thus, we obtain (b).

Let

$$H = (A_1)^\dagger B_{11}(A_1^\dagger)^* + (A_1)^\dagger B_{12}(A_2^\dagger)^* + (A_2)^\dagger B_{12}^*(A_1^\dagger)^* + (A_2)^\dagger B_{22}(A_2^\dagger)^*,$$

and rewrite (2.7) as

$$\begin{aligned} & X - X_1 - X_2 - X_2^* - X_3 \\ & = X - H - [\, F_{A_1},\, F_{A_2},\, F_{A_1},\, F_{A_2} \,] \begin{bmatrix} V_1 \\ U_2^* \\ U_1 \\ V_2 \end{bmatrix} - [\, V_1^*,\, U_2,\, U_1^*,\, V_2^* \,] \begin{bmatrix} F_{A_1} \\ F_{A_2} \\ F_{A_1} \\ F_{A_2} \end{bmatrix}. \end{aligned}$$

Applying (1.20) to this expression gives

$$\begin{aligned} & \min_{X_1+X_2+X_2^*+X_3 \in \mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3 \,) \\ & = r \begin{bmatrix} X-H & F_{A_1} & F_{A_2} & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 \end{bmatrix} - 2r[\, F_{A_1}\, F_{A_2},\;\, F_{A_1}\, F_{A_2} \,]. \end{aligned} \tag{2.18}$$

Applying (1.37) to the 5×5 block matrix in (2.18) gives

$$\begin{aligned} & \max_{V} r \left(\begin{bmatrix} G & F_{A_1} & F_{A_2} & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} F_A \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V[\, I_n,\, 0,\, 0,\, 0,\, 0 \,] \right. \\ & \qquad \left. + \begin{bmatrix} I_n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V^*[F_A,\, 0,\, 0,\, 0,\, 0\,] \right) \\ & = \min \left\{ r \begin{bmatrix} G & F_{A_1} & F_{A_2} & F_{A_1} & F_{A_2} & I_n \\ F_{A_1} & 0 & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\;\; r \begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} \right\} \\ & = \min \left\{ n + r[\, F_{A_1},\, F_{A_2},\, F_{A_1},\, F_{A_2} \,],\, r \begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} \right\} \\ & = \min \left\{ 2n + r(A) - r(A_1) - r(A_2),\, 2n + r \begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} - 2r(A_1) - 2r(A_2) \right\}. \end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19) yields

$$\begin{aligned} & \max_{X \in \mathcal{S}} \min_{X_1+X_2+X_2^*+X_3 \in \mathcal{T}} r(\, X - X_1 - X_2 - X_2^* - X_3 \,) \\ & \quad = \min \left\{ r(A_1) + r(A_2) - r(A),\;\, r \begin{bmatrix} 0 & A^* \\ A & P \end{bmatrix} - 2r(A) \right\}. \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.6) gives (c). □

3. Inequalities for solutions of $AXA^* = B$ and the small-size equations

Theorem 3.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be defined as in Theorem 2.1. Also let*

$$Q = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix}.$$

Then, the following hold.

(a) *There exist $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ such that $X > X_1+X_2+X_2^*+X_3$ if and only if*

$$i_+\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A_1) + r(A_2). \tag{3.1}$$

(b) *There exist $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ such that $X < X_1+X_2+X_2^*+X_3$ if and only if*

$$i_-\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A_1) + r(A_2). \tag{3.2}$$

(c) *All $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ satisfy $X \geqslant X_1 + X_2 + X_2^* + X_3$ if and only if*

$$i_-\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} + n = r(A_1) + r(A_2). \tag{3.3}$$

(d) *All $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ satisfy $X \leqslant X_1 + X_2 + X_2^* + X_3$ if and only if*

$$i_+\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} + n = r(A_1) + r(A_2). \tag{3.4}$$

(e) *There exist $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ such that $X \geqslant X_1+X_2+X_2^*+X_3$ if and only if*

$$i_-\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A). \tag{3.5}$$

(f) *There exist $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ such that $X \leqslant X_1+X_2+X_2^*+X_3$ if and only if*

$$i_+\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A). \tag{3.6}$$

(g) *All $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ satisfy $X > X_1 + X_2 + X_2^* + X_3$ if and only if*

$$i_+\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A) + n. \tag{3.7}$$

(h) *All* $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ *satisfy* $X < X_1 + X_2 + X_2^* + X_3$ *if and only if*

$$i_-\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} = r(A) + n. \tag{3.8}$$

Proof. Let

$$G_1 = \frac{1}{2}A_1^\dagger B_{11} + \frac{1}{2}A_2^\dagger B_{12}^*(A_1^\dagger)^* A_1^*, \quad G_2 = \frac{1}{2}A_2^+ B_{22} + \frac{1}{2}A_1^\dagger B_{12}(A_2^\dagger)^* A_2^*.$$

Note that $X > X_1 + X_2 + X_2^* + X_3$ ($X < X_1 + X_2 + X_2^* + X_3$) for some $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ is equivalent to

$$\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_+(\, X - X_1 - X_2 - X_2^* - X_3\,) = n \tag{3.9}$$
$$\left(\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_-(\, X - X_1 - X_2 - X_2^* - X_3\,) = n\right);$$

$X \geqslant X_1 + X_2 + X_2^* + X_3$ ($X \leqslant X_1 + X_2 + X_2^* + X_3$) holds for some $X \in \mathcal{S}$, $X_1 + X_2 + X_2^* + X_3 \in \mathcal{T}$ is equivalent to

$$\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_-(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0 \tag{3.10}$$
$$\left(\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_+(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0\right);$$

$X > X_1+X_2+X_2^*+X_3$ ($X < X_1+X_2+X_2^*+X_3$) for all $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ is equivalent to

$$\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_+(\, X - X_1 - X_2 - X_2^* - X_3\,) = n \tag{3.11}$$
$$\left(\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_-(\, X - X_1 - X_2 - X_2^* - X_3\,) = n\right);$$

$X \geqslant X_1+X_2+X_2^*+X_3$ ($X \leqslant X_1+X_2+X_2^*+X_3$) for all $X \in \mathcal{S}$, $X_1+X_2+X_2^*+X_3 \in \mathcal{T}$ is equivalent to

$$\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_-(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0 \tag{3.12}$$
$$\left(\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_+(\, X - X_1 - X_2 - X_2^* - X_3\,) = 0\right).$$

From (2.7), (1.41) and (1.42), we have

$$\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_{\pm}(\, X - X_1 - X_2 - X_2^* - X_3\,)$$
$$= \min_{W} i_{\pm}(\, G + HW + W^*H^*)$$
$$= i_{\pm}\begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} - r(H), \tag{3.13}$$
$$\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_{\pm}(\, X - X_1 - X_2 - X_2^* - X_3\,)$$
$$= \max_{W} i_{\pm}(\, G + HW + W^*H^*)$$
$$= i_{\pm}\begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix}. \tag{3.14}$$

From (2.11) and the given assumptions, we obtain

$$i_{\pm}\begin{bmatrix} G & H \\ H^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} G & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} G & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 \\ F_{A_2} & 0 & 0 \end{bmatrix}$$
$$= i_{\pm}\begin{bmatrix} G & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A_1^* & 0 \\ I_n & 0 & 0 & 0 & A_2^* \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{bmatrix} - r(A_1) - r(A_2)$$
$$= i_{\pm}\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & I_n & G_1 & G_2 \\ I_n & 0 & 0 & A_1^* & 0 \\ I_n & 0 & 0 & 0 & A_2^* \\ G_1^* & A_1 & 0 & 0 & 0 \\ G_2^* & 0 & A_2 & 0 & 0 \end{bmatrix} - r(A_1) - r(A_2)$$
$$= i_{\pm}\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & -A_1^* & 0 \\ 0 & 0 & 0 & A_1^* & A_2^* \\ 0 & -A_1 & A_1 & -B_{11} & B_{12} \\ 0 & 0 & A_2 & B_{12}^* & -B_{22} \end{bmatrix} - r(A_1) - r(A_2)$$
$$= i_{\pm}\begin{bmatrix} A^{\dagger}B(A^{\dagger})^* & I_n & 0 & 0 \\ I_n & 0 & 0 & A^*\begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & A^* \\ 0 & \begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} A & A & \begin{bmatrix} -B_{11} & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix} \end{bmatrix} - r(A_1) - r(A_2)$$

$$
= i_{\pm}\begin{bmatrix} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & A^*\begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & A^* \\ 0 & \begin{bmatrix} -I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}A & A & \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & -B_{22} \end{bmatrix} \end{bmatrix} - r(A_1) - r(A_2) \tag{3.15}
$$

$$
= n + i_{\pm}\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} - r(A_1) - r(A_2). \tag{3.16}
$$

Substituting (3.16) and (2.9) into (3.13) and (3.14) gives

$$
\min_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_{\pm}(\, X - X_1 - X_2 - X_2^* - X_3\,)
$$

$$
= i_{\pm}\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} - r(A), \tag{3.17}
$$

$$
\max_{X\in\mathcal{S},\, X_1+X_2+X_2^*+X_3\in\mathcal{T}} i_{\pm}(\, X - X_1 - X_2 - X_2^* - X_3\,)
$$

$$
= n + i_{\pm}\begin{bmatrix} 0 & A^* \\ A & Q \end{bmatrix} - r(A_1) - r(A_2). \tag{3.18}
$$

Substituting (3.17) and (3.18) into (3.9)–(3.12) yields (a)–(h). □

In addition to (1.5) and (1.6), other two possible equalities for the solutions of (1.1) and (1.4) are

$$
X = (X_1 + X_3)/2, \tag{3.19}
$$

$$
X = (X_1 + X_2 + X_2^* + X_3)/4, \tag{3.20}
$$

(3.21)

while eight possible equalities in the Löwner partial ordering for the solutions of (1.1) and (1.4) are

$$
X > (\geqslant, <, \leqslant)\ (X_1 + X_3)/2, \tag{3.22}
$$

$$
X > (\geqslant, <, \leqslant)\ (X_1 + X_2 + X_2^* + X_3)/4. \tag{3.23}
$$

It is no doubt that closed-form formulas calculating the maximal and minimal ranks and inertias can be established for the difference of both sides of (3.19)–(3.23), while necessary and sufficient conditions for the equality and inequalities to hold can be derived, as demonstrated in the previous two sections, from the rank and inertia formulas.

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