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ORTHOGONALLY ADDITIVE AND ORTHOGONALLY MULTIPLICATIVE HOLOMORPHIC FUNCTIONS OF MATRICES

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. Let $H : M_m \to M_m$ be a holomorphic function of the algebra M_m of complex $m \times m$ matrices. Suppose that H is orthogonally additive and orthogonally multiplicative on self-adjoint elements. We show that either the range of H consists of zero trace elements, or there is a scalar sequence $\{\lambda_n\}$ and an invertible S in M_m such that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in M_m,$$

or

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in M_m.$$

Here, x^t is the transpose of the matrix x. In the latter case, we always have the first representation form when H also preserves zero products. We also discuss the cases where the domain and the range carry different dimensions.

1. INTRODUCTION

Let E and F be real or complex Banach spaces, and n a positive integer. A map $P : E \to F$ is called a *bounded n-homogeneous polynomial* if there is a bounded symmetric n-linear operator $T : E \times \cdots \times E \to F$ such that

$$P(x) = T(x, \ldots, x), \quad \forall x \in E.$$

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In this case, we have

$$T(x_1,\dots,x_n)=\frac{1}{2^n n!}\sum_{\epsilon_i=\pm 1}\epsilon_1\cdots\epsilon_n P\left(\sum_{i=1}^{n}\epsilon_i x_i\right),\quad \forall x_1,\dots,x_n\in E.$$

A map $H:U\to F$ is said to be *holomorphic* on a nonempty open subset U of E if for each a in U there exist an open ball $B_E(a;r)\subseteq U$, centered at a with radius $r>0$, and a unique sequence of bounded n-homogeneous polynomials $P_n:E\to F$ such that

$$H(x)=\sum_{n=0}^{\infty}P_n(x-a)$$

uniformly for all x in $B_E(a;r)$.

To study holomorphic functions, we might assume, after translation, $a=0$. A holomorphic function $H:B_E(0;r)\to F$ has its Taylor series at zero:

$$H(x)=\sum_{n=0}^{\infty}P_n(x) \tag{1.1}$$

uniformly for all x in $B_E(0;r)$. In the complex case, we have the Cauchy integral formulae:

$$P_n(x)=\frac{1}{2\pi i}\int_{|\lambda|=1}\frac{H(\lambda x)}{\lambda^{n+1}}\,d\lambda,\qquad n=0,1,2,\dots. \tag{1.2}$$

For the general theory of homogeneous polynomials and holomorphic functions, we refer to [11, 17].

When E,F are Banach algebras, a function $\Phi:E\to F$ is said to be *orthogonally additive* if

$$fg=gf=0\quad\text{implies}\quad\Phi(f+g)=\Phi(f)+\Phi(g),\quad\forall f,g\in E,$$

and *orthogonally multiplicative* if

$$fg=gf=0\quad\text{implies}\quad\Phi(f)\Phi(g)=0,\quad\forall f,g\in E.$$

The notions of orthogonally additive and orthogonally multiplicative transformations have been studied by many authors, for example, [2, 13, 14, 12, 9, 1, 15, 20, 3, 7, 18, 8, 4, 19, 16].

Our goal is to study orthogonally additive and orthogonally multiplicative holomorphic functions between C^*-algebras. Every abelian C^*-algebra is the algebra $C_0(X)$ of continuous functions of a locally compact Hausdorff space X vanishing at infinity. In general, a C^*-algebra can be embedded into $B(H)$ as a norm closed self-adjoint subalgebra. When E,F are algebras of continuous functions, it is established in [5] the following nice representation.

Proposition 1.1 ([5]). *Let $H:B_{C_0(X)}(0;r)\to C_0(Y)$ be a bounded orthogonally additive and orthogonally multiplicative holomorphic function. Then there exist a sequence $\{h_n\}$ of bounded scalar continuous functions in $C(Y)$ and a map $\varphi:Y\to X$ such that*

$$H(f)(y)=\sum_{n\geq 1}h_n(y)(f(\varphi(y)))^n,\quad\forall y\in Y,$$

uniformly for all f in $B_{C_0(X)}(0;r)$. Here, φ is continuous wherever any h_n is nonvanishing.

Talking orthogonality of a pair of elements a, b in a general C^*-algebra (in $B(H)$), people usually refers one of the following situations.

(i) Zero product: $ab = 0$.
(ii) Two side zero product: $ab = ba = 0$.
(iii) Range orthogonality: $a^*b = 0$.
(iv) Initial space or domain orthogonality: $ab^* = 0$.
(v) Range and domain orthogonality: $a^*b = ab^* = 0$.

In the abelian case, however, all these concepts coincide. They coincide in general when both a, b are self-adjoint.

Some partial results concerning the structures of homogeneous polynomials between general C^*-algebras are also given in [5]. For example, we have

Proposition 1.2 ([5]). *Let H be a complex Hilbert space of arbitrary dimension. Let $P : B(H) \to B(H)$ be a bounded n-homogeneous polynomial, which is additive and multiplicative on pairs of orthogonal self-adjoint elements. Suppose that $P(1)$ is invertible or $P(B(H)) \supset B(H)^+$. Then there is a nonzero scalar λ and an invertible operator S in $B(H)$ such that either*

$$P(x) = \lambda S^{-1} x^n S, \ \forall x \in B(H),$$

or

$$P(x) = \lambda S^{-1} (x^t)^n S, \ \forall x \in B(H).$$

Here, x^t is the transpose of a bounded linear operator x in $B(H)$ with respect to some arbitrary but fixed orthogonal basis of the Hilbert space H. For a matrix $x = (x_{ij})$, we simply define $x^t = (x_{ji})$ to be the transpose of x.

However, results in [5] usually assume a rather strong hypothesis that $P(1)$ is invertible or $P(A) \supset B^+$. It is not very likely every summand P_n in the Taylor series (1.1) of a holomorphic function $H : B_A(0;r) \to B$ would satisfy one of these conditions. Thus, a general structure result about such holomorphic functions is still far away from reaching.

In this paper, we will establish another important case. We will give a description of orthogonally additive and orthogonally multiplicative holomorphic function $H : M_m \to M_m$ of complex matrix algebras.

In the following we say that a map H between complex matrices is *orthogonally additive* (resp. *multiplicative*) *on self-adjoint elements* if $H(a+b) = H(a) + H(b)$ (resp. $H(a)H(b) = 0$) whenever a, b are self-adjoint complex matrices in its domain with $ab = 0$.

Theorem 1.3. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $H : B_{M_m}(0;r) \to M_s$ be a holomorphic function between complex matrix algebras. Assume H is orthogonally additive and orthogonally multiplicative on self-adjoint elements. Then either*

(A) the range of H consists of zero trace elements (this case occurs whenever $s < m$), or

(B) there exist a scalar sequence $\{\lambda_n\}$ *(some* λ_n *can be zero) and an invertible* $m \times m$ *matrix* S *such that*

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in B_{M_m}(0; r), \tag{‡}$$

or

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in B_{M_m}(0; r).$$

In the case (B), we always have the representation (‡) when H *also preserves zero products, i.e.,*

$$ab = 0 \quad \Longrightarrow \; H(a)H(b) = 0, \qquad \forall a, b \in B_{M_m}(0; r).$$

The proof of Theorem 1.3 will be given in the next section. The following example shows that the exception case in Theorem 1.3 can occur when $s = m$.

Example 1.4. Let E_{ij} be the matrix unit with '1' at the (i, j)th entry and '0' elsewhere. Consider the linear map $T : M_2 \to M_2$ defined by $T(E_{11}) = E_{12}$, and $T(E_{ij}) = 0$ for all other i, j. Then T is an orthogonally multiplicative, and linear (and thus holomorphic) map. It is plain that the range of T consists of nilpotent matrices.

On the other hand, we can have other possibilities when the range have larger dimension than the domain.

Example 1.5. Consider $\theta : M_k \to M_{k+2}$ defined by

$$\left(a_{ij}\right) \mapsto \begin{pmatrix} 0 & a_{11} & a_{12} & \dots & a_{1k} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_{11} \\ 0 & 0 & 0 & \dots & 0 & a_{21} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{k1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then θ is linear (and thus holomorphic), and orthogonally multiplicative on self-adjiont elements. Note that the range of θ does not have trivial multiplication, since $\theta(E_{11})^2 = E_{1,k+2}$. However, θ cannot be written as the form $c\varphi$ for any fixed element c in M_{k+2} and any homomorphism or anti-homomorphism $\varphi : M_k \to M_{k+2}$. Assume on the contrary that $\theta = c\varphi$. Then we arrive at a contradiction

$$\begin{aligned} E_{1,k+2} &= \theta(E_{11})^2 = \theta(E_{11})c\varphi(E_{11}) = \theta(E_{11})c(\varphi(E_{12})\varphi(E_{21})) \\ &= \theta(E_{11})(c\varphi(E_{12}))\varphi(E_{21})) = \theta(E_{11})\theta(E_{12})\varphi(E_{21}) = 0\varphi(E_{21}) = 0. \end{aligned}$$

Example 1.6. Consider $\phi : M_k \to M_{2k+2}$ defined by

$$\left(a_{ij}\right) \mapsto \left(a_{ij}\right) \oplus \theta\left(a_{ij}\right).$$

Here, θ is the map defined in Example 1.5. Again θ is linear (and thus holomorphic) and orthogonally multiplicative. However, θ cannot be written in any form stated in Theorem 1.3(B), although its range contains elements of nonzero trace.

The infinite dimensional case can be more complicated.

Example 1.7. Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_n : n = 1, 2, \ldots\}$.

(a) Consider $\theta : B(H) \to B(H)$ defined by

$$\left(a_{ij}\right) \mapsto J^2 \left(a_{ij}\right) J^{*2} + \begin{pmatrix} 0 & a_{11} \\ 0 & 0 \end{pmatrix} \oplus 0.$$

Here, $J : B(H) \to B(H)$ is the unilateral shift operator sending e_n to e_{n+1} for $n = 1, 2, \ldots$. Then θ is not of the standard form, while its range contains elements of nonzero trace.

(b) Let E and F be the isometries in $B(H)$ such that $E(e_n) = e_{2n}$ and $F(e_n) = e_{2n-1}$ for $n = 1, 2, \ldots$, respectively. Define a holomorphic function $H : B(H) \to B(H)$ by

$$H(a) = EaE^* + F(a^t)^2 F^*, \quad \forall a \in B(H).$$

Then H is orthogonally additive and orthogonally multiplicative, but not zero product preserving. (Readers can make up one preserving zero products easily.) The range of H contains the identity $H(1) = 1$. However, it cannot be written in any form stated in Theorem 1.3(B).

2. The proofs

We begin with an observation.

Lemma 2.1. *Let $H : B_E(0; r) \to F$ be a holomorphic function between C^*-algebras with Taylor series at zero $H = \sum_{n=0}^{\infty} P_n$.*

(a) If H is orthogonally additive on self-adjoint elements then each P_n is also orthogonally additive on self-adjoint elements.

(b) If H is orthogonally multiplicative on self-adjoint elements then each P_n is also orthogonally multiplicative on self-adjoint elements. Indeed, for orthogonal self-adjoint elements x, y in $B_E(0; r)$ we have

$$xy = 0 \quad \Longrightarrow \quad P_m(x)P_n(y) = 0, \quad m, n = 0, 1, 2, \ldots.$$

Proof. Let $\{x, y\}$ be an orthogonal pair of self-adjoint elements in $B_E(0; r)$. Suppose first that H is orthogonally additive. For sufficiently small scalar α, we have

$$H(\alpha x + \alpha y) = \sum_n P_n(\alpha x + \alpha y) = \sum_n \alpha^n P_n(x + y)$$

$$= H(\alpha x) + H(\alpha y) = \sum_n (P_n(\alpha x) + P_n(\alpha y)) = \sum_n \alpha^n (P_n(x) + P_n(y)).$$

As α can be arbitrary (but small), we see that

$$P_n(x + y) = P_n(x) + P_n(y), \quad n = 0, 1, 2, \ldots.$$

Suppose then H is orthogonally multiplicative. For sufficiently small scalars α, β, we have

$$0 = H(\alpha x)H(\beta y) = \sum_{m,n} P_m(\alpha x)P_n(\beta y) = \sum_{m,n} \alpha^m \beta^n P_m(x)P_n(y).$$

As α, β can be arbitrary (but small), we see that

$$P_n(x)P_m(y) = 0, \quad n, m = 0, 1, 2, \dots.$$

□

It follows from Lemma 2.1 that if H is orthogonally additive then $P_0 = 0$.

The following linearization of orthogonally additive n-homogeneous polynomials of matrix algebras is an important tool of us. The result for general C^*-algebras is given by C. Palazuelos, A. M. Peralta and I. Villanueva [18], and M. Burgosy, F. J. Fernández-Poloz, J. J. Garcésx and A. M. Peralta [6], which extend the commutative version of D. Perez-Garcia and I. Villanueva [20] (see also [18]).

Lemma 2.2. *Let F be a complex Banach space, and $P : M_m \to F$ an n-homogeneous polynomial. If P is orthogonally additive on self-adjoint elements then there exists a linear operator $T : M_m \to F$ such that*

$$P(x) = T(x^n), \quad \forall x \in M_m.$$

Recall that we say a map θ between rings preserving zero products if $\theta(x)\theta(y) = 0$ whenever $xy = 0$. We say that a set Z of a ring has trivial products, if $xy = 0$ for all x, y in Z.

Lemma 2.3 ([9, Corollary 2.4])**.** *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and $\theta : M_m(\mathbb{F}) \to M_s(\mathbb{F})$ a linear map preserving zero products. Then either the range of θ has trivial multiplication, or $m = s$ and there exist an invertible matrix S in $M_m(\mathbb{F})$ and a nonzero scalar c such that*

$$\theta(x) = cS^{-1}xS \quad \forall x \in M_m(\mathbb{F}).$$

Note that the orthogonal multiplicity of an orthogonal additive polynomial P does not guarantee its linearization T preserving zero products. So we cannot apply Lemma 2.3 directly. But when x, y are idempotents with $xy = 0$, we have $T(x)T(y) = P(x)P(y) = 0$. This suggests the following two lemmas. Fortunately, they are sufficient for our proof of Theorem 1.3.

Lemma 2.4. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\theta : M_m \to M_s$ be a complex linear map. Assume that*

$$\theta(p)\theta(q) = 0 \quad \textit{whenever} \quad p, q \textit{ are orthogonal rank one projections.} \tag{2.1}$$

Then either

(A) the range of θ consists of nilpotent elements (this happens whenever $s < m$), or

(B) $m = s$ and there exist an invertible matrix S in M_m and a nonzero scalar λ such that

$$\theta(x) = \lambda S^{-1}xS, \quad \forall x \in M_m,$$

or

$$\theta(x) = \lambda S^{-1}x^tS, \quad \forall x \in M_m.$$

Proof. Note that (2.1) holds indeed for all orthogonal pairs of self-adjoint matrices (through spectral decompositions). For any projection p in M_m, we have

$$(1-p)p = p(1-p) = 0.$$

The assumption implies that

$$\theta(1-p)\theta(p) = \theta(p)\theta(1-p) = 0.$$

Hence

$$\theta(1)\theta(p) = \theta(p)\theta(1) = \theta(p)^2$$

holds for all projections p in M_m. By the spectral theory, we have

$$\theta(1)\theta(a) = \theta(a)\theta(1) \tag{2.2}$$

for all self-adjoint, and thus for all, a in M_m. Moreover,

$$\theta(1)\theta(a^2) = \theta(a^2)\theta(1) = \theta(a)^2$$

holds for all self-adjoint elements a in M_m. Considering $(a+b)^2$ for two self-adjoint elements a and b, we have

$$\theta(1)\theta(ab+ba) = \theta(a)\theta(b) + \theta(b)\theta(a).$$

Since very complex matrix a in M_m can be written as $a = b + \sqrt{-1}c$ for two self-adjoint matrices b and c, we see that

$$\theta(1)\theta(a^2) = \theta(a^2)\theta(1) = \theta(a)^2, \quad \forall a \in M_m. \tag{2.3}$$

It follows further that

$$\theta(1)\theta(ab+ba) = \theta(a)\theta(b) + \theta(b)\theta(a), \quad \forall a, b \in M_m. \tag{2.4}$$

Suppose that there is an x in M_m such that $\theta(x)$ is not nilpotent. It follows from (2.2) and (2.3) that

$$\theta(1)^s\theta(x^2)^s = (\theta(1)\theta(x^2))^s = \theta(x)^{2s} \neq 0.$$

Consequently, $\theta(1)$ is not nilpotent, and thus its spectrum contains a complex number $\lambda \neq 0$. Using the Riesz functional calculus, we have an idempotent e in M_s such that $e\theta(1) = \theta(1)e = \lambda e \neq 0$ and e commutes with every matrix commuting with $\theta(1)$ (see, e.g., [10, Prop. 4.11]). Since $\theta(1)$ commutes with all $\theta(a)$'s, so does e. Define $\Psi : M_m \to M_s$ by $\Psi(a) = e\theta(a)/\lambda$. It follows from (2.4) that

$$\Psi(ab+ba) = \Psi(a)\Psi(b) + \Psi(b)\Psi(a), \quad \forall a, b \in M_m.$$

By the well-known theorem of Herstein, we see that either $\Psi = 0$, or Ψ is an injective homomorphism or anti-homomorphism. But the first case implies the contradiction $e = \Psi(1) = 0$. Hence, the latter case occurs, and we must have $s = m$. This forces $e = 1$ and thus $\theta(1) = \lambda$. It follows from the Noether-Skolem theorem that we have one of the expected representations of θ. This completes the proof. □

Lemma 2.5. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\theta : M_m \to M_s$ be a complex linear map. Assume that*

$$\theta(e)\theta(f) = 0 \quad \textit{whenever} \quad e, f \textit{ are rank one idempotents with } ef = 0.$$

If the range of θ does not consist of nilpotent elements then $s = m$, and there exist an invertible matrix S in M_m and a nonzero scalar λ such that

$$\theta(x) = \lambda S^{-1} x S, \quad \forall x \in M_n.$$

Proof. Since projections are idempotents, it follows from Lemma 2.4 that $m = s$ and there is an invertible matrix S and a nonzero scalar λ such that $\frac{1}{\lambda} S\theta(x)S^{-1}$ is either always x or always x^t. Consider the idempotents a, b in M_m with the top left 2×2 blocks given below and zero elsewhere, respectively:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $ab = 0$ but $ba \neq 0$. Since θ preserves zero products, so does $\frac{1}{\lambda} S\theta S^{-1}$. Because $a^t b^t = (ba)^t \neq 0$, we conclude that $\frac{1}{\lambda} S\theta(x)S^{-1} = x$ for all x in M_m. This gives us the desired assertion. □

Proof of Theorem 1.3. It is not difficult to see that the two cases stated in the conclusions are exclusive. Assume from now on $H(d)$ is a matrix in M_s of nonzero trace for some d in $B_{M_m}(0; r)$.

Lemma 2.1 ensures that each summand P_n is an orthogonally additive and orthogonally multiplicative n-homogeneous polynomial, and the constant term $P_0 = H(0) = 0$. Lemma 2.2 provides a linear map $T_n : M_m \to M_s$ for each n such that

$$P_n(x) = T_n(x^n), \quad \forall x \in M_m.$$

Inherited from $\{P_n\}$, the family $\{T_n\}$ satisfies the orthogonality preserving property stated in (2.1).

Since $H(d) = \sum_n T_n(d^n)$, the continuity of the trace functional ensures that some $T_k(d^k)$ in the sum has nonzero trace. In particular, $T_k(d^k)$ is not a nilpotent. Lemma 2.4 ensures $m = s$ and provides an invertible matrix S_k and a nonzero scalar λ_k such that

$$P_k(x) = \lambda_k S_k^{-1} x^k S_k, \quad \forall x \in B_{M_m}(0; r),$$

or

$$P_k(x) = \lambda_k S_k^{-1} (x^t)^k S_k, \quad \forall x \in B_{M_m}(0; r).$$

We claim that all other P_n either carries a similar form or constantly zero. Redefining $H(x)$ with $\frac{1}{\lambda_k} S_k H(x) S_k^{-1}$ or $\frac{1}{\lambda_k} S_k H(x^t) S_k^{-1}$, we can assume

$$P_k(x) = x^k, \quad \forall x \in B_{M_m}(0; r).$$

Suppose that with a nonzero scalar λ_n and an invertible S_n in M_m we have

$$P_n(x) = \lambda_n S_n^{-1} x^n S_n, \quad \forall x \in B_{M_m}(0; r),$$

or

$$P_n(x) = \lambda_n S_n^{-1} (x^t)^n S_n, \quad \forall x \in B_{M_m}(0; r).$$

By Lemma 2.1(b),

$$xS_n^{-1}yS_n = S_n^{-1}yS_nx = 0, \quad \text{whenever } x, y \text{ are orthogonal projections.}$$

This forces

$$S_n^{-1}yS_n = \alpha_y y$$

with some scalar α_y for every rank one projection y in M_m. In particular, every nonzero vector is an eigenvector of S_n. Thus $S_n = \alpha I$ for some nonzero scalar α. In other words, we can assume that $P_n(x) = \lambda_n x^n$ for all x in $B_{M_m}(0;r)$, or $P_n(x) = \lambda_n (x^t)^n$ for all x in $B_{M_m}(0;r)$. However, Lemma 2.1(b) ruins out the possibility of the second case. For example, try the pair of orthogonal projections

$$a = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \oplus 0 \quad \text{and} \quad b = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \oplus 0.$$

While $ab = 0$, we have $ab^t \neq 0$.

Suppose next that there is a P_n whose range consists of nilpotent elements. We will verify that $P_n = 0$. Arguing as above, we have

$$xT_n(y) = T_n(y)x = 0, \quad \text{whenever } x, y \text{ are orthogonal projections.}$$

This forces

$$T_n(y) = \alpha_y y$$

with some scalar α_y for every rank one projection y in M_m. Since $\beta^{mn}T_n(y)^m = P_n(\beta y)^m = 0$ for all small scalar $\beta > 0$, we see that $\alpha_y = 0$. Since every self-adjoint matrix is an orthogonal sum of rank one projections, the linear map $T_n = 0$, and thus $P_n = 0$, on M_m.

The claim is established. It follows that there is a scalar sequence $\{\lambda_n\}$ and an invertible S in M_m such that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1}x^n S, \quad \forall x \in B_{M_m}(0;r).$$

Translating back to the original situation, there is also another possible case that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1}(x^t)^n S, \quad \forall x \in B_{M_m}(0;r).$$

Finally, assume that H also preserves zero products, and thus so does every P_k. Consequently, the linearization T_k sends two rank one idempotents with zero products to a pair of elements with zero products. By Lemma 2.5, we have $T_k(x) = \lambda_k S^{-1}x^k S$ for all x in M_m. This forces H carries the first form as in (‡). This completes the proof. □

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