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GENERAL MULTIPLE OPIAL-TYPE INEQUALITIES FOR THE CANAVATI FRACTIONAL DERIVATIVES

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ABSTRACT. In this paper we establish some general multiple Opial-type inequalities involving the Canavati fractional derivatives. In some cases the best possible constants are discussed.

1. INTRODUCTION AND PRELIMINARIES

In 1960, Opial [7] proved the following inequality:
Let $f \in C^1[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$. Then

$$\int_0^h |f(x)\, f'(x)|\, dx \le \frac{h}{4} \int_0^h [f'(x)]^2\, dx\,, \tag{1.1}$$

where $h/4$ is the best possible.
This inequality has been generalized and extended over the last 50 years in several directions, and used in many applications in differential equations (for more details see [1], [9]). The aim of our research is an Opial-type inequality for fractional derivatives, which has the general form

$$\int_a^b w_1(t) \left(\prod_{i=1}^N \left| D^{\beta_i} f(t) \right|^{r_i} \right)^p |D^\alpha f(t)|^q\, dt \le C \left(\int_a^b w_2(t)\, |D^\alpha f(t)|^{p+q}\, dt \right)^{\frac{pr+q}{p+q}},$$

where w_1 and w_2 are weight functions, $r = \sum_{i=1}^N r_i$ and $D^\gamma f$ denotes the Canavati fractional derivative of f of order γ.

First we survey some facts about the fractional integrals and derivatives needed in this paper. For more details see the monographs [8, Chapter 1] and [3].

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By $C^n[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in AC[a,b]$.

By $L_p[a,b]$, $1 \leq p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f|^p$ is Lebesgue integrable on $[a,b]$, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geq 1$.

Let $x \in [a,b]$, $\alpha > 0$, $n = [\alpha]+1$, $[\alpha]$ denotes the integral part of α and Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$. For $f \in L_1[a,b]$ the *Riemann-Liouville fractional integrals* $J_{a+}^\alpha f$ (left-sided) and $J_{b-}^\alpha f$ (right-sided) of order α are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, f(t)\, dt\,,$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}\, f(t)\, dt\,.$$

The subspaces $C_{a+}^\alpha[a,b]$ and $C_{b-}^\alpha[a,b]$ of $C^{n-1}[a,b]$ are defined by

$$C_{a+}^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] \colon J_{a+}^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\},$$

$$C_{b-}^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] \colon J_{b-}^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\}.$$

For $f \in C_{a+}^\alpha[a,b]$ and $g \in C_{b-}^\alpha[a,b]$ the *Canavati fractional derivatives* $D_{a+}^\alpha f$ (left-sided) and $D_{b-}^\alpha g$ (right-sided) of order α are defined by

$$D_{a+}^\alpha f(x) = \frac{d}{dx} J_{a+}^{n-\alpha} f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1}\, f^{(n-1)}(t)\, dt\,,$$

$$D_{b-}^\alpha g(x) = (-1)^n \frac{d}{dx} J_{b-}^{n-\alpha} g^{(n-1)}(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{n-\alpha-1}\, g^{(n-1)}(t)\, dt\,.$$

In addition, we stipulate

$$D_{a+}^0 f := f =: J_{a+}^0 f\,,$$

$$D_{b-}^0 g := g =: J_{b-}^0 g\,.$$

If $\alpha \in \mathbb{N}$ then $D_{a+}^\alpha f = f^{(\alpha)}$ and $D_{b-}^\alpha g = (-1)^\alpha g^{(\alpha)}$, the ordinary α-order derivatives.

The composition identity for the Canavati left-sided fractional derivatives comes from [5], and will be used in all presented Opial-type inequalities. Notice that we relaxed some conditions on parameters and a function, comparing to the analogous identity given in [3].

Theorem 1.1. [5, Theorem 2.1] *Let* $\alpha > \beta \geq 0$, $n = [\alpha]+1$, $m = [\beta]+1$. *Let* $f \in C_{a+}^\alpha[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m-1, \ldots, n-2$. *Then* $f \in C_{a+}^\beta[a,b]$ *and*

$$D_{a+}^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, D_{a+}^\alpha f(t)\, dt\,, \quad x \in [a,b]\,. \tag{1.2}$$

Our goal is to give general multiple Opial-type inequalities for the Canavati fractional derivatives. The starting point is next Opial-type inequality for the Riemann-Liouville[1] left-sided fractional derivatives $\mathbf{D}_{a+}^{\alpha} f$ that comes from [6].

Theorem 1.2. [6, Theorem 4.1] *Let $p_i, q, \beta_i, \alpha, r$ $(i = 1, \dots N)$ be real numbers such that $p_i \geq 0$, $p := \sum_{i=1}^{N} p_i > 0$, $q > 0$, $\alpha > \beta_i + 1 \geq 0$ for all $i = 1, \dots, N$, and $r > \max\{1, q, (\alpha - \beta_i)^{-1} : i = 1, \dots, N\}$. Suppose $f \in L_1[a,b]$ has an integrable left-sided fractional derivative $\mathbf{D}_{a+}^{\alpha} f \in L_\infty[a,b]$ and $\mathbf{D}_{a+}^{\alpha-j} f(a) = 0$ for $j = 1, \dots, [\alpha]+1$. Then for any $w_1, w_2 \in C[a,b]$ with $w_1 \geq 0$ and $w_2 > 0$,*

$$\int_a^x w_1(t) \left|\mathbf{D}_{a+}^{\alpha} f(t)\right|^q \prod_{i=1}^{N} \left|\mathbf{D}_{a+}^{\beta_i} f(t)\right|^{p_i} dt \leq A(x) \left(\int_a^x w_2(t) \left|\mathbf{D}_{a+}^{\alpha} f(t)\right|^r dt\right)^{\frac{p+q}{r}},$$

where

$$A(x) = \left(\frac{q}{p+q}\right)^{\frac{q}{r}} \left[\int_a^x w_1(t)^{\frac{r}{r-q}} w_2(t)^{-\frac{q}{r-q}} \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r-q}} dt\right]^{\frac{r-q}{r}},$$

$$P_i(t) = \int_a^t w_2(\tau)^{-\frac{1}{r-1}} K_i(t,\tau)^{\frac{r}{r-1}} d\tau, \quad a \leq t \leq x,$$

$$K_i(t,\tau) = \frac{(t-\tau)_+^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}, \quad a \leq t, \tau \leq x,$$

$$(t-\tau)_+ = \max\{t-\tau, 0\}.$$

We will give two-weighted, one-weighted and non-weighted versions of this theorem involving the Canavati left-sided fractional derivatives. Also we will give versions of those inequalities which include decreasing or bounded weight functions.

The right-sided versions of all inequalities in this paper can be established and proven analogously.

2. Two-weighted case

First theorem is the Canavati fractional derivatives analogy of Theorem 1.2, with relaxed conditions on the function (here the role of p_i and r from Theorem 1.2 have $r_i p$ and $p+q$ respectively).

Theorem 2.1. *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m = \min\{[\beta_i]+1 : i = 1, \dots, N\}$ and $n = [\alpha]+1$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$. Let w_1 and w_2 be continuous weight functions on $[a,x]$ with $w_1 \geq 0$ and $w_2 > 0$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$*

[1] Riemann-Liouville left-sided fractional derivative is defined by $\mathbf{D}_{a+}^{\alpha} f(x) = \frac{d^n}{dx^n} J_{a+}^{n-\alpha} f(x)$.

and let $\alpha > \beta_i + \sigma$ *for* $i = 1, \dots, N$. *Let also* $D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. *Then*

$$\begin{aligned}
&\int_a^x w_1(t) \left(\prod_{i=1}^N \left| D^{\beta_i}_{a+} f(t) \right|^{r_i} \right)^p \left| D^{\alpha}_{a+} f(t) \right|^q dt \\
&\le \;\; C_1 \left(\int_a^x w_2(t) \left| D^{\alpha}_{a+} f(t) \right|^{p+q} dt \right)^{\frac{rp+q}{p+q}} \\
&\quad \cdot \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\alpha-\beta_i-1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p},
\end{aligned} \tag{2.1}$$

where

$$C_1 = \prod_{i=1}^N [\Gamma(\alpha - \beta_i)]^{-r_i p} \left(\frac{q}{rp+q} \right)^{\sigma q}. \tag{2.2}$$

Proof. Let $q \neq 0$, $\delta_i = \alpha - \beta_i - 1$, $i = 1, \dots, N$. Using composition identity (1.2), the triangle inequality and Hölder's inequality for $\frac{1}{1-\sigma}$ and $\frac{1}{\sigma}$, for $t \in [a,x]$ follows

$$\begin{aligned}
&|D^{\beta_i}_{a+} f(t)| \\
&\le \;\; \frac{1}{\Gamma(\delta_i + 1)} \int_a^t [w_2(\tau)]^{-\sigma} \, [w_2(\tau)]^{\sigma} \, (t-\tau)^{\delta_i} \, |D^{\alpha}_{a+} f(\tau)| \, d\tau \\
&\le \;\; \frac{1}{\Gamma(\delta_i + 1)} \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{1-\sigma} \left(\int_a^t w_2(\tau) \left| D^{\alpha}_{a+} f(\tau) \right|^{\frac{1}{\sigma}} d\tau \right)^{\sigma}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D^{\beta_i}_{a+} f(t)|^{r_i p} \, |D^{\alpha}_{a+} f(t)|^q \, dt \\
&\le \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \int_a^x w_1(t) \, [w_2(t)]^{-\sigma q} \, [w_2(t)]^{\sigma q} \, |D^{\alpha}_{a+} f(t)|^q \\
&\cdot \left(\int_a^t w_2(\tau) \, |D^{\alpha}_{a+} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{(1-\sigma) r_i p} dt \,.
\end{aligned} \tag{2.3}$$

Applying Hölder's inequality for $\frac{1}{\sigma p}$, $\frac{1}{\sigma q}$ and simple integration, we get

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\
&\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \\
&\quad \cdot \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} \, [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} dt \right]^{\sigma p} \\
&\quad \cdot \left[\int_a^x w_2(t) \, |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} \left(\int_a^t w_2(\tau) \, |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} \, d\tau \right)^{\frac{pr}{q}} dt \right]^{\sigma q} \\
&= \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \left(\frac{q}{rp+q} \right)^{\sigma q} \left(\int_a^x w_2(t) \, |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} \, dt \right)^{\sigma(rp+q)} \qquad (2.4) \\
&\quad \cdot \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} \, [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} dt \right]^{\sigma p},
\end{aligned}$$

which gives us inequality (2.1).
If $q = 0$ ($\sigma = \frac{1}{p}$), then inequality (2.3) has the form

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, dt \\
&\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \left(\int_a^x w_2(\tau) \, |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} \, d\tau \right)^{r} \\
&\quad \cdot \int_a^x w_1(t) \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{(1-\sigma) r_i p} dt,
\end{aligned}$$

from which we get inequality (2.1) for $q = 0$. □

Next results complement Theorem 2.1. To obtain inequality (2.5) we need a monotonicity of w_1 and w_2.

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 hold. Suppose also that w_1 is an increasing and w_2 is a decreasing functions. Then*

$$\begin{aligned} &\int_a^x w_1(t) \left(\prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\ &\leq C_2 \, w_1(x) \, [w_2(x)]^{-\sigma(rp+q)} \, (x-a)^{(\rho+\sigma)p} \left(\int_a^x w_2(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{rp+q}{p+q}}, \end{aligned} \tag{2.5}$$

where

$$C_2 = \frac{C_1 \, \sigma^{\sigma p} \, (1-\sigma)^{(1-\sigma)rp}}{(\rho+\sigma)^{\sigma p} \prod_{i=1}^N (\alpha-\beta_i-\sigma)^{r_i(1-\sigma)p}} \tag{2.6}$$

and C_1 is defined by (2.2).

Proof. We start the proof with obtained inequality (2.1) form Theorem 2.1. By monotonicity of w_1 and w_2 follows

$$\begin{aligned} &\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}} \, [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\ &\leq w_1(x) \, [w_2(x)]^{-\sigma(rp+q)} \left[\int_a^x \prod_{i=1}^N \left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\ &= w_1(x) \, [w_2(x)]^{-\sigma(rp+q)} \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N (\delta_i+1-\sigma)^{(1-\sigma)r_i p}} \, (x-a)^{(\rho+\sigma)p} \, \frac{\sigma^{\sigma p}}{(\rho+\sigma)^{\sigma p}} . \end{aligned} \tag{2.7}$$

Inequality (2.5) now follows from (2.4) and (2.7).
For $q = 0$, we proceed the same as in the proof of Theorem 2.1. □

For the next theorem we suppose that weight functions are bounded.

Theorem 2.3. *Suppose that the assumptions of Theorem 2.1 hold. Suppose also $w_1(t) \leq B$ and $A \leq w_2(t)$ for $t \in [a,x]$. Then*

$$\begin{aligned} &\int_a^x w_1(t) \left(\prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\ &\leq C_2 \, B \, A^{-\sigma(rp+q)} \, (x-a)^{(\rho+\sigma)p} \left(\int_a^x w_2(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{rp+q}{p+q}}, \end{aligned} \tag{2.8}$$

where C_2 is defined by (2.6).

Proof. The proof of (2.8) is the same as the one for (2.5), except one change: instead of inequalities $w_1(t) \leq w_1(x)$, $w_2(t) \geq w_2(x)$ we use $w_1(t) \leq B$, $w_2(t) \geq A$ respectively. □

With extra parameters s_1, s_2 and s_3 we can extract expressions containing just weight functions to get inequality (2.9).

Theorem 2.4. *Suppose that the assumptions of Theorem 2.1 hold. Suppose also that* $s_k > 1$ *and* $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ *for* $k = 1, 2, 3$. *Then*

$$\begin{aligned}
&\int_a^x w_1(t) \left(\prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\
&\quad \leq \; C_3\, P(x)\, Q(x)\, R(x)\, (x-a)^{\rho p + \frac{\sigma p}{s_2 s_3} - \frac{(1-\sigma) r p}{s_1'}} \left(\int_a^x w_2(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{rp+q}{p+q}},
\end{aligned} \tag{2.9}$$

where

$$C_3 = \frac{C_1\,(1-\sigma)^{\frac{(1-\sigma) r p}{s_1}} (\sigma s_1)^{\frac{\sigma p}{s_2 s_3}}}{\prod_{i=1}^N \left[s_1(\alpha - \beta_i - 1) + 1 - \sigma \right]^{\frac{(1-\sigma) r_i p}{s_1}} \left[\sum_{i=1}^N \left[s_1(\alpha - \beta_i - 1) + 1 - \sigma \right] r_i s_2 s_3 + \sigma s_1 \right]^{\frac{\sigma p}{s_2 s_3}}}$$

and

$$\begin{aligned}
P(x) &= \left(\int_a^x [w_2(t)]^{-\frac{\sigma}{1-\sigma} s_1'} \, dt \right)^{\frac{(1-\sigma) r p}{s_1'}}, \\
Q(x) &= \left(\int_a^x [w_1(t)]^{\frac{s_2'}{\sigma p}} \, dt \right)^{\frac{\sigma p}{s_2'}}, \\
R(x) &= \left(\int_a^x [w_2(t)]^{-\frac{q}{p} s_2 s_3'} \, dt \right)^{\frac{\sigma p}{s_2 s_3'}}.
\end{aligned} \tag{2.10}$$

Proof. We start the proof with obtained inequality (2.1) form Theorem 2.1. By Hölder's inequality we have

$$\begin{aligned}
&\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \\
&\leq \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s_1'} \, d\tau \right)^{\frac{1}{s_1'}} \left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma} s_1} \, d\tau \right)^{\frac{1}{s_1}} \\
&\leq \left(\int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s_1'} \, d\tau \right)^{\frac{1}{s_1'}} \left(\frac{1-\sigma}{\delta_i s_1 + 1 - \sigma} \right)^{\frac{1}{s_1}} (t-a)^{\frac{\delta_i}{1-\sigma} + \frac{1}{s_1}}.
\end{aligned}$$

Now follows

$$\begin{aligned}
&\prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} \\
&\leq \left(\int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s_1'} \, d\tau \right)^{\frac{(1-\sigma) r}{\sigma s_1'}} \frac{(1-\sigma)^{\frac{(1-\sigma) r}{\sigma s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma) r_i}{\sigma s_1}}} (t-a)^{\frac{\sum_{i=1}^N (\delta_i s_1 + 1 - \sigma) r_i}{\sigma s_1}}.
\end{aligned}$$

Let $\varepsilon = \sum_{i=1}^{N}(\delta_i s_1 + 1 - \sigma) r_i$. Applying Hölder's inequalities we get

$$\begin{aligned}
&\left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^{N} \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p} \\
&\leq P(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^{N}(\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left[\int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} (t-a)^{\frac{\varepsilon}{\sigma s_1}} dt\right]^{\sigma p} \\
&\leq P(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^{N}(\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left(\int_a^x [w_1(t)]^{\frac{s_2'}{\sigma p}} dt\right)^{\frac{\sigma p}{s_2'}} \\
&\quad \cdot \left(\int_a^x [w_2(t)]^{-\frac{q s_2}{p}} (t-a)^{\frac{\varepsilon}{\sigma s_1} s_2} dt\right)^{\frac{\sigma p}{s_2}} \\
&\leq P(x)\, Q(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^{N}(\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \\
&\quad \cdot \left(\int_a^x [w_2(t)]^{-\frac{q}{p} s_2 s_3'} dt\right)^{\frac{\sigma p}{s_2 s_3'}} \left(\int_a^x (t-a)^{\frac{\varepsilon}{\sigma s_1} s_2 s_3} dt\right)^{\frac{\sigma p}{s_2 s_3}} \\
&= P(x)\, Q(x)\, R(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^{N}(\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left(\frac{\sigma s_1}{\varepsilon s_2 s_3 + \sigma s_1}\right)^{\frac{\sigma p}{s_2 s_3}} (x-a)^{\frac{\varepsilon p}{s_1} + \frac{\sigma p}{s_2 s_3}} .
\end{aligned} \tag{2.11}$$

Inequality (2.9) now follows from (2.4) and (2.11).
For $q = 0$, we proceed the same as in the proof of Theorem 2.1. □

If we choose a convenient parameter s_3, then we get next corollary.

Corollary 2.5. *Suppose that the assumptions of Theorem 2.4 hold. Suppose also that* $s_3 = \frac{\sigma p s_1'}{\sigma p s_1' - q(1-\sigma)s_2} > 1$. *Then*

$$\begin{aligned}
&\int_a^x w_1(t) \left(\prod_{i=1}^{N} \left|D_{a+}^{\beta_i} f(t)\right|^{r_i}\right)^p \left|D_{a+}^{\alpha} f(t)\right|^q dt \\
&\leq \widetilde{C}_3\, \widetilde{P}(x)\, Q(x)\, (x-a)^{\rho p + \frac{\sigma p}{s_2} - \frac{(1-\sigma)(rp+q)}{s_1'}} \left(\int_a^x w_2(t) \left|D_{a+}^{\alpha} f(t)\right|^{p+q} dt\right)^{\frac{rp+q}{p+q}},
\end{aligned}$$

where Q is defined by (2.10) and

$$\widetilde{C}_3 = \frac{C_1 (1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^{N}[(\alpha-\beta_i-1)s_1 + 1 - \sigma]^{\frac{(1-\sigma)r_i}{s_1}}} \left[\frac{\sigma p s_1' - q(1-\sigma)s_2}{(\rho s_2 + \sigma) p s_1' - (1-\sigma)(rp+q)s_2}\right]^{\frac{\sigma p}{s_2} - \frac{q(1-\sigma)}{s_1'}},$$

$$\widetilde{P}(x)=\left(\int_a^x w_2(t)^{-\frac{\sigma}{1-\sigma}s_1'}\,dt\right)^{\frac{(1-\sigma)(rp+q)}{s_1'}}.$$

3. One-weighted case

First result is a direct consequence of Theorem 2.1.

Theorem 3.1. *Let $N\in\mathbb{N}$, $\alpha>\beta_i\geq 0$, $m=\min\{[\beta_i]+1\colon i=1,\dots,N\}$ and $n=[\alpha]+1$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\dots,n-2$. Let w be continuous positive weight function on $[a,x]$. Let $r_i\geq 0$, $r=\sum_{i=1}^N r_i>0$. Let $p>0$, $q\geq 0$, $\sigma=\frac{1}{p+q}<1$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i)-r\sigma$ and let $\alpha>\beta_i+\sigma$ for $i=1,\dots,N$. Let also $D^{\alpha}_{a+}f\in L_{p+q}[a,b]$. Then*

$$\begin{aligned}
&\int_a^x w(t)\left(\prod_{i=1}^N\left|D^{\beta_i}_{a+}f(t)\right|^{r_i}\right)^p\left|D^{\alpha}_{a+}f(t)\right|^q dt\\
&\leq\ C_1\left(\int_a^x w(t)\left|D^{\alpha}_{a+}f(t)\right|^{p+q}dt\right)^{\frac{rp+q}{p+q}}\\
&\quad\cdot\left[\int_a^x w(t)\prod_{i=1}^N\left(\int_a^t[w(\tau)]^{-\frac{\sigma}{1-\sigma}}(t-\tau)^{\frac{\alpha-\beta_i-1}{1-\sigma}}d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right]^{\sigma p},
\end{aligned}$$

where C_1 is defined by (2.2) .

If we have a decreasing weight function, then we need the assumption $r\geq 1$.

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 hold. Suppose also that $r\geq 1$ and w is a decreasing function. Then*

$$\begin{aligned}
&\int_a^x w(t)\left(\prod_{i=1}^N\left|D^{\beta_i}_{a+}f(t)\right|^{r_i}\right)^p\left|D^{\alpha}_{a+}f(t)\right|^q dt\\
&\leq\ C_2\,[w(x)]^{(1-r)\sigma p}(x-a)^{(\rho+\sigma)p}\left(\int_a^x w(t)\left|D^{\alpha}_{a+}f(t)\right|^{p+q}dt\right)^{\frac{pr+q}{p+q}},
\end{aligned}\tag{3.1}$$

where C_2 is defined by (2.6) .

Proof. Let $q\neq 0$, $\delta_i=\alpha-\beta_i-1$, $i=1,\dots,N$. Since w is decreasing, then

$$1\leq\left[\frac{w(\tau)}{w(t)}\right]^{\sigma},\quad \tau\leq t.\tag{3.2}$$

Using composition identity (1.2), the triangle inequality and Hölder's inequality for $\frac{1}{1-\sigma}$ and $\frac{1}{\sigma}$, for $t \in [a,x]$ follows

$$\begin{aligned}
\prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p} &\leq \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \prod_{i=1}^{N} \left[\int_a^t (t-\tau)^{\delta_i} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right]^{r_i p} \\
&\leq \frac{[w(t)]^{-\sigma r p}}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \prod_{i=1}^{N} \left[\int_a^t (t-\tau)^{\delta_i} [w(\tau)]^{\sigma} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right]^{r_i p} \\
&\leq \frac{[w(t)]^{-\sigma r p} (1-\sigma)^{(1-\sigma) r p}}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)(\delta_i+1-\sigma)^{1-\sigma}]^{r_i p}} \\
&\quad \cdot (t-a)^{\sum_{i=1}^{N} (\delta_i+1-\sigma) r_i p} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma r p}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^{\alpha} f(t)|^q\, dt \\
&\leq \prod_{i=1}^{N} \left[\left(\frac{1-\sigma}{\delta_i+1-\sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\delta_i+1)} \right]^{r_i p} \qquad (3.3) \\
&\quad \cdot \int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t-a)^{\sum_{i=1}^{N} (\delta_i+1-\sigma) r_i p} \qquad (3.4) \\
&\quad \cdot \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma r p} dt.
\end{aligned}$$

Applying Hölder's inequality for $\frac{1}{\sigma p}$ and $\frac{1}{\sigma q}$ with $\rho p = \sum_{i=1}^{N} (\delta_i+1-\sigma) r_i p$, we obtain

$$\begin{aligned}
&\int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t-a)^{\rho p} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma r p} dt \\
&\leq \left(\int_a^x (t-a)^{\frac{\rho}{\sigma}}\, dt \right)^{\sigma p} \\
&\quad \cdot \left(\int_a^x [w(t)]^{\frac{1-\sigma r p-\sigma q}{\sigma q}} w(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\frac{rp}{q}} dt \right)^{\sigma q} \\
&\leq (x-a)^{(\rho+\sigma)p} \left(\frac{\sigma}{\rho+\sigma} \right)^{\sigma p} [w(x)]^{\frac{p(1-r)}{p+q}} \\
&\quad \cdot \left(\frac{q}{rp+q} \right)^{\sigma q} \left(\int_a^x w(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}}\, dt \right)^{\sigma(rp+q)}. \qquad (3.5)
\end{aligned}$$

The inequality (3.1) now follows from (3.3) and (3.5).
For $q = 0$, we proceed the same as in the proof of Theorem 2.1. □

If $r = 1$ then we have Alzer's inequality [2, Theorem 1] for the Canavati left-sided fractional derivatives:

Corollary 3.3. *Suppose that the assumptions of Theorem 3.2 hold and let* $r = 1$. *Then*

$$\int_a^x w(t) \left(\prod_{i=1}^{N} \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt$$
$$\leq \widetilde{C}_2 \, (x-a)^{\sum_{i=1}^{N} r_i(\alpha-\beta_i)p} \int_a^x w(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt,$$

where

$$\widetilde{C}_2 = \sigma \, q^{\sigma q} \left[\sum_{i=1}^{N} r_i(\alpha-\beta_i) \right]^{-\sigma p} \prod_{i=1}^{N} \left[\left(\frac{1-\sigma}{\alpha-\beta_i-\sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\alpha-\beta_i)} \right]^{r_i p}.$$

For the next theorem we suppose that weight function is bounded.

Theorem 3.4. *Suppose that the assumptions of Theorem 3.1 hold. Suppose also that* $r \geq 1$ *and* $A \leq w(t) \leq B$ *for* $t \in [a, x]$. *Then*

$$\int_a^x w(t) \left(\prod_{i=1}^{N} \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt$$
$$\leq C_2 \left(\frac{B}{A^r} \right)^{\sigma p} (x-a)^{(\rho+\sigma)p} \left(\int_a^x w(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{3.6}$$

where C_2 *is given by* (2.6).

Proof. The proof of (3.6) is the same as the one for (3.1), except two changes. Instead of inequality (3.2) we use $1 \leq (w(\tau)/A)^{\sigma}$. Moreover, in (3.4) we apply the inequality $w(t) = [w(t)]^{\sigma p} [w(t)]^{\sigma q} \leq B^{\sigma p} [w(t)]^{\sigma q}$. These two changes lead to the inequality (3.6). ☐

4. Non-weighted case

The last result is a non-weighted case of previous theorems. Here we also give a case with a best possible solution.

Proposition 4.1. *Let* $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m = \min\{[\beta_i] + 1 \colon i = 1, \ldots, N\}$ *and* $n = [\alpha] + 1$. *Let* $f \in C_{a+}^{\alpha}[a, b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m-1, \ldots, n-2$. *Let* $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. *Let* $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$ *and let* $\alpha > \beta_i + \sigma$ *for* $i = 1, \ldots, N$. *Let also* $D_{a+}^{\alpha} f \in L_{p+q}[a, b]$. *Then*

$$\int_a^x \left(\prod_{i=1}^{N} \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt$$
$$\leq C_2 \, (x-a)^{(\rho+\sigma)p} \left(\int_a^x \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{4.1}$$

where C_2 *is defined by* (2.6).

Inequality (4.1) is sharp if and only if $\alpha = \beta_i + 1$, $i = 1, \ldots, N$ and $q = 1$. The equality in this case is attained for a function f such that $D^{\alpha}_{a+}f(t) = 1$, $t \in [a, x]$.

Proof. Let $q \neq 0$, $\delta_i = \alpha - \beta_i - 1$, $i = 1, \ldots, N$. As in the proof of Theorem 2.1, using composition identity (1.2), the triangle inequality and Hölder's inequality for $\frac{1}{1-\sigma}$ and $\frac{1}{\sigma}$, for $t \in [a, x]$ follows

$$\begin{aligned} &\prod_{i=1}^{N} |D^{\beta_i}_{a+}f(t)|^{r_i p} \\ &\leq \frac{1}{\prod\limits_{i=1}^{N} [\Gamma(\delta_i + 1)]^{r_i p}} \prod_{i=1}^{N} \left[\left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{1-\sigma} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}} \, d\tau \right)^{\sigma} \right]^{r_i p} \qquad (4.2) \\ &= \frac{(1-\sigma)^{(1-\sigma)rp} \, (t-a)^{\sum_{i=1}^{N}(\delta_i + 1 - \sigma) r_i p}}{\prod\limits_{i=1}^{N} \left[\Gamma(\delta_i + 1) \, (\delta_i + 1 - \sigma)^{1-\sigma} \right]^{r_i p}} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}} \, d\tau \right)^{\sigma r p} . \end{aligned}$$

Now we have

$$\begin{aligned} &\int_a^x \prod_{i=1}^{N} |D^{\beta_i}_{a+}f(t)|^{r_i p} \, |D^{\alpha}_{a+}f(t)|^q \, dt \\ &\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod\limits_{i=1}^{N} \left[\Gamma(\delta_i + 1) \, (\delta_i + 1 - \sigma)^{1-\sigma} \right]^{r_i p}} \\ &\quad \cdot \int_a^x (t-a)^{\sum_{i=1}^{N}(\delta_i + 1 - \sigma) r_i p} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}} \, d\tau \right)^{\sigma r p} |D^{\alpha}_{a+}f(t)|^q \, dt . \end{aligned}$$

By Hölder's inequality for $\frac{1}{\sigma p}$ and $\frac{1}{\sigma q}$ and a simple integration inequality (4.1) follows.
For $q = 0$, we proceed as in the proof of Theorem 2.1.

Using the equality condition in Hölder's inequality we have equality in (4.2) if and only if $|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}} = \lambda(t-\tau)^{\frac{\delta_i}{1-\sigma}}$, $i = 1, \ldots, N$, which implies (since $D^{\alpha}_{a+}f(\tau)$ depends only on τ) that $\delta_i = 0$, that is $\alpha = \beta_i + 1$ for $i = 1, \ldots, N$. Due to homogeneous property of inequality (4.1) we can take $D^{\alpha}_{a+}f(\tau) = 1$, which gives $D^{\beta_i}_{a+}f(\tau) = D^{\alpha-1}_{a+}(\tau) = \tau - a$, $i = 1, \ldots, N$. Substituting this in equality (4.1) for the left side we get

$$\int_a^x \prod_{i=1}^{N} (t-a)^{r_i p} \, dt = \int_a^x (t-a)^{rp} \, dt = \frac{(x-a)^{rp+1}}{rp+1} .$$

For the right side, with $\rho = r - r\sigma$, follows

$$C_2 \, (x-a)^{(\rho+\sigma)p} \left(\int_a^x dt \right)^{\frac{pr+q}{p+q}} = \frac{\sigma^{\sigma p} \, q^{\sigma q}}{(r - r\sigma + \sigma)^{\sigma p} \, (rp+q)^{\sigma q}} \, (x-a)^{rp+1} .$$

Hence,

$$\frac{1}{rp+1}=\frac{q^{\frac{q}{p+q}}}{[r(p+q)-r+1]^{\frac{p}{p+q}}\ (rp+q)^{\frac{q}{p+q}}}$$

which is equivalent with

$$[r(p+q)-r+1]^{p}\,[rp+q]^{q}=q^{q}(rp+1)^{p+q}\,. \tag{4.3}$$

For $q=1$ equality (4.3) obviously holds. For $q=0$ equality (4.3) implies $r=0$, which gives trivial identity in (4.1). By simple rearrangements equation (4.3) becomes

$$\left[1+r\frac{q-1}{rp+1}\right]^{p}\left[1+r\frac{p}{q}\frac{1-q}{rp+1}\right]^{q}=1. \tag{4.4}$$

For $p=q$ the left-hand side of equation (4.4) is equal to $\left[1-\left(r\frac{1-p}{rp+1}\right)^{2}\right]^{p}$, which is strictly less then 1, except in trivial cases. For $0<p<q$, $q\neq 1$, $r>0$, using the Bernoulli inequality, we have

$$\left[1+r\frac{q-1}{rp+1}\right]^{\frac{p}{q}}\left[1+r\frac{p}{q}\frac{1-q}{rp+1}\right]<\left[1+r\frac{p}{q}\frac{q-1}{rp+1}\right]\left[1+r\frac{p}{q}\frac{1-q}{rp+1}\right],$$

which is obviously strictly less then 1. For $0<q<p$, $q\neq 1$, $r>0$, using the Bernoulli inequality, we have

$$\left[1+r\frac{q-1}{rp+1}\right]\left[1+r\frac{p}{q}\frac{1-q}{rp+1}\right]^{\frac{p}{q}}<\left[1+r\frac{q-1}{rp+1}\right]\left[1+r\frac{1-q}{rp+1}\right],$$

which is again obviously strictly less then 1. It follows that (4.3) holds if and only if $q=1$. □

Remark 4.2. Let $N=1$, $\alpha=1$, $\beta_1=0$, $r_1=r=1$, $p=q=1$, $a=0$ and $x=h$. Then inequality (4.1) becomes Beesack's inequality [4]

$$\int_0^h |f(t)\,f'(t)|\,dt\leq\frac{h}{2}\int_0^h\left[f'(t)\right]^2\,dt\,. \tag{4.5}$$

He proved that inequality (4.5) is valid for any function f absolutely continuous on $[0,h]$ satisfying single boundary condition $f(0)=0$.

In order to get classical Opial's inequality (1.1) we need right-sided version of inequality (4.1) for $N=1$, $r_1=r=1$ and $p=q=1$:

$$\int_x^b |D_{b-}^{\beta_1}f(t)|\,|D_{b-}^{\alpha}f(t)|\,dt\leq C_2\,(b-x)\int_x^b |D_{b-}^{\alpha}f(t)|^2\,dt \tag{4.6}$$

satisfying $f(b)=0$. Observe the inequality

$$\int_a^{\frac{a+b}{2}} |D_{a+}^{\beta_1}f(t)|\,|D_{a+}^{\alpha}f(t)|\,dt+\int_{\frac{a+b}{2}}^b |D_{b-}^{\beta_1}f(t)|\,|D_{b-}^{\alpha}f(t)|\,dt$$

$$\leq\ C_2\left(\frac{b-a}{2}\right)\left(\int_a^{\frac{a+b}{2}}\left|D_{a+}^{\alpha}f(t)\right|^2\,dt+\int_{\frac{a+b}{2}}^b\left|D_{b-}^{\alpha}f(t)\right|^2\,dt\right). \tag{4.7}$$

If we put $\alpha = 1$, $\beta_1 = 0$, $a = 0$ and $b = h$, then inequality (4.7) becomes Opial's inequality (1.1) (having boundary conditions $f(0) = f(h) = 0$).

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