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VECTOR VALUED FUNCTIONS OF BOUNDED BIDIMENSIONAL Φ-VARIATION

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ABSTRACT. In this article we present a generalization of the concept of function of bounded variation, in the sense of Riesz, for functions defined on a rectangle in $\mathbb{R}^2$, which take values in a Banach space. As applications, we obtain generalizations of some results due to Chistyakov and a counterpart of the classical Riesz's Lemma.

1. INTRODUCTION

Let us recall that a function $u : [a, b] \to \mathbb{R}$ is of bounded variation if

$$V(u, [a, b]) := \sup\left\{ \sum_{i=1}^{m} |u(t_i) - u(t_{i-1})| : \{t_i\}_{i\in\mathbb{N}} \in \pi[a, b] \right\} < \infty$$

where $\pi[a, b] :\ a = t_0 < t_1 < \cdots < t_n = b$.

The class of all real valued functions of bounded variation was introduced by Jordan in 1881 ([10]), who established the relation between these and the class of all monotone functions; namely,

A function $f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation if and only it is the difference of two monotone functions.

This fact has important implications: in the first place, every function of bounded variation, defined on an closed interval $[a, b]$, has one-sided limits at every point of (a, b) and, moreover, the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist; on the other hand, by a celebrated theorem of Lebesgue (see e.g., [15, page 112]) a function of bounded variation is differentiable almost everywhere on $[a, b]$.

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Historically one of the most important implications of Jordan's characterization is that it permits to extend the Dirichlet's criterium (for the convergence of the Fourier series of piecewise monotone functions) to the class of functions of bounded variation.

The interest generated by this notion has lead to some generalizations of the concept (see, e.g., [2, 7, 17]). Many of these generalizations are mainly intended to the search of a bigger class of functions whose elements have pointwise convergent Fourier series. As in the classical case, this generalizations have found also many applications in the study of certain differential and integral equations (see, e.g., [3]).

In 1910, Riesz ([14]) introduces the class of functions of bounded p-variation on $[a,b]$ ($1 \leq p < \infty$); it consists of those functions $u : [a,b] \to \mathbb{R}$ such that

$$V_p^R(u;[a,b]) = \sup\left\{ \sum_{i=1}^{n} \frac{|u(t_i) - u(t_{i-1})|^p}{|t_i - t_{i-1}|^{p-1}} : \{t_i\}_{i\in\mathbb{N}} \in \pi[a,b] \right\} < \infty,$$

where $\pi[a,b]$ denotes the set of all partitions $\{t_k\}_{k=0}^n$ of $[a,b]$; that is, $a = t_0 < t_1 < \cdots < t_n = b$.
He proved that in the case $1 < p < \infty$, this class coincides with the class of those functions u that are absolutely continuous and whose derivative $u' \in L_p[a,b]$. In fact, the renowned *Riesz's Lemma* establishes that

$$V_p^R(u;[a,b]) < \infty \iff V_p^R(u;[a,b]) = \|u'\|^p_{L^p[a,b]}.$$

In this paper we will use the following standard notation (see [5]): $\mathcal{N}$ will denote the set of all continuous convex functions $\Phi : [0,+\infty) \to [0,+\infty)$ such that $\Phi(\rho) = 0$ if and only if $\rho = 0$, and $\mathcal{N}_\infty$ the set of all functions $\Phi \in \mathcal{N}$, for which the Orlicz condition (also called ∞_1) holds: $\lim\limits_{\rho\to\infty} \dfrac{\Phi(\rho)}{\rho} = +\infty$. Any function $\Phi \in \mathcal{N}$ is strictly increasing, and so, its inverse Φ^{-1} is continuous and concave; besides, the functions $\rho \longmapsto \dfrac{\Phi(\rho)}{\rho}$ and $\rho \longmapsto \rho\Phi^{-1}\left(\dfrac{1}{\rho}\right)$ are nondecreasing for $\rho > 0$.

One says that a function $\Phi \in \mathcal{N}$ satisfies a condition Δ_2, and writes $\Phi \in \Delta_2$, if there are constants $K > 2$ and $t_0 \in \mathbb{R}$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all} \quad t \geq t_0. \tag{1.1}$$

For instance, if $\Phi(x) := t^p$, $p > 1$, one may chooses the optimal constant $K = 2^p$.

In 1953, Medvedev ([18]), gives a generalization of the notion of Riesz bounded p-variation: given an $\Phi \in \mathcal{N}_\infty$, *a function* $u : [a,b] \to \mathbb{R}$ *is said to be of bounded* Φ*-variation on* $[a,b]$*, in the sense of Riesz if:*

$$V_\Phi^R(u;[a,b]) := \sup\left\{ \sum_{i=1}^{n} \Phi\left(\frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|}\right) |t_i - t_{i-1}| : \{t_i\}_{i\in\mathbb{N}} \in \pi[a,b] \right\} < \infty.$$

The class of all such functions is denoted by $RV_{\Phi}[a,b]$. Cybertowicz and Matuszewska ([8]) also gets the following generalization of the Riesz's lemma:

Proposition 1.1. *Let* $\Phi \in \mathcal{N}_{\infty}$ *such that* $\lim_{\rho\to 0^+}\frac{\Phi(\rho)}{\rho} = 0$. *A function* $u \in RV_{\Phi}[a,b]$ *if and only if,* u *is absolutely continuous on* $[a,b]$ *and* $\int_a^b \Phi(|u'(t)|)dt < \infty$. *In this case we have,* $V_{\Phi}^R(u;[a,b]) = \int_a^b \Phi(|u'(t)|)dt$.

In 1998, Chistyakov ([7]), assuming that (X,d) is a metric space, introduced the notion of Φ-variation in the sense of Jordan, Riesz and Orlicz:

A function $u : [a,b] \to X$ *is said to be of (total) bounded* Φ*-variation, in the sense of Jordan, Riesz and Orlicz* if

$$V_{\Phi,d}(u;[a,b]) := \left\{ \sum_{k=1}^{m} \Phi\left(\frac{d(u(t_k),u(t_{k-1}))}{t_k - t_{k-1}}\right)(t_k - t_{k-1}) : \{t_k\}_{k\in\mathbb{N}} \in \pi[a,b] \right\} < \infty.$$

The set of all such functions is denoted by $BV_{\Phi,d}([a,b];X)$.

The following result is due to Chystiakov ([5, Theorem 3.3]).

Theorem 1.2. *Let* $(X,\|\cdot\|)$ *be a reflexive Banach space,* $\Phi \in \mathcal{N}_{\infty}$ *and suppose that* $f : [a,b] \to X$ *is a mapping of bounded* Φ*-variation, then* f *is strongly differentiable a.e. on* $[a,b]$*, its derivative* f' *is strongly measurable and Bochner integrable on* $[a,b]$*,* f *is represented as* $f(t) = f(a) + \int_a^t f'(\tau)d\tau$ *for all* $t \in [a,b]$ *and the following integral formula for the* Φ*-variation holds*

$$V_{\Phi,d}(f,[a,b]) = \int_a^b \Phi(\|f'(t)\|)dt.$$

A multidimensional extension of this notion requires an appropriate generalization of total variation for functions of several variables.

Given two points $\mathbf{a} = (a_1,a_2)$ and $\mathbf{b} = (b_1,b_2)$ in $\mathbb{R}^2$ we will denote a rectangle $[a_1,b_1]\times[a_2,b_2]$ by $I_{\mathbf{a}}^{\mathbf{b}}$. The notation $\pi([x,y])$ will stand for the set of all partitions of a closed interval $[x,y] \subset \mathbb{R}$. Accordingly, the set of all partitions of the form $\xi \times \eta$ of $I_{\mathbf{a}}^{\mathbf{b}}$ will be denoted as $\pi(I_{\mathbf{a}}^{\mathbf{b}})$.

Recently, Aziz [1] (see also [2]) introduces the notion of functions of bounded bidimensional Φ-variation (in the sense of Riesz), for real valued functions defined on a rectangle $I_{\mathbf{a}}^{\mathbf{b}} \subseteq \mathbb{R}^2$. In this article, and inspired in [1] and [2], we present a generalization of this last notion introducing the class $RBV_{\Phi}(I_{\mathbf{a}}^{\mathbf{b}},X)$, of X-valued functions of bounded Φ-variation, where X is a Banach space. In particular, we get the following generalizations of Chistyakov's result:

Theorem 5.1 Let $(X,\|\cdot\|)$ be a reflexive Banach space and suppose that $\Phi \in \mathcal{N}_{\infty}$ and $f \in RBV_{\Phi}(I_{\mathbf{a}}^{\mathbf{b}})$. Then, for all $0 \neq x^* \in X^*$

$$\begin{aligned} TV_{\Phi}^R(x^*\circ f, I_{\mathbf{a}}^{\mathbf{b}};\mathbb{R}) &= \int_{a_1}^{b_1} \Phi\left(\left|\frac{\partial(x^*\circ f)(x,a_2)}{\partial x}\right|\right)dx + \int_{a_2}^{b_2} \Phi\left(\left|\frac{\partial(x^*\circ f)(a_1,y)}{\partial y}\right|\right)dy \\ &\quad + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right)dydx. \end{aligned}$$

Corollary 5.3 Let $\Phi \in \mathcal{N}_\infty$ and suppose that $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$, where X is a reflexive Banach space. If $f \in C^2(I_{\mathbf{a}}^{\mathbf{b}})$ and $f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ then

$$\begin{aligned} TV_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, X) &= \int_{a_1}^{b_1} \Phi\left(\left\|\frac{\partial f(x,a_2)}{\partial x}\right\|\right) dx + \int_{a_2}^{b_2} \Phi\left(\left\|\frac{\partial f(a_1,y)}{\partial y}\right\|\right) dy \\ &+ \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi\left(\left\|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right\|\right) dydx. \end{aligned}$$

2. Bidimensional Φ-variation in the sense of Riesz

In the sequel X will denote a normed linear space and, as usual, $\mathbb{N}$ will denote the set of all natural numbers (positive integers).

Given $I_{\mathbf{a}}^{\mathbf{b}}$ and a function $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$ we introduce the following notations:

(1) If $\xi := \{t_i\}_{i=1}^m \in \pi[a_1, b_1]$

$$V_\Phi^R(f, [a_1, b_1], \xi) := \sum_{i=1}^m \Phi\left[\frac{\|f(t_i, a_2) - f(t_{i-1}, a_2)\|}{t_i - t_{i-1}}\right](t_i - t_{i-1})$$

(2) If $\eta := \{s_j\}_{j=1}^n \in \pi[a_2, b_2]$

$$V_\Phi^R(f, [a_2, b_2], \eta) := \sum_{j=1}^n \Phi\left[\frac{\|f(a_1, s_j) - f(a_1, s_{j-1})\|}{s_j - s_{j-1}}\right](s_j - s_{j-1}).$$

(3) The Vitali Difference of a rectangle $[t_1, t_2] \times [s_1, s_2]$ is defined as:

$$\Delta(f, [t_1, t_2] \times [s_1, s_2]) := f(t_1, s_1) - f(t_1, s_2) - f(t_2, s_1) + f(t_2, s_2). \tag{2.1}$$

Finally,

If $\xi := \{t_i\}_{i=0}^n \in \pi[a_1, b_1]$ and $\eta := \{s_j\}_{j=0}^m \in \pi[a_2, b_2]$, define

$$V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta) := \sum_{i=1}^n\sum_{j=1}^m \Phi\left[\frac{\|\Delta(f, [t_{i-1}, t_i] \times [s_{j-1}, s_j])\|}{(t_i - t_{i-1})(s_j - s_{j-1})}\right](t_i - t_{i-1})(s_j - s_{j-1}).$$

Remark 2.1. Given multi-indexes $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in (\mathbb{N} \cup \{0\})^2$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ define:

$$|\alpha| := \alpha_1 + \alpha_2, \quad \alpha \pm \beta := (\alpha_1 \pm \beta_1,\, \alpha_2 \pm \beta_2) \quad \text{and} \quad \alpha\mathbf{x} := (\alpha_1 x_1,\, \alpha_2 x_2).$$

Then, if $\mathbf{x}_{ij} := (t_{i-1}, s_{j-1})$, $\mathbf{y}_{ij} := (t_i, s_j)$ and $[\mathbf{x}_{ij}, \mathbf{y}_{ij}] = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$ the expression $\Delta(f, [\mathbf{x}_{ij}, \mathbf{y}_{ij}])$ can be written as

$$\sum_{|\alpha| \le \mathbf{2}} (-1)^{|\alpha|} f(\alpha\mathbf{x}_{ij} + (\mathbf{1} - \alpha)\mathbf{y}_{ij}), \tag{2.2}$$

where, the symbol $\mathbf{1}$ stands for the multi-index $(1, 1)$.

Definition 2.2. Let X be a Banach space, $\Phi \in \mathcal{N}$, $I_{\mathbf{a}}^{\mathbf{b}}$ be a rectangle in $\mathbb{R}^2$ and $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$. Define

(a) the $R\Phi$-variation of the function $f(\cdot, a_2)$ as

$$V_{10,\Phi}^R(f, [a_1, b_1]) := \sup_{\xi \in \pi[a_1, b_1]} V_\Phi^R(f, [a_1, b_1], \xi).$$

(b) Similarly, the $R\Phi$-variation of the function $f(a_1,\cdot)$ is defined as

$$V^R_{01,\Phi}(f,[a_2,b_2]) := \sup_{\eta\in\pi[a_2,b_2]} V^R_\Phi(f,[a_2,b_2],\eta).$$

Finally,

(c) the $R\Phi$-bidimensional variation of f is defined by

$$V^R_{11,\Phi}(f,I^{\mathbf{b}}_{\mathbf{a}}) := \sup_{\substack{\xi\in\pi[a_1,b_1]\\ \eta\in\pi[a_2,b_2]}} V^R_\Phi(f,I^{\mathbf{b}}_{\mathbf{a}},\xi,\eta).$$

We will say that f has (total) **bounded bi-dimensional Φ-variation, in the sense of Riesz**, on $I^{\mathbf{b}}_{\mathbf{a}}$, if

$$TV^R_\Phi(f,I^{\mathbf{b}}_{\mathbf{a}},X) := V^R_{10,\Phi}(f,[a_1,b_1]) + V^R_{01,\Phi}(f,[a_2,b_2]) + V^R_{11,\Phi}(f,I^{\mathbf{b}}_{\mathbf{a}}) < \infty.$$

The class of such functions will be denoted by $RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$.

When the role of the space X is clear and unambiguous we simply use the notation $TV^R_\Phi(f,I^{\mathbf{b}}_{\mathbf{a}})$ (instead of $TV^R_\Phi(f,I^{\mathbf{b}}_{\mathbf{a}},X)$).

Example 2.3. Let $f:[0,1]\times[0,1]\to\mathcal{C}_0$ be defined as

$$f(x,y) := \left(\frac{x-y}{n}\right)_{n\in\mathbb{N}}.$$

Then

$$V^R_{10,\Phi}(f,[0,1]) = V^R_{01,\Phi}(f,[0,1]) = \Phi(1).$$

On the other hand (see 2.2)

$$\sum_{|\alpha|\le \mathbf{2}} (-1)^{|\alpha|} f(\alpha\mathbf{x}_{ij} + (\mathbf{1}-\alpha)\mathbf{y}_{ij}) = (0)_{\mathbb{N}}.$$

Consequently,

$$\begin{aligned} & V^R_\Phi(f,[0,1]\times[0,1]) \\ := {} & \sup_{(\xi,\eta)}\sum_{i=1}^m\sum_{j=1}^n \Phi\left[\frac{\left\|\sum_{|\alpha|\le\mathbf{2}}(-1)^{|\alpha|} f(\alpha\mathbf{x}_{ij}+(\mathbf{1}-\alpha)\mathbf{y}_{ij})\right\|}{(t_i-t_{i-1})(s_j-s_{j-1})}\right](t_i-t_{i-1})(s_j-s_{j-1}) = 0, \end{aligned}$$

which implies that $TV^R_\Phi(f,[0,1]\times[0,1],\mathcal{C}_0) = 2\Phi(1) < \infty$; that is, $f\in RBV_\Phi([0,1]\times[0,1])$.

Next, we present some properties that are satisfied by the functions in the class $RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$. These are counterparts of several results given in [2] in the real case.

Proposition 2.4. *$RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$ as a subset of the linear space $X^{I^{\mathbf{b}}_{\mathbf{a}}}$ is a convex set, and $TV^R_\Phi(\cdot,I^{\mathbf{b}}_{\mathbf{a}},X): RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})\to[0,\infty)$ is a convex function.*

Proof. The results are consequence of the fact that both Φ and the norm $\|\cdot\|_X$ are convex functions. □

Proposition 2.5. *Let* $\xi = \{t_i\}_{i=0}^n$ *and* $\eta = \{s_j\}_{j=0}^m$ *partitions of* $[a_1, b_1]$ *and* $[a_2, b_2]$ *respectively, and suppose that* $(t, s) \in I_{\mathbf{a}}^{\mathbf{b}} \setminus \{\xi \times \eta\}$. *Then*

$$V_\Phi^R(f, [a_1, b_1], \xi) \leq V_\Phi^R(f, [a_1, b_1], \xi \cup \{t\}), \tag{2.3}$$

$$V_\Phi^R(f, [a_2, b_2], \eta) \leq V_\Phi^R(f, [a_2, b_2], \eta \cup \{s\}), \tag{2.4}$$

$$V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta) \leq V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi \cup \{t\}, \eta), \tag{2.5}$$

$$V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta) \leq V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta \cup \{s\}), \tag{2.6}$$

and

$$V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta) \leq V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi \cup \{t\}, \eta \cup \{s\}). \tag{2.7}$$

Proof. Fix $k \in \{1, 2, \ldots, n\}$ and $r \in \{1, 2, \ldots, m\}$ such that $t_{k-1} < t < t_k$ and $s_{r-1} < s < s_r$.

It is readily seen that (2.3) and (2.4) follow from the fact that both, Φ and the norm $\|\cdot\|_X$, are convex functions. On the other hand, for all $1 \leq j \leq m$, if $\mathbf{x}_{kj} := (t_{k-1}, s_{j-1})$ and $\mathbf{y}_{kj} := (t_k, s_j)$ then we have

$$\begin{aligned}
&\Phi\left(\frac{\left\|\sum_{|\alpha|\leq \mathbf{2}} (-1)^{|\alpha|} f(\alpha \mathbf{x}_{kj} + (\mathbf{1} - \alpha)\mathbf{y}_{kj})\right\|}{(t_k - t_{k-1})(s_j - s_{j-1})}\right)(t_k - t_{k-1})(s_j - s_{j-1}) \\
\leq\; &\Phi\left(\frac{\|f(t, s_j) - f(t, s_{j-1}) - f(t_{k-1}, s_j) + f(t_{k-1}, s_{j-1})\|}{(t - t_{k-1})(s_j - s_{j-1})}\right)(t - t_{k-1})(s_j - s_{j-1}) \\
&+\Phi\left(\frac{\|f(t_k, s_j) - f(t_k, s_{j-1}) - f(t, s_j) + f(t, s_{j-1})\|}{(t_k - t)(s_j - s_{j-1})}\right)(t_k - t)(s_j - s_{j-1}).
\end{aligned} \tag{2.8}$$

Now, define $\xi \cup \{t\} := \{x_i\}_{i=0}^{n+1}$, where

$$x_i := \begin{cases} t_i & 0 \leq i \leq k-1 \\ t & i = k \\ t_{i-1} & k+1 \leq i \leq n+1. \end{cases}$$

Then,

$$\begin{aligned}
&V_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, \xi, \eta) \\
= &\sum_{i=1}^{k-1}\sum_{j=1}^{m} \Phi\left[\frac{\|f(t_i, s_j) - f(t_i, s_{j-1}) - f(t_{i-1}, s_j) + f(t_{i-1}, s_{j-1})\|}{(t_i - t_{i-1})(s_j - s_{j-1})}\right](t_i - t_{i-1})(s_j - s_{j-1}) \\
&+\sum_{j=1}^{m} \Phi\left[\frac{\|f(t_k, s_j) - f(t_k, s_{j-1}) - f(t_{k-1}, s_j) + f(t_{k-1}, s_{j-1})\|}{(t_k - t_{k-1})(s_j - s_{j-1})}\right](t_k - t_{k-1})(s_j - s_{j-1}) \\
&+\sum_{i=k+1}^{n}\sum_{j=1}^{m} \Phi\left[\frac{\|f(t_i, s_j) - f(t_i, s_{j-1}) - f(t_{i-1}, s_j) + f(t_{i-1}, s_{j-1})\|}{(t_i - t_{i-1})(s_j - s_{j-1})}\right](t_i - t_{i-1})(s_j - s_{j-1})
\end{aligned}$$

which, by (2.8) is

$$
\begin{aligned}
&\leq \sum_{i=1}^{k}\sum_{j=1}^{m}\Phi\left[\frac{\|f(x_i,s_j)-f(x_i,s_{j-1})-f(x_{i-1},s_j)+f(x_{i-1},s_{j-1})\|}{(x_i-x_{i-1})(s_j-s_{j-1})}\right](x_i-x_{i-1})(s_j-s_{j-1})\\
&\quad+\sum_{i=k}^{n}\sum_{j=1}^{m}\Phi\left[\frac{\|f(x_{i+1},s_j)-f(x_{i+1},s_{j-1})-f(x_i,s_j)+f(x_i,s_{j-1})\|}{(x_{i+1}-x_i)(s_j-s_{j-1})}\right](x_{i+1}-x_i)(s_j-s_{j-1})\\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{m}\Phi\left[\frac{\left\|\sum_{|\alpha|\leq \mathbf{2}}(-1)^{|\alpha|}f(\alpha\mathbf{x}_{ij}+(\mathbf{1}-\alpha)\mathbf{y}_{ij})\right\|}{(x_i-x_{i-1})(s_j-s_{j-1})}\right](x_i-x_{i-1})(s_j-s_{j-1})\\
&= V_\Phi^R(f,I_{\mathbf{a}}^{\mathbf{b}},\xi\cup\{t\},\eta).
\end{aligned}
$$

This proves (2.5).

Proceeding in a similar fashion when $a_2 < s < b_2$ one gets (2.6). Finally, by combining (2.5) and (2.6) we obtain (2.7). □

Lemma 2.6. *Let* $\Phi \in \mathcal{N}$*, and suppose* $a_1 \leq x_1 < u_1 < u_2 < x_2 \leq b_1$*,* $a_2 \leq y_1 < v_1 < v_2 < y_2 \leq b_2$*, and* $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$*. Then*

(a) $V_{10,\Phi}^R(f,[u_1,u_2]) \leq V_{10,\Phi}^R(f,[x_1,x_2])$;
(b) $V_{01,\Phi}^R(f,[v_1,v_2]) \leq V_{01,\Phi}^R(f,[y_1,y_2])$;
(c) $V_{11,\Phi}^R(f,[u_1,u_2]\times[v_1,v_2]) \leq V_{11,\Phi}^R(f,[x_1,x_2]\times[y_1,y_2])$.
(d) $$\|f(u_2,a_2)-f(u_1,a_2)\| \leq \Phi^{-1}(u_2-u_1)\left(\frac{V_{10,\Phi}^R(f,[a_1,b_1])}{u_2-u_1}\right);$$
(e) $$\|f(a_1,v_2)-f(a_1,v_1)\| \leq \Phi^{-1}(v_2-v_1)\left(\frac{V_{01,\Phi}^R(f,[a_2,b_2])}{v_2-v_1}\right);$$
(f) $$\|\Delta(f,[u_1,u_2]\times[v_1,v_2])\| \leq \Phi^{-1}\left(\frac{V_{11,\Phi}^R(f,I_{\mathbf{a}}^{\mathbf{b}})}{(u_2-u_1)(v_2-v_1)}\right)(u_2-u_1)(v_2-v_1).$$

Proof. Let $\xi := \{t_i\}_{i=0}^n$ be a partition of $[u_1,u_2] \subset [x_1,x_2]$. Then

$$
\begin{aligned}
V_\Phi^R(f,[u_1,u_2],\xi) &\leq \Phi\left[\frac{\|f(u_1,a_2)-f(x_1,a_2)\|}{(u_1-x_1)}\right](u_1-x_1)\\
&\quad+\sum_{i=1}^{n}\Phi\left[\frac{\|f(t_i,a_2)-f(t_{i-1},a_2)\|}{(t_i-t_{i-1})}\right](t_i-t_{i-1})\\
&\quad+\Phi\left[\frac{\|f(x_2,a_2)-f(u_2,a_2)\|}{(x_2-u_2)}\right](x_2-u_2)\\
&\leq V_{10,\Phi}^R(f,[x_1,x_2]).
\end{aligned}
$$

Consequently $V_{10,\Phi}^R(f,[u_1,u_2]) \leq V_{10,\Phi}^R(f,[x_1,x_2])$. Similarly one proves (b).

Now we will show part (c). Let ξ be as above and let $\eta = \{s_i\}_{i=0}^m$ be a partition of $[v_1, v_2]$ and $\mathbf{x}_{ij} = (t_{i-1}, s_{j-1})$, $\mathbf{y}_{ij} = (t_j, s_j)$. Then

$$\begin{aligned}
& V_\Phi^R(f, [u_1, u_2] \times [v_1, v_2], \xi, \eta) \\
\leq\ & \sum_{j=1}^m \Phi\left[\frac{\|f(u_1, s_j) - f(u_1, s_{j-1}) - f(x_1, s_j) + f(x_1, s_{j-1})\|}{(u_1 - x_1)(s_j - s_{j-1})}\right](u_1 - x_1)(s_j - s_{j-1}) \\
& + \sum_{i=1}^n \left\{ \sum_{j=1}^m \Phi\left[\frac{\left\| \sum_{|\alpha| \leq \mathbf{2}} (-1)^{|\alpha|} f(\alpha \mathbf{x}_{ij} + (\mathbf{1} - \alpha)\mathbf{y}_{ij}) \right\|}{(t_i - t_{i-1})(s_j - s_{j-1})}\right](t_i - t_{i-1}) \right\} (s_j - s_{j-1}) \\
& + \sum_{j=1}^m \Phi\left[\frac{\|f(x_2, s_j) - f(x_2, s_{j-1}) - f(u_2, s_j) + f(u_2, s_{j-1})\|}{(x_2 - u_2)(s_j - s_{j-1})}\right](x_2 - u_2)(s_j - s_{j-1}) \\
=\ & V_\Phi^R(f, [x_1, x_2] \times [v_1, v_2], \{x_1\} \cup \xi \cup \{x_2\}, \eta).
\end{aligned}$$

Put $\zeta = \{t^i\}_{i=0}^{n+2} := \{x_1\} \cup \xi \cup \{x_2\} = \begin{cases} x_1, & i = 0 \\ t_{i-1}, & 1 \leq i \leq n+1 \\ x_2, & i = n+2; \end{cases}$

and $\varrho = \{s^j\}_{j=0}^{m+2} := \{y_1\} \cup \eta \cup \{y_2\} = \begin{cases} y_1, & j = 0 \\ s_{j-1}, & 1 \leq j \leq m+1 \\ y_2, & j = m+2. \end{cases}$

Then

$$\begin{aligned}
& V_\Phi^R(f, [u_1, u_2] \times [v_1, v_2], \xi, \eta) \\
\leq\ & \sum_{i=1}^{n+2} \sum_{j=1}^m \Phi\left[\frac{\|f(t^i, s_j) - f(t^i, s_{j-1}) - f(t^{i-1}, s_j) + f(t^{i-1}, s_{j-1})\|}{(t^i - t^{i-1})(s_j - s_{j-1})}\right](t^i - t^{i-1})(s_j - s_{j-1}) \\
\leq\ & \sum_{i=1}^{n+2} \left\{ \Phi\left[\frac{\|f(t^i, y_2) - f(t^i, v_2) - f(t^{i-1}, y_2) + f(t^{i-1}, v_2)\|}{(t^i - t^{i-1})(y_2 - v_2)}\right](t^i - t^{i-1})(y_2 - v_2) \right. \\
& + \sum_{j=1}^m \Phi\left[\frac{\|f(t^i, s_j) - f(t^i, s_{j-1}) - f(t^{i-1}, s_j) + f(t^{i-1}, s_{j-1})\|}{(t^i - t^{i-1})(s_i - s_{i-1})}\right](t^i - t^{i-1})(s_i - s_{i-1}) \\
& \left. + \Phi\left[\frac{\|f(t^i, v_1) - f(t^i, y_1) - f(t^{i-1}, v_1) + f(t^{i-1}, y_1)\|}{(t^i - t^{i-1})(v_1 - y_1)}\right](t^i - t^{i-1})(v_1 - y_1) \right\} \\
=\ & \sum_{i=1}^{n+2} \sum_{j=1}^{m+2} \Phi\left[\frac{\|f(t^i, s^j) - f(t^i, s^{j-1}) - f(t^{i-1}, s^j) + f(t^{i-1}, s^{j-1})\|}{(t^i - t^{i-1})(s^j - s^{j-1})}\right](t^i - t^{i-1})(s^j - s^{j-1}) \\
=\ & V_\Phi^R(f, [x_1, x_2] \times [y_1, y_2], \zeta, \varrho) \\
\leq\ & V_{11,\Phi}^R(f, [x_1, x_2] \times [y_1, y_2]),
\end{aligned}$$

which completes the proof of (c).

To prove (d), notice that

$$V_{10,\Phi}^R(f, [a_1, b_1]) \ \geq\ \Phi\left[\frac{\|f(u_2, a_2) - f(u_1, a_2)\|}{u_2 - u_1}\right](u_2 - u_1);$$

and hence

$$\Phi^{-1}\left[\frac{V^R_{10,\Phi}(f,[a_1,b_1])}{u_2-u_1}\right](u_2-u_1) \quad \geq \quad \|f(u_2,a_2)-f(u_1,a_2)\|.$$

Similarly, we can obtain (e).

Finally, if $\xi=\{t_i\}_{i=0}^3 := \{a_1,u_1,u_2,b_1\}$ and $\eta=\{s_i\}_{i=0}^3 := \{a_2,v_1,v_2,b_2\}$, then

$$\begin{aligned} V^R_{11,\Phi}(f,I_{\mathbf{a}}^{\mathbf{b}}) &\geq \sum_{i=1}^3\sum_{j=1}^3 \Phi\left[\frac{\left\|\sum_{|\alpha|\leq\mathbf{2}}(-1)^{|\alpha|}f(\alpha\mathbf{x}_{ij}+(\mathbf{1}-\alpha)\mathbf{y}_{ij})\right\|}{(t_i-t_{i-1})(s_j-s_{j-1})}\right](t_i-t_{i-1})(s_j-s_{j-1}) \\ &\geq \Phi\left[\frac{\|f(u_2,v_2)-f(u_2,v_1)-f(u_1,v_2)+f(u_1,v_1)\|}{(u_2-u_1)(v_2-v_1)}\right](u_2-u_1)(v_2-v_1), \end{aligned}$$

where $\mathbf{x}_{ij}=(t_{i-1},s_{j-1})$ and $\mathbf{y}_{ij}=(t_i,s_j)$.

Therefore

$$\Phi^{-1}\left[\frac{V^R_{11,\Phi}(f,I_{\mathbf{a}}^{\mathbf{b}})}{(u_2-u_1)(v_2-v_1)}\right](u_2-u_1)(v_2-v_1)\geq\|f(u_2,v_2)-f(u_2,v_1)-f(u_1,v_2)+f(u_1,v_1)\|,$$

which proves (f). □

Theorem 2.7. *If* $a_1<t<b_1$ *and* $a_2<s<b_2$, *then*

$$TV^R_\Phi(f,I_{\mathbf{a}}^{\mathbf{b}}) = TV^R_\Phi(f,[a_1,t]\times[a_2,b_2])+TV^R_\Phi(f,[t,b_1]\times[a_2,b_2]), \tag{2.9}$$

$$TV^R_\Phi(f,[a_1,b_1]\times[a_2,b_2]) = TV^R_\Phi(f,[a_1,b_1]\times[a_2,s])+TV^R_\Phi(f,[a_1,b_1]\times[s,b_2]) \tag{2.10}$$

and

$$\begin{aligned} TV^R_\Phi(f,I_{\mathbf{a}}^{\mathbf{b}}) &= TV^R_\Phi(f,[a_1,t]\times[a_2,s])+TV^R_\Phi(f,[a_1,t]\times[s,b_2]) \\ &+ TV^R_\Phi(f,[t,b_1]\times[a_2,s])+TV^R_\Phi(f,[t,b_1]\times[s,b_2]). \end{aligned} \tag{2.11}$$

Proof. Suppose $\xi=\{t_i\}_{i=0}^n\in\pi[a_1,t]$ and $\zeta=\{\tau_i\}_{i=0}^m\in\pi[t,b_1]$. Then

$$V^R_\Phi(f,[a_1,t],\xi)+V^R_\Phi(f,[t,b_1],\zeta) = V^R_\Phi(f,[a_1,b_1],\xi\cup\zeta),$$

and therefore

$$V^R_{10,\Phi}(f,[a_1,t])+V^R_{10,\Phi}(f,[t,b_1]) = V^R_{10,\Phi}(f,[a_1,b_1]).$$

Similarly, we obtain

$$V^R_{01,\Phi}(f,[a_2,s])+V^R_{01,\Phi}(f,[s,b_2]) = V^R_{01,\Phi}(f,[a_2,b_2]).$$

Assume now that ξ,ζ,η are partitions of $[a_1,t],[t,b_1]$ and $[a_2,b_2]$ respectively. Then

$$V^R_\Phi(f,[a_1,t]\times[a_2,b_2],\xi,\eta)+V^R_\Phi(f,[t,b_1]\times[a_2,b_2],\zeta,\eta) = V^R_\Phi(f,I_{\mathbf{a}}^{\mathbf{b}},\xi\cup\zeta,\eta),$$

which implies (2.9).

Furthermore, for $\xi \in \pi[a_1, b_1]$, $\varrho \in \pi[a_2, s]$ and $\chi \in \pi[s, b_2]$:

$$\begin{aligned} & V_\Phi^R(f, [a_1, b_1] \times [a_2, s], \xi, \varrho) + V_\Phi^R(f, [a_1, b_1] \times [s, b_2], \xi, \chi) \\ = \; & V_\Phi^R(f, [a_1, b_1] \times [a_2, b_2], \xi, \varrho \cup \chi), \end{aligned}$$

which yields (2.10).

Finally, applying (2.10) to (2.9) we get (2.11). □

Theorem 2.8. *Let* $\{\Phi_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{N}$ *and* $\{f_n\}_{n\geq 1} \subseteq X^{I_{\mathbf{a}}^{\mathbf{b}}}$. *Suppose that*

$$\lim_{n\to\infty} \|f_n(t, s) - f(t, s)\| = 0, \;\; \forall \;\, (t, s) \in I_{\mathbf{a}}^{\mathbf{b}},$$

and

$$\lim_{n\to\infty} \Phi_n(\rho) = \Phi(\rho), \;\; \forall \;\, \rho \in [0, +\infty).$$

Then

$$V_{10,\Phi}^R(f, [a_1, b_1]) \leq \liminf_{n\to\infty} V_{10,\Phi_n}^R(f_n, [a_1, b_1]),$$

$$V_{01,\Phi}^R(f, [a_2, b_2]) \leq \liminf_{n\to\infty} V_{01,\Phi_n}^R(f_n, [a_2, b_2]), \;\; \textit{and} \tag{2.12}$$

$$V_{11,\Phi}^R(f, I_{\mathbf{a}}^{\mathbf{b}}) \leq \liminf_{n\to\infty} V_{11,\Phi_n}^R(f_n, I_{\mathbf{a}}^{\mathbf{b}}). \tag{2.13}$$

Proof. Define

$$\psi(t) := f(t, a_2)$$

and, for $n = 1, 2, \cdots$, put

$$\psi_n(t) := f_n(t, a_2).$$

Then, for each $t \in [a_1, b_1]$

$$\lim_{n\to\infty} \|\psi_n(t) - \psi(t)\| = \lim_{n\to\infty} \|f_n(t, a_2) - f(t, a_2)\| = 0\,.$$

Thus, by Lemma 3.1(d) of [5]

$$V_\Phi^R(\psi, [a_1, b_1]) \leq \liminf_{n\to\infty} \;\; V_{\Phi_n}(\psi_n, [a_1, b_1]),$$

which, in turn, implies that

$$V_{10,\Phi}^R(f, [a_1, b_1]) \;\; \leq \;\; \liminf_{n\to\infty} V_{10,\Phi_n}^R(f_n, [a_1, b_1]).$$

Inequality (2.12) can be shown similarly.

To prove the last part, we proceed as in [6, Lemma 2.1, (c)]: consider partitions $\xi = \{t_i\}_{i=0}^k \in \pi[a_1, b_1]$ and $\eta = \{s_j\}_{j=0}^m \in \pi[a_2, b_2]$. Fix i, j and put

$$\rho_n := \frac{\|\Delta(f_n, [t_{i-1}, t_i] \times [s_{j-1}, s_j])\|}{(t_i - t_{i-1})(s_j - s_{j-1})}$$

and

$$\rho := \frac{\|\Delta(f, [t_{i-1}, t_i] \times [s_{j-1}, s_j])\|}{(t_i - t_{i-1})(s_j - s_{j-1})}.$$

Then $\Phi_n(\rho_n) \to \Phi(\rho)$ as $n \to +\infty$.
Indeed, let $\rho > 0$ and $\epsilon > 0$. By continuity of Φ there is $0 < \delta = \delta(\epsilon) < \rho$ such that

$$|\Phi(r) - \Phi(\rho)| < \frac{\epsilon}{2} \quad \text{para todo} \quad r \geq 0 \ \text{ con } \ |r - \rho| \leq \delta.$$

Since $\rho_n \to \rho$ and $\Phi_n \to \Phi$ pointwise as $n \to +\infty$, there exists an $N(\epsilon) \in \mathbb{N}$ such that $n \geq N(\epsilon)$ implies $\rho - \delta < \rho_n < \rho + \delta$ and

$$|\Phi_n(\rho - \delta) - \Phi(\rho - \delta)| \leq \frac{\epsilon}{2} \qquad |\Phi_n(\rho + \delta) - \Phi(\rho + \delta)| \leq \frac{\epsilon}{2}.$$

Thus,

$$\Phi_n(\rho_n) < \Phi_n(\rho + \delta) \leq \Phi(\rho + \delta) + \frac{\epsilon}{2} \leq \Phi(\rho) + \epsilon$$
$$\Phi_n(\rho_n) > \Phi_n(\rho - \delta) \geq \Phi(\rho - \delta) - \frac{\epsilon}{2} \geq \Phi(\rho) - \epsilon$$

or equivalently $|\Phi_n(\rho_n) - \Phi(\rho)| < \epsilon$. The case $\rho = 0$ can be treated similarly.

Now, by definition of $V^R_{11,\Phi_n}(f_n, I^{\mathbf{b}}_{\mathbf{a}})$ we have

$$\sum_{i=1}^{k}\sum_{j=1}^{m}\Phi_n(\rho_n) = V^R_{\Phi_n}(f_n, I^{\mathbf{b}}_{\mathbf{a}}, \xi, \eta) \quad \leq \quad V^R_{11,\Phi_n}(f_n, I^{\mathbf{b}}_{\mathbf{a}}),$$

thus, by taking lim inf in both sides of this inequality we get

$$\sum_{i=1}^{k}\sum_{j=1}^{m}\Phi(\rho) = V^R_{\Phi}(f, I^{\mathbf{b}}_{\mathbf{a}}, \xi, \eta) \quad \leq \quad \liminf_{n\to\infty} \ V^R_{11,\Phi_n}(f_n, I^{\mathbf{b}}_{\mathbf{a}})$$

which implies

$$V^R_{11,\Phi}(f, I^{\mathbf{b}}_{\mathbf{a}}) \quad \leq \quad \liminf_{n\to\infty} \ V^R_{11,\Phi_n}(f_n, I^{\mathbf{b}}_{\mathbf{a}}).$$

□

3. Linearity

A natural question that remains to be answered is: *Under what conditions is* $RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$ *a linear space?*.

The answer is: Φ must satisfy a Δ_2 condition. Actually, this condition is also necessary. Indeed:

Suppose that $\Phi \in \Delta_2$, X is a real (complex) Banach space, $c \in \mathbb{R}$ (resp. $\mathbb{C}$) and $f, g \in RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$.

If $|c| \leq 1$, the convexity of $RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$ ensures that $cf \in RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$. If, on the contrary, $|c| > 1$, then the fact that Φ satisfies a Δ_2 condition ensures that we can find positive constants $K(c)$ and ρ_0 such that

$$\Phi(|c|\rho) \leq K(c)\Phi(\rho) \quad \text{ for all } \quad \rho \geq \rho_0.$$

Next, observe that the relation $\Psi(\rho) := \Phi(|c|\rho)$ defines a function $\Psi \in \mathcal{N}$ such that

$$\lim_{\rho\to+\infty} \frac{\Psi(\rho)}{\Phi(\rho)} < \infty.$$

It readily follows that

$$RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}}) \subseteq RBV_\Psi(I_{\mathbf{a}}^{\mathbf{b}})$$

and

$$TV_\Phi^R(cf, I_{\mathbf{a}}^{\mathbf{b}}) = TV_\Psi^R(f, I_{\mathbf{a}}^{\mathbf{b}}) < \infty.$$

On the other hand,

$$TV_\Phi^R(f+g, I_{\mathbf{a}}^{\mathbf{b}}) \leq \frac{1}{2} TV_\Phi^R(2f, I_{\mathbf{a}}^{\mathbf{b}}) + \frac{1}{2} TV_\Phi^R(2g, I_{\mathbf{a}}^{\mathbf{b}}) < \infty.$$

Thus, $RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ is a linear space whenever $\Phi \in \Delta_2$.

To prove the reciprocal implication, notice that if $RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ is a linear space then: $f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ implies $2f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$, which, in turn, implies that $RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}}) \subseteq RBV_\Psi(I_{\mathbf{a}}^{\mathbf{b}})$, where $\Psi(\rho) := \Phi(2\rho)$, $\rho \geq 0$. Therefore, there are constants $C > 0$ and $\rho_0 > 0$ such that $\Phi(2\rho) = \Psi(\rho) \leq C\Phi(\rho)$ for all $\rho \geq \rho_0$. This means that $\Phi \in \Delta_2$.

The following proposition summarizes the facts we have just discussed.

Proposition 3.1. $RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ *is a linear space if and only if* $\Phi \in \Delta_2$.

4. Absolutely continuous functions

Now we define the concept of absolute continuity for vectorial functions defined on a rectangle $\mathbb{I} \subseteq \mathbb{R}^2$. To this end, as in [2], we base ourselves in the definition given by Carathéodory [4], in 1918, and in its recent reinterpretation due to Jiří Šremr [16].

If $\mathbb{I} \subseteq I_{\mathbf{a}}^{\mathbf{b}}$ then, we denote by $|\,\mathbb{I}\,|$ its area. We say that the rectangles $\mathbb{I}_1$ and $\mathbb{I}_2$ are **adjacent** if they do not overlap and $\mathbb{I}_1 \cup \mathbb{I}_2 \in \pi(I_{\mathbf{a}}^{\mathbf{b}})$.

Definition 4.1. A map $F : \pi(I_{\mathbf{a}}^{\mathbf{b}}) \to X$ is said to be **rectangle-additive** if, given any pair of adjacent rectangles $\mathbb{I}_1$ and $\mathbb{I}_2 \in \pi(I_{\mathbf{a}}^{\mathbf{b}})$, the identity

$$F(\mathbb{I}_1 \cup \mathbb{I}_2) = F(\mathbb{I}_1) + F(\mathbb{I}_2)$$

holds.

It readily follow from the definition that if $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$ is any map and we define

$$F_f([t_1, t_2] \times [s_1, s_2]) := f(t_1, s_1) - f(t_1, s_2) - f(t_2, s_1) + f(t_2, s_2), \tag{4.1}$$

for any $[t_1, t_2] \times [s_1, s_2] \in \pi(I_{\mathbf{a}}^{\mathbf{b}})$, then F_f is a rectangle-additive function. We will refer to the map F_f as the f-induced rectangle map.

Definition 4.2. A rectangle-additive map $F : \pi(I_{\mathbf{a}}^{\mathbf{b}}) \to X$ is said to be **absolutely continuous**, in the sense of Carathéodory, if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that: for any finite collection of non-overlapping rectangles $I_1, \cdots, I_k \in \pi(I_{\mathbf{a}}^{\mathbf{b}})$,

$$\sum_{j=1}^{k} |I_j| \leq \delta \implies \sum_{j=1}^{k} \|F(I_j)\| \leq \epsilon.$$

Definition 4.3. A function $F : I_{\mathbf{a}}^{\mathbf{b}} \to X$ is said to be **absolutely continuous**, in the sense of Carathéodory, if it satisfies the following conditions:

(a) The $f-$induced rectangle map F_f is absolutely continuous;
(b) The functions $f(a_1, \cdot) : [a_2, b_2] \to X$ and $f(\cdot, a_2) : [a_1, b_1] \to X$ are absolutely continuous.

We denote the set of all absolutely continuous functions, on $I_{\mathbf{a}}^{\mathbf{b}}$, as $AC(I_{\mathbf{a}}^{\mathbf{b}}; X)$

Example 4.4. Let $f : I_{\mathbf{a}}^{\mathbf{b}} \to \mathcal{C}_0$ be defined as

$$f(t,s) := \left(\frac{t+s}{n}\right)_{n\in\mathbb{N}}, \text{ where } I_{\mathbf{a}}^{\mathbf{b}} =: [0,1] \times [0,1].$$

It is readily seen that the functions $f(\cdot, 0)$ and $f(0, \cdot)$ are absolutely continuous on $[0, 1]$. On the other hand, if $\mathbb{I}_i := [t_i, t_i'] \times [s_i, s_i'],\ i = 1, 2, \ldots, m,$ then

$$\sum_{i=1}^{m} \|f(t_i, s_i) - f(t_i, s_i') - f(t_i', s_i) + f(t_i', s_i')\|_\infty = \sum_{i=1}^{m} \left\| \left(\frac{0}{n}\right)_{n\in\mathbb{N}} \right\|_\infty = 0 < \epsilon.$$

Consequently, f is absolutely continuous on $I_{\mathbf{a}}^{\mathbf{b}}$. Notice that this example also shows that the The $f-$induced rectangle map F_f is a rectangle-additive absolutely continuous map.

Now, we are going to show that every function of bounded bi-dimensional Φ-variation, in the sense of Riesz, is an absolutely continuous function in the sense of Carathéodory. We will need the following property of an $\mathcal{N}_\infty-$function Φ, which we have already mentioned at the beginning of section 2.

$$\lim_{r\to 0^+} r\Phi^{-1}(c/r) = c \lim_{v\to\infty} v/\Phi(v) = 0 \ \forall \ c \in [0, +\infty). \tag{4.2}$$

Lemma 4.5. *Suppose that* $\Phi \in \mathbb{R}^{(0,\infty)}$ *is a convex function, and that* $\{\rho_i\}_{i=1}^n$ *and* $\{p_i\}_{i=1}^n$, $n \in \mathbb{N}$, *are any two finite sequences of real numbers such that* $\rho_i \geq 0$, *and* $p_i > 0$, $\forall\ 1 \leq i \leq n$. *Then*

$$\Phi\left(\frac{\sum_{i=1}^n \rho_i}{\sum_{i=1}^n p_i}\right) \leq \sum_{i=1}^{n} \frac{p_i}{\sum_{i=1}^n p_i} \Phi\left(\frac{\rho_i}{p_i}\right).$$

Proof. Indeed, if we put $k := \sum_{i=1}^{n} p_i$ then

$$\Phi\left(\frac{\sum_{i=1}^n \rho_i}{k}\right) = \Phi\left(\sum_{i=1}^{n} \frac{\rho_i}{k}\right) = \Phi\left(\sum_{i=1}^{n} \frac{p_i\left(\dfrac{\rho_i}{p_i}\right)}{k}\right) \leq \sum_{i=1}^{n} \frac{p_i}{k} \Phi\left(\frac{\rho_i}{p_i}\right),$$

that was to be proved. □

As a consequence of the last lemma we have the following useful estimates. If $\{(t_i,s_i)\}_{i=1}^m$ and $\{(x_i,y_i)\}_{i=1}^m$ are mutually disjoint open intervals contained in $[a_1,b_1]$ and $[a_2,b_2]$, respectively, then

$$\sum_{i=1}^m \|f(s_i,a_2)-f(t_i,a_2)\| \;\le\; \Phi^{-1}\left(\frac{V^R_{10,\Phi}(f,[a_1,b_1])}{\sum_{i=1}^m (s_i-t_i)}\right)\sum_{i=1}^m (s_i-t_i). \quad (4.3)$$

$$\sum_{i=1}^m \|f(a_1,y_i)-f(a_1,x_i)\| \;\le\; \Phi^{-1}\left(\frac{V^R_{01,\Phi}(f,[a_2,b_2])}{\sum_{i=1}^m (y_i-x_i)}\right)\sum_{i=1}^m (y_i-x_i).$$

and

$$\sum_{i=1}^m \|f(x_i,y_i)-f(x_i,s_i)-f(t_i,y_i)+f(t_i,s_i)\| \le \Phi^{-1}\left(\frac{V^R_{11,\Phi}(f,I^{\mathbf{b}}_{\mathbf{a}})}{\sum_{i=1}^m (s_i-t_i)(y_i-x_i)}\right)\sum_{i=1}^m (s_i-t_i)(y_i-x_i). \quad (4.4)$$

Proposition 4.6. *If $\Phi\in\mathcal{N}_\infty$, and $f\in RBV_\Phi(I^{\mathbf{b}}_{\mathbf{a}})$, then $f(\cdot,a_2), f(a_1,\cdot)$ are absolutely continuous (in the one dimensional classical sense), and the induced rectangle map F_f is absolutely continuous in the sense of Carathéodory.*

Proof. It is readily seen that both $f(\cdot,a_2)$ and $f(a_1,\cdot)$ are absolutely continuous (see, e. g., [5, Corollary 3.4(a)]. Here, we limit ourselves to show that F_f is absolutely continuous.

Let $\epsilon>0$. Taking $c=V^R_\Phi(f,I^{\mathbf{b}}_{\mathbf{a}})$ in (4.2) we have that there exists $\delta>0$ such that

$$0<r<\delta \quad \text{implies} \quad \left| r\Phi^{-1}\left(\frac{V^R_{11,\Phi}(f,I^{\mathbf{b}}_{\mathbf{a}})}{r}\right)\right| < \epsilon.$$

Now, consider a finite family of rectangles
$\mathbb{I}_{ij} := [t_i,x_i]\times[s_j,y_j]\subset I^{\mathbf{b}}_{\mathbf{a}},\ \ i=1,\cdots,n,\ \ j=1,2,\cdots,m,$ such that

$$t_1\le x_1\le t_2\le x_2\le\cdots\le t_n\le x_n,\quad s_1\le y_1\le s_2\le y_2\le\cdots\le s_m\le y_m,$$

and

$$\sum_{i=1}^n\sum_{j=1}^m (x_i-t_i)(y_j-s_j)\le\delta.$$

Then, by (4.4) we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\|f(x_i,y_j)-f(x_i,s_j)-f(t_i,y_j)+f(t_i,s_j)\|$$
$$\leq \Phi^{-1}\left(\frac{V^R_{11,\Phi}(f,I_{\mathbf{a}}^{\mathbf{b}})}{\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-t_i)(y_j-s_j)}\right)\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-t_i)(y_j-s_j)<\epsilon.$$

The proof is complete. □

5. Riesz's type Lemma

Notice that if $f\in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ and X is a reflexive Banach space, the functions $f(\cdot,a_2)$ and $f(a_1,\cdot)$, are absolutely continuous and they admit strong derivatives a. e. on $[a_1,b_1]$, and $[a_2,b_2]$, respectively. As usual, denote these derivatives by $\frac{\partial f(x,a_2)}{\partial x}$ and $\frac{\partial f(a_1,y)}{\partial y}$, respectively. Then, by Theorem 3.3 in [5]:

$$V^R_{10,\Phi}(f,[a_1,b_1]) = \int_{a_1}^{b_1}\Phi\left(\left|\frac{\partial f(x,a_2)}{\partial x}\right|\right)dx$$
$$V^R_{01,\Phi}(f,[a_2,b_2]) = \int_{a_2}^{b_2}\Phi\left(\left|\frac{\partial f(a_1,y)}{\partial y}\right|\right)dy.$$

In our next result, we establish a dual (weak) counterpart, for maps of bounded bi-dimensional Φ-variation, of the classical Riesz Lemma ([14], see also [1]).

In the sequel X^* will denote the dual space of X. The integrals to be considered are understood in the Lebesgue sense in the real valued case and Bochner integrals in the vector valued case.

Theorem 5.1. *Let $(X,\|\cdot\|)$ be a reflexive Banach space and suppose that $\Phi\in\mathcal{N}_\infty$ and $f\in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$. Then, for all $0\neq x^*\in X^*$*

$$TV^R_\Phi(x^*\circ f,I_{\mathbf{a}}^{\mathbf{b}};\mathbb{R}) = \int_{a_1}^{b_1}\Phi\left(\left|\frac{\partial(x^*\circ f)(x,a_2)}{\partial x}\right|\right)dx+\int_{a_2}^{b_2}\Phi\left(\left|\frac{\partial(x^*\circ f)(a_1,y)}{\partial y}\right|\right)dy$$
$$+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right)dydx.$$

Proof. Let $x^*\in X^*$. From Proposition 4.6 and the continuity of x^*, it is readily seen that the function

$$x^*\circ f: I_{\mathbf{a}}^{\mathbf{b}}\to\mathbb{R}$$

is a real-valued absolutely continuous function in the sense of Carathéodory.

Consequently, by [16, Theorem 2.1 (3)], every such function $x^*\circ f$ satisfies the conditions:

(a) $(x^*\circ f)(x,\cdot)\in AC([a_2,b_2];\mathbb{R})$ for every $x\in[a_1,b_1]$, $(x^*\circ f)(\cdot,y)\in AC([a_1,b_1];\mathbb{R})$ for every $y\in[a_2,b_2]$;

(b) $\dfrac{\partial(x^*\circ f)(x,\cdot)}{\partial x}\in AC([a_2,b_2];\mathbb{R})$ for almost every $x\in[a_1,b_1]$;

(c) $\dfrac{\partial^2(x^*\circ f)}{\partial x\partial y}\in L_1(I_{\mathbf{a}}^{\mathbf{b}})$.

Thus, by [5, Theorem. 3.3] we have

$$V^R_{10,\Phi}((x^*\circ f),[a_1,b_1]) = \int_{a_1}^{b_1}\Phi\left(\left|\frac{\partial(x^*\circ f)(x,a_2)}{\partial x}\right|\right)dx \tag{5.1}$$

$$V^R_{01,\Phi}((x^*\circ f),[a_2,b_2]) = \int_{a_2}^{b_2}\Phi\left(\left|\frac{\partial(x^*\circ f)(a_1,y)}{\partial y}\right|\right)dy. \tag{5.2}$$

In particular, property (a), above, implies that $x^*\circ f$ is a measurable function on $I_{\mathbf{a}}^{\mathbf{b}}$ (c.f., [13]) and therefore if $0<h_1<b_1-a_1$ and $0<h_2<b_2-a_2$ the functions

$$(x,y)\to|\Delta(x^*\circ f,[x,y]\times[x+h_1,y+h_2])|$$

and

$$V(x,y):=V^R_{11,\Phi}((x^*\circ f),[a_1,x]\times[a_2,y])$$

are both Lebesgue integrable functions on each $[a_1,b_1-h_1]\times[a_2,b_2-h_2]$.

In the first place, suppose that $\xi=\{x_i\}_{i=0}^n$ and $\eta=\{y_j\}_{j=0}^m$ are partitions of $[a_1,b_1]$ and $[a_2,b_2]$ respectively. If $I_{ij}:=[x_{i-1},x_i]\times[y_{j-1},y_j]$ and $|I_{ij}|:=(x_i-x_{i-1})(y_j-y_{j-1})$, then by the integral representation of $(x^*\circ f)$ given by [16, Theorem 2.1 (2)] and using Jensen integral inequality, we obtain

$$\begin{aligned} V^R_{\Phi}((x^*\circ f),[a_1,b_1]\times[a_2,b_2],\xi,\eta) &\le \sum_{i=1}^n\sum_{j=1}^m\Phi\left(\frac{1}{|I_{ij}|}\int_{I_{ij}}\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|dydx\right)|I_{ij}| \\ &\le \sum_{i=1}^n\sum_{j=1}^m\int_{I_{ij}}\Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right)dydx \\ &= \int_{I_{\mathbf{a}}^{\mathbf{b}}}\Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right)dydx, \end{aligned}$$

thus

$$V(b_1,b_2)) \le \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right)dydx \tag{5.3}$$

Observe that (5.3) implies, in particular, that

$$\lim_{x\to a_1^+}\lim_{y\to a_2^+}V(x,y)=0. \tag{5.4}$$

To prove the other inequality notice by the definition and Theorem 2.7

$$\begin{aligned} &\Phi\left(\frac{|\Delta(x^*\circ f,[x,y]\times[x+h_1,y+h_2])|}{h_1h_2}\right) \qquad (5.5)\\ \le\ & \frac{1}{h_1h_2}[V^R_{11,\Phi}((x^*\circ f),[a_1,x+h_1]\times[a_2,y+h_2])-V^R_{11,\Phi}(x^*\circ f),[a_1,x]\times[a_2,y]) \\ -\ & V^R_{11,\Phi}(x^*\circ f),[a_1,x]\times[y,y+h_2])-V^R_{11,\Phi}(x^*\circ f),[x,x+h_1]\times[a_2,a_2+h_2])]. \end{aligned}$$

Integrating over $[a_1, b_1 - h_1] \times [a_2, b_2 - h_2]$ and changing variables appropriately we get, for h_1, h_2 small enough

$$E_1)\ \int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2}\{V(x+h_1,y+h_2)-V(x,y)\}\,dydx = \int_{a_1+h_1}^{b_1}\int_{a_2+h_2}^{b_2} V(x,y)\,dydx$$
$$-\int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2} V(x,y)\,dydx$$
$$E_2)\ \int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2} V(x,y+h_2)-V(x,y)\,dydx = \int_{a_1}^{b_1-h_1}\left(\int_{b_2-h_2}^{b_2} - \int_{a_2}^{a_2+h_2}\right)V(x,y)\,dydx$$
$$E_3)\ \int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2} V(x+h_1,y)-V(x,y)\,dydx = \left(\int_{b_1-h_1}^{b_1} - \int_{a_1}^{a_1+h_1}\right)\int_{a_2}^{b_2-h_2} V(x,y)\,dydx$$

Integrating now both sides of inequality (5.5), over $[a_1, b_1 - h_1] \times [a_2, b_2 - h_2]$, and taking into account E_1, E_2 and E_3, the additivity properties of the integral and the fact that V is non negative and monotone, we obtain, for h_1, h_2 small enough

$$
\begin{aligned}
& h_1h_2 \int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2} \Phi\left(\frac{|\Delta(x^*\circ f,[x,y]\times[x+h_1,y+h_2])|}{h_1h_2}\right) dydx \\
\leq\ & \int_{a_1}^{a_1+h_1}\int_{a_2}^{a_2+h_2} V(x,y)\,dydx + \int_{b_1}^{b_1-h_1}\int_{b_2}^{b_2-h_2} V(x,y)\,dydx \\
-\ & \int_{a_1}^{a_1+h_1}\int_{b_2-h_2}^{b_2} V(x,y)\,dydx - \int_{b_1-h_1}^{b_1}\int_{a_2}^{a_2+h_2} V(x,y)\,dydx \\
\leq\ & \int_{a_1}^{a_1+h_1}\int_{a_2}^{a_2+h_2} V(x,y)\,dydx + \int_{b_1}^{b_1-h_1}\int_{b_2}^{b_2-h_2} V(x,y)\,dydx \\
\leq\ & h_1h_2\,[V(a_1+h_1,a_2+h_2)+V(b_1,b_2)].
\end{aligned}
$$

Since $\dfrac{\partial(x^*\circ f)}{\partial x}(x,\cdot)$, $\dfrac{\partial(x^*\circ f)}{\partial y}(\cdot,y)$ and $\dfrac{\partial^2}{\partial x\partial y}(x^*\circ f)$ exist a.e., from inequality (5.5), Fatou's lemma and property (5.4) it follows that

$$
\begin{aligned}
& \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right) dydx \qquad (5.6)\\
\leq\ & \liminf_{h_1\to 0}\liminf_{h_2\to 0}\int_{a_1}^{b_1-h_1}\int_{a_2}^{b_2-h_2} \Phi\left(\frac{|\Delta(x^*\circ f,[x,y]\times[x+h_1,y+h_2])|}{h_1h_2}\right) dydx \\
\leq\ & V^R_{11,\Phi}((x^*\circ f),[a_1,b_1]\times[a_2,b_2]).
\end{aligned}
$$

From (5.6) and (5.3) we get

$$V^R_{11,\Phi}((x^*\circ f),[a_1,b_1]\times[a_2,b_2]) \quad = \quad \int_{I_{\mathbf{a}}^{\mathbf{b}}} \Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right) dydx \quad (5.7)$$

Finally, combining (5.1), (5.2) and (5.7), we obtain

$$\begin{aligned} TV_{\Phi}^{R}((x^*\circ f), I_{\mathbf{a}}^{\mathbf{b}}, \mathbb{R}) &= \int_{a_1}^{b_1} \Phi\left(\left|\frac{\partial(x^*\circ f)(x,a_2)}{\partial x}\right|\right) dx + \int_{a_2}^{b_2} \Phi\left(\left|\frac{\partial(x^*\circ f)(a_1,y)}{\partial y}\right|\right) dy \\ &\quad + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi\left(\left|\frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y}\right|\right) dydx. \end{aligned}$$

□

Lemma 5.2. *Let $\Phi \in \mathcal{N}_\infty$ and suppose that $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$, where X is a reflexive Banach space. If $f \in C^2(I_{\mathbf{a}}^{\mathbf{b}})$ (strongly) and $f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ then f admits integral representation*

$$f(x,y) = e + \int_{a_1}^{x} u(t)dt + \int_{a_2}^{y} v(s)ds + \int_{a_1}^{x}\int_{a_2}^{y} h(t,s)dsdt$$

where $e \in X$, $u \in L([a_1,b_1],X)$, $v \in L([a_2,b_2],X)$ y $h \in L([a_1,b_1]\times[a_2,b_2],X)$

Proof. Again, since $f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$, from Proposition 4.6 it follows that for all $0 \neq x^* \in X^*$, the functions

$$x^*\circ f : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$$

are real-valued absolutely continuous functions in the sense of Carathéodory. Thus, by [16, Theorem 2.1], every $x^*\circ f$ admits the integral representation

$$(x^*\circ f)(x,y) = r + \int_{a_1}^{x} u_{x^*}(t)dt + \int_{a_2}^{y} v_{x^*}(s)ds + \int_{a_1}^{x}\int_{a_2}^{y} h_{x^*}(t,s)dsdt$$

where $r \in \mathbb{R}$, $u_{x^*} \in L([a_1,b_1],\mathbb{R})$, $v_{x^*} \in L([a_2,b_2],\mathbb{R})$ y $h_{x^*} \in L([a_1,b_1]\times[a_2,b_2],\mathbb{R})$.

Now, since $f \in C^2(I_{\mathbf{a}}^{\mathbf{b}})$, it must be

$$\begin{aligned} \frac{\partial(x^*\circ f)(x,a_2)}{\partial x} = x^*\left(\frac{\partial f(x,a_2)}{\partial x}\right) &= u_{x^*}(x) \\ \frac{\partial(x^*\circ f)(a_1,y)}{\partial y} = x^*\left(\frac{\partial f(a_1,y)}{\partial y}\right) &= v_{x^*}(y) \\ \frac{\partial^2(x^*\circ f)(x,y)}{\partial x\partial y} = x^*\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right) &= h_{x^*}(x,y). \end{aligned}$$

Thus

$$\begin{aligned} x^*(f(x,y)) &= x^*(f(a_1,a_2)) + \int_{a_1}^{x} x^*\left(\frac{\partial f(t,a_2)}{\partial t}\right) dt + \int_{a_2}^{y} x^*\left(\frac{\partial f(a_1,s)}{\partial s}\right) ds \\ &\quad + \int_{a_1}^{x}\int_{a_2}^{y} x^*\left(\frac{\partial^2 f(t,s)}{\partial t\partial s}\right) dsdt \\ &= x^*\left(f(a_1,a_2) + \int_{a_1}^{x} \frac{\partial f(t,a_2)}{\partial t}dt + \int_{a_2}^{y} \frac{\partial f(a_1,s)}{\partial s}ds \right. \\ &\quad \left. \int_{a_1}^{x}\int_{a_2}^{y} \frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right). \end{aligned}$$

By the Hahn-Banach theorem, we conclude that

$$f(x,y) \quad = \quad f(a_1,a_2) + \int_{a_1}^{x} u(t)dt + \int_{a_2}^{y} v(s)ds + \int_{a_1}^{x}\int_{a_2}^{y} h(t,s)dsdt$$

where $u \in L([a_1,b_1],X)$, $v \in L([a_2,b_2],X)$ and $h \in L([a_1,b_1]\times[a_2,b_2],X)$. □

As an consequence of the last theorem, and of its proof, we obtain the following corollary

Corollary 5.3. *Let $\Phi \in \mathcal{N}_\infty$ and suppose that $f : I_{\mathbf{a}}^{\mathbf{b}} \to X$, where X is a reflexive Banach space. If $f \in C^2(I_{\mathbf{a}}^{\mathbf{b}})$ and $f \in RBV_\Phi(I_{\mathbf{a}}^{\mathbf{b}})$ then*

$$\begin{aligned} TV_\Phi^R(f, I_{\mathbf{a}}^{\mathbf{b}}, X) \quad &= \quad \int_{a_1}^{b_1} \Phi\left(\left\|\frac{\partial f(x,a_2)}{\partial x}\right\|\right)dx + \int_{a_2}^{b_2} \Phi\left(\left\|\frac{\partial f(a_1,y)}{\partial y}\right\|\right)dy \\ &+ \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi\left(\left\|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right\|\right)dydx. \end{aligned}$$

Proof. By [5, Theorem 3.3])

$$V_{10,\Phi}^R(f,[a_1,b_1]) \quad = \quad \int_{a_1}^{b_1} \Phi\left(\left\|\frac{\partial f(x,a_2)}{\partial x}\right\|\right)dx$$

and

$$V_{01,\Phi}^R(f,[a_2,b_2]) \quad = \quad \int_{a_2}^{b_2} \Phi\left(\left\|\frac{\partial f(a_1,y)}{\partial y}\right\|\right)dy.$$

To show that

$$V_{11,\Phi}^R(f,[a_1,b_1]\times[a_2,b_2]) \quad \geq \quad \int_{I_{\mathbf{a}}^{\mathbf{b}}} \Phi\left(\left\|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right\|\right)dydx$$

we proceed as in the proof of theorem 5.1 by integrating both sides of a strongly version of inequality (5.5) (replacing $x^* \circ f$ by f). Likewise, the reverse inequality follows from Lemma 5.2 after applying Jensen inequality, just as we did in the proof of Theorem 5.1. □

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