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KWONG MATRICES AND OPERATOR MONOTONE FUNCTIONS ON $(0,1)$

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. In this paper we study positive operator monotone functions on $(0,1)$ which have some differences from those on $(0,\infty)$: we show that for a concave operator monotone function f on $(0,1)$, the Kwong matrices $K_f(s_1,\ldots,s_n)$ are positive semidefinite for all n and $s_i \in (0,1)$, and $f(s^p)^{1/p}$ for $0 < p \leqq 1$ and $s/f(s)$ are operator monotone. We also give a sufficient condition for the Kwong matrices to be positive semidefinite.

1. INTRODUCTION

Let f be a real-valued C^1 function on an interval (a,b). For n distinct real numbers $t_1,\ldots,t_n \in (a,b)$ a *Loewner* (or *Pick*) *matrix* $L_f(t_1,\ldots,t_n)$ associated with f is the $n \times n$ matrix defined as

$$L_f(t_1,\ldots,t_n) = \left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right],$$

where the diagonal entries are understood as the first derivatives $f'(t_i)$. In the case where $(a,b) \subseteqq (0,\infty)$, a *Kwong* (or an *anti-Loewner*) *matrix* $K_f(t_1,\ldots,t_n)$ associated with f is the $n \times n$ matrix defined by

$$K_f(t_1,\ldots,t_n) = \left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right].$$

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A real-valued function f on an interval (a, b) is said to be *matrix monotone of order* n if $A \geqq B$ implies $f(A) \geqq f(B)$ for $n \times n$ Hermitian matrices A and B with eigenvalues in (a, b). If f is matrix monotone of every order then f is said to be *operator monotone.* In operator/matrix theory of great importance are operator monotone functions, which also play an essential role in related fields: for instance, in quantum information theory.

In this paper we study positive operator monotone functions on $(0, 1)$ which give some differences from those on $(0, \infty)$: we show that the similar results of Loewner/Kwong matrices do not hold in this case, but for a concave operator monotone function f on $(0, 1)$, the Kwong matrices $K_f(s_1, \ldots, s_n)$ are positive semidefinite for all n and $s_i \in (0, 1)$, and $f(s^p)^{1/p}$ for $0 < p \leqq 1$ and $s/f(s)$ are operator monotone. We also give a sufficient condition of f on $(0, 1)$ for the Kwong matrices associated with f to be positive semidefinite.

These observations come from our preceding studies on Loewner and Kwong matrices [4, 5, 9, 10, 14]. In the remainder of this section, we review some of them: on Loewner matrices, Bhatia and the second-named author of this paper [4] present a characterization for operator convexity of a positive function f on $(0, \infty)$ in terms of the conditional negative definiteness of the Loewner matrices $L_f(t_1, \ldots, t_n)$. Moreover, Hiai and Sano [9] give this generalization by considering matrix monotonicity/convexity. On the other hand, Kwong [11] shows that if f is a non-negative operator monotone function on $(0, \infty)$ then the Kwong matrices $K_f(t_1, \ldots, t_n)$ are positive semidefinite for all n and $t_i \in (0, \infty)$. Audenaert [2] gives a characterization of f with $K_f(t_1, \ldots, t_n)$ positive semidefinite for all n and $t_i \in (0, \infty)$. By using this characterization, Hidaka and Sano [10] study the conditional negative definiteness of the Kwong matrices, which is given by Bhatia and Sano for operator convex functions and more. For a positive integer m and a positive operator monotone function f on $(0, \infty)$, Tachibana and Sano [14] show the positive semidefiniteness of the matrices

$$\left[\frac{f(t_i)^m + f(t_j)^m}{t_i^m + t_j^m}\right] \quad \text{and} \quad \left[\frac{f(t_i)^m - f(t_j)^m}{t_i^m - t_j^m}\right].$$

2. Operator monotone functions on $(0, 1)$

In this section, we consider positive operator monotone functions on $(0, 1)$. First we recall basic facts on operator monotone functions; we refer the reader to [3, 6]: it is known by Löwner [12] that f is matrix monotone on (a, b) of order n if and only if the $n \times n$ Loewner matrices $L_f(t_1, \ldots, t_n)$ are positive semidefinite for all $t_1, \ldots, t_n \in (a, b)$; therefore, f is operator monotone on (a, b) if and only if the Loewner matrices $L_f(t_1, \ldots, t_n)$ are positive semidefinite for all n and $t_1, \ldots, t_n \in (a, b)$. Another characterization by Löwner is that f has an analytic continuation to the upper half-plane which maps the upper half-plane into itself.

The following is easy to see by direct computations but useful in our argument.

Lemma 2.1.

$$
\begin{aligned}
(1)\quad K_f(t_1,\ldots,t_n)+L_f(t_1,\ldots,t_n) &= 2\left[\frac{t_if(t_i)-t_jf(t_j)}{t_i^2-t_j^2}\right]\\
&= 2\,C\circ L_{tf(t)}(t_1,\ldots,t_n)\\
&= 2\,L_{\sqrt{t}f(\sqrt{t})}(s_1,\ldots,s_n), \qquad (2.1)
\end{aligned}
$$

where C is the Cauchy matrix $\left[\dfrac{1}{t_i+t_j}\right]$, $\circ$ stands for the Hadamard or Schur product and $s_i=t_i^2$.

$$
\begin{aligned}
(2)\quad K_f(t_1,\ldots,t_n)-L_f(t_1,\ldots,t_n) &= 2\left[\frac{t_if(t_j)-t_jf(t_i)}{t_i^2-t_j^2}\right]\\
&= 2\,D\left[\frac{t_i/f(t_i)-t_j/f(t_j)}{t_i^2-t_j^2}\right]D\\
&= 2\,C\circ\Big(DL_{t/f(t)}(t_1,\ldots,t_n)D\Big) \qquad (2.2)\\
&= 2\,DL_{\sqrt{t}/f(\sqrt{t})}(s_1,\ldots,s_n)D, \qquad (2.3)
\end{aligned}
$$

where C and s_i are the same as in (1) *and D is the diagonal matrix given as $D=diag\,(f(t_1),\ldots,f(t_n))$.*

For our study we prepare the representation of positive operator monotone functions f on $(0,1)$. We follow the observation as in [8, p.183]. Let $\psi(t)$ be the function from $(-1,1)$ onto $(0,1)$ defined by $\psi(t)=(t+1)/2=s$. Since the function $g(t):=f(\psi(t))$ is operator monotone on $(-1,1)$, by [7, Theorem 4.4] $g(t)$ is of the form

$$g(t)=g(0)+g'(0)\int_{[-1,1]}\frac{t}{1-\lambda t}\,d\mu(\lambda),\quad t\in(-1,1)$$

for a probability measure μ on $[-1,1]$. Since $g(-1):=\lim_{t\downarrow -1}g(t)=\lim_{s\downarrow 0}f(s)\geqq 0$ by assumption, it follows that

$$\int_{[-1,1]}\frac{1}{1+\lambda}\,d\mu(\lambda)<\infty.$$

In particular, $\mu(\{-1\})=0$ and

$$g(t)-g(-1)=g'(0)\int_{(-1,1]}\frac{1+t}{(1-\lambda t)(1+\lambda)}\,d\mu(\lambda).$$

Hence, putting $t=\psi^{-1}(s)$ and $\lambda=\psi^{-1}(\zeta)$, we have

$$f(s)-f(0)=\int_{(0,1]}\frac{s}{s+\zeta-2s\zeta}\,dm_0(\zeta),$$

where m_0 is the measure on $(0,1]$ defined as $m_0=\tilde{\mu}\circ\psi^{-1}$ where $d\tilde{\mu}(\lambda)=g'(0)/(1+\lambda)d\mu(\lambda)$, and if we define the measure m on $[0,1]$ as $m=f(0)\delta_0+m_0$ then we have:

Theorem 2.2. *A positive operator monotone function $f(s)$ on $(0,1)$ is of the form*

$$f(s) = \int_{[0,1]} \frac{s}{s+\zeta-2s\zeta}\, dm(\zeta),$$

where m is a positive measure on $[0,1]$.

For $0 \leqq \zeta \leqq 1$ we consider the positive operator monotone function on $(0,1)$

$$f_\zeta(s) := \frac{s}{(1-2\zeta)s+\zeta} = \frac{s}{s+\zeta-2s\zeta}. \tag{2.4}$$

Theorem 2.3. *Let $f_\zeta(s)$ be the function in (2.4). Then $s/f_\zeta(s)$ is operator monotone if and only if $\zeta \leqq 1/2$.*

Proof. It suffices to determine when $-f_\zeta(s)/s$ is operator monotone, which is equivalent to that $1-2\zeta \geqq 0$. □

Corollary 2.4. *Let $f(s)$ be a positive operator monotone function on $(0,1)$ which is of the form*

$$f(s) = \int_{[0,1/2]} f_\zeta(s)\, dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1-2\zeta)s+\zeta}\, dm(\zeta), \tag{2.5}$$

where m is a positive measure on $[0,1/2]$. Then $s/f(s)$ is operator monotone on $(0,1)$.

The following corresponds to Kwong [11].

Theorem 2.5. *If $f(s)$ is the operator monotone function in (2.5), then all Kwong matrices associated with f are positive semidefinite.*

Proof. By assumption and Corollary 2.4, Loewner matrices associated with $f(s)$ and $s/f(s)$ are positive semidefinite; therefore, (2.2) and Schur's Theorem yield the conclusion. Note that when $s/f(s)$ is operator monotone so is $\sqrt{s}/f(\sqrt{s})$; hence, (2.3) also implies the assertion. □

We remark that similar argument for operator monotone functions on $(0,\infty)$ is given by Nakamura [13]. For the functions $f_\zeta(s)$, we could say more:

Theorem 2.6. *Let $f_\zeta(s)$ be the function in (2.4). Then all Kwong matrices associated with f_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.*

For the proof, we recall the following characterization:

Proposition 2.7. ([10, Proposition 3.1]) *For a non-negative function $f(s)$ on $(0,1)$, $K_f(s_1,s_2)$ are positive semidefinite for all $s_1,s_2 \in (0,1)$ if and only if $f(s)/s$ is decreasing and $sf(s)$ is increasing.*

Proof of Theorem 2.6. The if part follows from Theorem 2.5. On the other hand, for the only if part, Proposition 2.7 implies that $f_\zeta(s)/s = 1/\{(1-2\zeta)s+\zeta\}$ on $(0,1)$ should be decreasing, hence $\zeta \leqq 1/2$; therefore the proof is complete. □

The following is a counterpart to Audenaert [2].

Theorem 2.8. *Let $f(s)$ be a positive function on $(0,1)$. If $\sqrt{s}f(\sqrt{s})$ or $\sqrt{s}/f(\sqrt{s})$ is the operator monotone function in (2.5), then all Kwong matrices associated with f are positive semidefinite.*

Proof. Since $\dfrac{s}{\sqrt{s}f(\sqrt{s})} = \dfrac{\sqrt{s}}{f(\sqrt{s})}$ or $\dfrac{s}{\sqrt{s}/f(\sqrt{s})} = \sqrt{s}f(\sqrt{s})$, the assumption and Corollary 2.4 yield the operator monotonicity of both functions. Hence, by (2.1) and (2.3), $K_f(s_1,\dots,s_n) \pm L_f(s_1,\dots,s_n)$ are positive semidefinite for any n and $s_i \in (0,1)$. By adding them, $K_f(s_1,\dots,s_n)$ are positive semidefinite for any n and $s_i \in (0,1)$. Therefore we get the conclusion. ☐

For $0 \leqq \zeta \leqq 1$ we consider the function on $(0,1)$

$$g_\zeta(s) := \frac{f_\zeta(s^2)}{s} = \frac{s}{(1-2\zeta)s^2+\zeta}. \tag{2.6}$$

Theorem 2.9. *Let $g_\zeta(s)$ be the function in (2.6). Then $g_\zeta(s)$ is operator monotone if and only if $1/2 \leqq \zeta$, and all Kwong matrices associated with g_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.*

Proof. We first show the second statement: note that $\sqrt{s}g_\zeta(\sqrt{s}) = f_\zeta(s)$ and $\dfrac{\sqrt{s}}{g_\zeta(\sqrt{s})} = \dfrac{s}{f_\zeta(s)}$ are operator monotone when $\zeta \leqq 1/2$ by Theorem 2.3. Hence, by Theorem 2.8, the if part is proved. The only if part follows from Proposition 2.7 since $g_\zeta(s)/s$ should be decreasing.

For $\alpha := 1-2\zeta$, by the identity

$$\begin{aligned}\frac{1}{a-b}\left(\frac{a}{\alpha a^2+\zeta} - \frac{b}{\alpha b^2+\zeta}\right) &= \frac{1}{a-b}\,\frac{(a-b)\zeta - \alpha ab(a-b)}{(\alpha a^2+\zeta)(\alpha b^2+\zeta)} \\ &= \frac{\zeta - \alpha ab}{(\alpha a^2+\zeta)(\alpha b^2+\zeta)},\end{aligned}$$

we have

$$L_{g_\zeta}(s_1,\dots,s_n) = \zeta D_1ED_1 + (-\alpha)D_2ED_2 \tag{2.7}$$

where D_1 and D_2 are the diagonal matrices defined as

$$D_1 = \operatorname{diag}\left(\frac{1}{\alpha s_1^2+\zeta},\dots,\frac{1}{\alpha s_n^2+\zeta}\right), \quad D_2 = \operatorname{diag}\left(\frac{s_1}{\alpha s_1^2+\zeta},\dots,\frac{s_n}{\alpha s_n^2+\zeta}\right),$$

and E is the matrix with all its entries equal to 1. If $\zeta \geqq 1/2$ or $\alpha \leqq 0$, then by (2.7) $L_{g_\zeta}(s_1,\dots,s_n)$ is positive semidefinite since E is positive semidefinite; that is, $g_\zeta(s)$ is operator monotone. We also note that

$$g_\zeta(s) = \frac{1}{2\sqrt{\zeta}}\Big(\frac{s}{\sqrt{\zeta}-\sqrt{-\alpha}s} + \frac{s}{\sqrt{\zeta}+\sqrt{-\alpha}s}\Big),$$

which is the sum of operator monotone functions when $\zeta \geqq 1/2$. By (2.7),

$$D_1^{-1}L_{g_\zeta}(s_1,s_2)D_1^{-1} = \left[\zeta - \alpha s_is_j\right],$$

and

$$\det D_1^{-1}L_{g_\zeta}(s_1,s_2)D_1^{-1} = -\alpha\zeta(s_1-s_2)^2 \leqq 0,$$

if $\zeta < 1/2$. Hence in this case $L_{g_\zeta}(s_1, s_2)$ is not positive semidefinite; therefore $g_\zeta(s)$ is not operator monotone, and the proof is complete. □

Example 2.10. Let $h(s) := \tan(\pi/2)s$ on $(0, 1)$, which is a well-known operator monotone function. Since $h(s)/s$ is increasing, it follows from Proposition 2.7 that Kwong matrices associated with h is not positive semidefinite. Similarly Kwong matrices associated with $h(s^2)/s$ is not positive semidefinite.

Furthermore, we see:

Theorem 2.11. *If $f(s)$ is the operator monotone function in (2.5), then $f(s^p)^{1/p}$ is operator monotone on $(0, 1)$ for $0 < p \leqq 1$.*

Proof. We give a proof as in [1, p. 216]; suppose that $f(s)$ is of the form

$$f(s) = \int_{[0,1/2]} f_\zeta(s)\, dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1-2\zeta)s + \zeta}\, dm(\zeta).$$

Then f has an analytic continuation $f(z)$ to the upper half-plane which maps the upper half-plane into itself. Since $\operatorname{Arg} f_\zeta(z) \leqq \operatorname{Arg} z, \operatorname{Arg} f(z) \leqq \operatorname{Arg} z$. Hence, the analytic function $f(z^p)^{1/p}$ is well-defined and maps the upper half-plane into itself; therefore $f(s^p)^{1/p}$ is operator monotone. □

In particular Theorem 2.11 implies:

Corollary 2.12. *Let $f(s)$ be the operator monotone function in (2.5). Then for any positive integer m,*

$$\left[\frac{f(s_i)^m - f(s_j)^m}{s_i^m - s_j^m}\right]$$

are positive semidefinite for all n and $s_1, \ldots, s_n$ in $(0, 1)$.

Note that under the same assumption we can prove by the similar argument as in [14] that for any positive integer m,

$$\left[\frac{f(s_i)^m + f(s_j)^m}{s_i^m + s_j^m}\right]$$

are positive semidefinite for all n and $s_1, \ldots, s_n$ in $(0, 1)$.

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References

1. T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products,* Linear Algebra Appl. **26** (1979), 203–241.
2. K.M.R. Audenaert, *A characterisation of anti-Löwner functions,* Proc. Amer. Math. Soc. **139** (2011), 4217–4223.

3. R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, 169, Springer, 1996.
4. R. Bhatia and T. Sano, *Loewner matrices and operator convexity,* Math. Ann. **344** (2009), 703–716.
5. R. Bhatia and T. Sano, *Positivity and conditional positivity of Loewner matrices,* Positivity **14** (2010), 421–430.
6. W.F. Donoghue, *Monotone Matrix Functions and Analytic Continuation,* Die Grundlehren der mathematischen Wissenschaften, Band 207, Springer, 1974.
7. F. Hansen and G.K. Pedersen, *Jensen's inequality for operators and Löwner's theory,* Math. Ann. **258** (1982), 229–241.
8. F. Hiai, *Matrix Analysis: Matrix monotone functions, matrix means, and majorization,* Interdiscip. Inform. Sci. **16** (2010), 139–248.
9. F. Hiai and T. Sano, *Loewner matrices of matrix convex and monotone functions,* J. Math. Soc. Japan **64** (2012), 343–364.
10. C. Hidaka and T. Sano, *Conditional negativity of anti-Loewner matrices,* Linear Multilinear Algebra **60** (2012), 1265–1270.
11. M.K. Kwong, *Some results on matrix monotone functions,* Linear Algebra Appl. **118** (1989), 129–153.
12. K. Löwner, *Über monotone Matrixfunctionen,* Math. Z. **38** (1934), 177–216.
13. Y. Nakamura, *Classes of operator monotone functions and Stieltjes functions,* Oper. Theory Adv. Appl. **41**, Birkhäuser (1989), 395–404.
14. T. Sano and S. Tachibana, *On Loewner and Kwong matrices,* Sci. Math. Jpn. **75** (2012), 335–338.

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