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A note on embedding and generating dual polar spaces

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Abstract. Generating sets of cardinality 2^n are constructed for the unitary dual polar space $\mathrm{DU}(2n+1, q^2)$ and the elliptic orthogonal dual polar space $\mathrm{DO}^-(2n+2, q)$. It is shown that the elliptic dual polar space $\mathrm{DO}^-(2n+2, q)$ has an embedding into $\mathbb{PG}(2^n-1, q^2)$ which is necessarily relatively universal. By a theorem of Kasikova and Shult ([9]) we conclude that this embedding is absolutely universal. A survey is included summarizing current knowledge of the generating rank and universal embedding spaces of dual polar spaces.

Key words. Geometry, near $2n$-gon, dual polar space, elliptic quadric, unitary space, generating set for a geometry, embedding for a geometry, relatively universal embedding, absolutely universal embedding.

1 Introduction, definitions and notation

This paper is a contribution to the program of determining the universal projective embeddings and the generating ranks of the dual polar spaces, an undertaking which is now nearly complete with the recent affirmative answer to the Brouwer conjecture for the dual polar spaces of symplectic type over $\mathbb{F}_2$ by Paul Li ([10]). Before proceeding to our results we begin with some basic definitions and notation.

1.1 Graphs, incidence systems, generation and embeddings. By a *graph* we mean a set $\mathscr{P}$ whose elements are called *vertices* together with a symmetric, antireflexive relation $\sim$ referred to as adjacency. The pairs $\{p, q\}$ from $\mathscr{P}$ with $p \sim q$ are called *edges*. A path between two points $p, q \in \mathscr{P}$ is a sequence $p = p_0, p_1, \ldots, p_d = q$ where $p_i \sim p_{i+1}$ for each $i = 0, 1, \ldots, d-1$. The length of such a path is the number d of adjacencies. The distance $d(p, q)$ between two points $p, q \in \mathscr{P}$ is the length of a minimal path joining them (which we call a *geodesic*), if a path exists, otherwise $d(p, q) = \infty$. The *diameter* of $(\mathscr{P}, \sim)$ is $\sup\{d(p, q) \mid p, q \in \mathscr{P}\}$.

An *incidence system* is a triple $(\mathscr{P}, \mathscr{L}, I)$ consisting of a set $\mathscr{P}$ whose elements are

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called *points*, a set $\mathscr{L}$ whose members are called *lines*, and a symmetric relation $I \subset (\mathscr{P} \times \mathscr{L}) \cup (\mathscr{L} \times \mathscr{P})$. If $p \in \mathscr{P}$, $L \in \mathscr{L}$ and $(p, L) \in I$ then we say p is *incident with* or *lies on* L. $(\mathscr{P}, \mathscr{L}, I)$ is said to be a *linear incidence system* or a *point-line geometry* if two points are incident with at most one line. In this case we may identify each line with its *shadow*, namely the set of points with which it is incident, and replace I with the symmetrization of the relation $\in$ and then we will write $(\mathscr{P}, \mathscr{L})$ in place of $(\mathscr{P}, \mathscr{L}, I)$. The *collinearity graph* of a linear incidence system $(\mathscr{P}, \mathscr{L})$ is the graph $(\mathscr{P}, \sim)$ where $p \sim q$ for $p, q \in \mathscr{P}$ if and only if p and q are collinear. For a point p we will denote the union of all lines on p by $p^{\perp}$. Thus, $p^{\perp}$ contains p and all points which are collinear with p. Γ is said to be *nondegenerate* if for no point p it is the case that $p^{\perp} = \mathscr{P}$.

By a *subspace* of a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ we mean a subset X of the point set $\mathscr{P}$ with the property that if a line meets X in at least two points then the line is entirely contained in X. Clearly the intersection of subspaces is a subspace. Consequently, for an arbitrary subset X of $\mathscr{P}$ we can define the *subspace generated* by X to be the intersection of all subspaces containing X and will denote it by $\langle X \rangle_{\Gamma}$. This is the unique minimal element (with respect to the ordering under inclusion) among the collection of subspaces which contain X. We will say that a subset X *generates* $\mathscr{P}$ if $\langle X \rangle_{\Gamma} = \mathscr{P}$. We define the generating rank of $\Gamma = (\mathscr{P}, \mathscr{L})$, $\mathrm{gr}(\Gamma)$, to be $\min\{|X| : \langle X \rangle_{\Gamma} = \mathscr{P}\}$. By a *singular subspace* we shall mean a subspace which is also a clique in the collinearity graph.

Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point-line geometry. By a *projective embedding* of Γ we mean an injective mapping $e : \mathscr{P} \to \mathbb{PG}(V)$, V a vector space over some division ring, such that

(i) the subspace of V spanned by $e(\mathscr{P})$ is all of V; and

(ii) for $L \in \mathscr{L}, e(L)$ is a full line of $\mathbb{PG}(V)$.

We say that Γ is *embeddable* if some projective embedding of Γ exists. Assume that $e : \mathscr{P} \to \mathbb{PG}(V)$ and $e' : \mathscr{P} \to \mathbb{PG}(V')$ are embeddings of Γ. A *morphism* from e to e' is a mapping $\Psi : \mathbb{PG}(V) \to \mathbb{PG}(V')$ induced by a semi-linear mapping $f : V \to V'$ such that $\Psi \circ e = e'$. Let $e : \mathscr{P} \to \mathbb{PG}(V)$ be an embedding of Γ. An embedding $\hat{e} : \mathscr{P} \to \mathbb{PG}(\hat{V})$ is said to be *universal relative to* e if there is a morphism $\hat{\Psi} : \hat{e} \to e$ such that for any other morphism Ψ from an embedding e' of Γ to e, $\hat{\Psi}$ factors through Ψ, that is, there is a morphism $\gamma : \hat{e} \to e'$ such that $\gamma \circ \hat{\Psi} = \Psi$. An embedding $e : \mathscr{P} \to \mathbb{PG}(V)$ is *relatively universal* if it is universal relative to itself. Finally, an embedding e of Γ is *absolutely universal* if it is universal relative to every embedding of Γ. This means that e is a universal source of the category of Γ-embeddings.

It is an immediate consequence of these definitions that if $e : \Gamma \to \mathbb{PG}(V)$ is an embedding then $\dim V \leqslant \mathrm{gr}(\Gamma)$ and, if $\dim V = \mathrm{gr}(\Gamma)$ then e is relatively universal. In this case a generating set X with $|X| = \mathrm{gr}(\Gamma)$ is called a *basis* (cf. [7]).

1.2 Polar spaces and dual polar spaces. For the purposes of this paper a *polar space* is a point-line geometry $(\mathscr{P}^*, \mathscr{L}^*)$ which satisfies

(**P**) For any point-line pair $(p, L) \in \mathscr{P}^* \times \mathscr{L}^*, p$ is collinear with one or all the points of L and

(**F**) There is an integer n, called the *rank* of $(\mathscr{P}^*, \mathscr{L}^*)$ such that any sequence $X_0 \subset X_1 \subset \cdots \subset X_m$ of distinct singular subspaces satisfies $m \leqslant n$.

A polar space of rank two is called a *generalized quadrangle*. If a generalized quadrangle $(\mathscr{P}^*, \mathscr{L}^*)$ is finite then it is said to be *regular with parameters* (s, t) if every line contains $s+1$ points and every point lies on $t+1$ lines.

Recall, if $(\mathscr{P}^*, \mathscr{L}^*)$ is a nondegenerate polar space then the *associated dual polar space* has as its points, $\mathscr{P}$, the collection of maximal singular subspaces of the polar space and as lines, $\mathscr{L}$, the shadows of the next to maximal singular subspaces.

The polar and dual polar spaces of type $\mathrm{O}^-(2n+2, q)$. An elliptic quadric of rank n (over a finite field $\mathbb{F}_q$) may be defined as follows: Let $aX^2 + bX + c$ be an irreducible quadratic over $\mathbb{F}_q$ (such quadratics always exist by an easy counting argument). Let V be a $(2n+2)$-dimensional vector space with basis $v_1, w_1, v_2, w_2, \ldots, v_n, w_n, v_{n+1}, w_{n+1}$ and define the mapping $Q : V \to \mathbb{F}$ by

$$Q\left(\sum_{i=1}^{n+1} X_i v_i + Y_i w_i\right) = \sum_{i=1}^{n} X_i Y_i + (aX_{n+1}^2 + bX_{n+1}Y_{n+1} + cY_{n+1}^2).$$

This is an elliptic quadratic form on V. Up to isometry there is only one such space. A subspace U is said to be *singular* if $Q(U) = \{0\}$. For the form defined above there exist subspaces U of dimension n such that $Q(U) = \{0\}$, for example $\langle v_1, v_2, \ldots, v_n \rangle$, but there do not exist such subspaces of dimension $n+1$. The isometry group of (V, Q),

$$G(V, Q) = \{T : V \to V \mid Q(Tv) = Q(v),\ \forall v \in V\}$$

is transitive on such subspaces. We say the *singular rank* of (V, Q) is n. We refer to the singular one spaces as *singular points* and the singular two spaces as *singular lines*. Let $\mathscr{P}^*$ be the collection of all singular points and $\mathscr{L}^*$ the collection of all singular lines. Then $(\mathscr{P}^*, \mathscr{L}^*)$ is the elliptic polar space of singular rank n which we denote by $\mathrm{O}^-(2n+2, q)$. The associated dual polar space will be denoted by $\mathrm{DO}^-(2n+2, q)$.

The polar space and dual polar spaces of type $\mathrm{U}(k, q^2)$. Let V be a vector space of dimension $k \geqslant 4$ over $\mathbb{F}_{q^2}$ with basis $v_1, v_2, \ldots, v_k$ and let $\tau : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ be the automorphism given by $\tau(x) = x^q$. We will usually denote images under this map by the "bar" notation: $\tau(x) = x^q = \bar{x}$. Now let $h : V \times V \to \mathbb{F}_q$ be the non-degenerate hermitian form given by

$$h\left(\sum_{i=1}^{k} X_i v_i, \sum_{i=1}^{k} Y_i v_i\right) = \sum_{i=1}^{k} X_i \overline{Y}_i.$$

A subspace U is *isotropic* if $h(U, U) = \{0\}$. The maximal dimension of an isotropic subspace is $\left[\frac{k}{2}\right]$ and all such subspaces are conjugate under the action of $G(V, h) = \{T : V \to V \mid h(Tv, Tw) = h(v, w),\ \forall v, w \in V\}$. Let $\mathscr{P}^*$ be the collection of isotropic one spaces and $\mathscr{L}^*$ the collection of isotropic two spaces. Then $(\mathscr{P}^*, \mathscr{L}^*)$ is the unitary polar space $\mathrm{U}(k, q^2)$. The associated dual polar space will be denoted by $\mathrm{DU}(k, q^2)$.

In section two of this paper we prove that the elliptic dual polar space $DO^-(2n+2,q)$ has an absolutely universal embedding of dimension 2^n. In the course of proving this we will show that this geometry can be generated by 2^n points. In light of this, the geometry $DO^-(2n+2,q)$ has a basis in the sense defined above. In section three we demonstrate that the unitary dual polar spaces in odd dimension, $DU(2n+1,q^2)$, are not embeddable in the sense defined above. Our main result in this section is that this geometry can be generated by 2^n points. We think that this is best possible but at this time are unable to prove this assertion. In section four we will show that any subgraph of the collinearity graph of the geometries $DU(2n+1,q^2)$ or $DO^-(2n+2,q)$ which is isomorphic to the n-hypercube generates the geometry and, moreover, that the respective automorphism groups of the geometries are transitive on such subgraphs. In section five we conclude with a survey of our current knowledge on absolutely universal embeddings and generating sets for dual polar spaces.

2 Generation and embedding of the elliptic dual polar spaces

In this section we will show that the elliptic dual polar space $DO^-(2n+2,q)$ can be generated by 2^n points. We then demonstrate that there exists an embedding into $\mathbb{PG}(X)$ where X is a space of dimension 2^n and prove that this embedding is absolutely universal.

2.1 A Generating Set for $DO^-(2n+2,q)$. Before proceeding to the specific result for $DO^-(2n+2,q)$ we prove a general lemma about generation of a generalized quadrangle with parameters (s,t) when $s>t$.

(2.1) Lemma. *Let $\Gamma=(\mathscr{P},\mathscr{L})$ be a generalized quadrangle with parameters (s,t) where $s>t$. Then Γ is generated by the four points of any circuit.*

Proof. Let (a,b,c,d,a) be a 4-circuit so that $M_1=ab$ and $M_2=cd$ are opposite lines. Let $\mathscr{X}=\{L_1,L_2,\dots,L_s,L_{s+1}\}$ be the collection of all lines joining a point of M_1 to a point of M_2. Then $\bigcup_{i=1}^{s+1}L_i\subset\langle a,b,c,d\rangle_\Gamma$. Choose an arbitrary point x not in $\bigcup L_i$. Then for each L_i, there is a unique line on x meeting L_i. This produces a mapping $f:\mathscr{X}\to\mathscr{L}_x$, the collection of lines on x. Since $s+1>t+1$, by the pigeonhole principle this map cannot be injective. Assume that $f(L_i)=f(L_j)$ and set $y=L_i\cap x^\perp$, $z=L_j\cap x^\perp$. Then $xy=xz=yz$. In particular, $x\in yz$. As $y,z\in\langle a,b,c,d\rangle$ it follows that $x\in\langle a,b,c,d\rangle$.

Now let V be a $(2n+2)$-dimensional vector space over $\mathbb{F}_q$ and Q an elliptic form of rank n on V. Let $\Gamma^*=(\mathscr{P}^*,\mathscr{L}^*)$ be the elliptic polar space of singular points and singular lines in V and let $\Gamma=(\mathscr{P},\mathscr{L})$ be the associated dual polar space. Recall that we may identify $\mathscr{P}$ with the maximal singular subspaces of V. For vectors $v,w\in V$ define $(v,w)=Q(v+w)-Q(v)-Q(w)$, the symmetric bilinear form associated with Q. For a vector w we will let $w^{\perp_Q}=\{u\in V:(w,u)=0\}$ and for a subspace W,

$$W^{\perp_Q} = \bigcap_{w \in W} w^{\perp_Q}.$$

Now let W be a (totally) singular subspace of V. Set $U(W) = \{p \in \mathscr{P} : W \subset p\}$. This is a convex subspace of Γ. Now the quotient space $\overline{W} = W^{\perp_Q}/W$ can be made into an elliptic space of dimension $(2n+2-2d)$ where $\dim W = d$ by defining $\overline{Q}(u+W) = Q(u)$ for a vector $u \in W^{\perp_Q}$. Moreover there is a one-to-one correspondence between the elements of $U(W)$ and the maximal singular subspaces of $\overline{W}$. This correspondence is an isomorphism of geometries and in this way we see that $U(W)$ is isomorphic to $\mathrm{DO}^-(2n+2-2d, q)$ where $\dim W = d$. In particular, for a singular point v of V, $U(v)$ is isomorphic to $\mathrm{DO}^-(2n, q)$. Our main result of this subsection, that $\mathrm{DO}^-(2n+2, q)$ can be generated by 2^n points, will be an immediate consequence of the following lemma:

(2.2) Lemma. *Let $x, y \in \mathscr{P}^*$, that is, singular points of V, and assume that $(x, y) \neq 0$. Then $\langle U(x), U(y) \rangle_\Gamma = \mathscr{P}$.*

Proof. We proceed by induction on $n \geqslant 2$. Suppose first that $n = 2$. In this case $\{U(x), U(y)\}$ is a pair of opposite lines in the generalized quadrangle $\mathrm{DO}^-(6, q) \cong \mathrm{U}(4, q^2)$ which has parameters (q^2, q) and therefore by (2.1) it follows that $\langle U(x), U(y) \rangle_\Gamma = \mathscr{P}$.

Now assume that the result holds for $n = k \geqslant 2$ and we must show that it holds for $n = k+1$. So assume that (V, Q) is an elliptic orthogonal space of dimension $2(k+1)+2 = 2k+4$ with $k \geqslant 2$ and let x, y be singular points of V, $(x, y) \neq 0$. We must show that $\langle U(x), U(y) \rangle_\Gamma = \mathscr{P}$. It suffices to show that for every singular point z in V that $U(z) \subset \langle U(x), U(y) \rangle_\Gamma$. Suppose first that $z \in x^{\perp_Q} \cap y^{\perp_Q}$. As previously noted, $U(z)$ is isomorphic to $\mathrm{DO}^-(2k+2, q)$. The subspaces $U(\langle z, x \rangle)$ and $U(\langle z, y \rangle)$ satisfy the hypotheses of the lemma and therefore by our inductive hypothesis

$$U(z) = \langle U(\langle z, x \rangle), U(\langle z, y \rangle) \rangle_\Gamma \subset \langle U(x), U(y) \rangle_\Gamma.$$

We have therefore shown that for every $z \in x^{\perp_Q} \cap y^{\perp_Q}$, $U(z) \subset \langle U(x), U(y) \rangle_\Gamma$.

Now suppose that $z_1, z_2 \in x^{\perp_Q} \cap y^{\perp_Q}$ are singular points of V and $(z_1, z_2) \neq 0$ and $z \in z_1^{\perp_Q} \cap z_2^{\perp_Q}$ is a singular point. Then by the above argument

$$U(z) \subset \langle U(z_1), U(z_2) \rangle_\Gamma$$

and in turn it follows that

$$U(z) \subset \langle U(x), U(y) \rangle_\Gamma.$$

Suppose now that z is any singular point, $z \neq x, y$. Then $z^{\perp_Q} \cap x^{\perp_Q} \cap y^{\perp_Q}$ is a hyperplane of $x^{\perp_Q} \cap y^{\perp_Q}$. $x^{\perp_Q} \cap y^{\perp_Q}$ has rank $k \geqslant 2$ and consequently there must be singular points $z_1, z_2 \in z^{\perp_Q} \cap x^{\perp_Q} \cap y^{\perp_Q}$, $(z_1, z_2) \neq 0$. It now follows from the above argument that $U(z) \subset \langle U(x), U(y) \rangle_\Gamma$ and the proof is complete.

We can now prove our main theorem of this subsection:

(2.3) Theorem. *For $n \geqslant 2$, $\mathrm{DO}^-(2n+2,q)$ can be generated by 2^n points.*

Proof. We prove this by induction on $n \geqslant 2$. When $n = 2$, $\mathrm{DO}^-(6,q)$ is a generalized quadrangle with parameters (q^2, q) and therefore can be generated by 4 points by (2.1). Now assume that $n > 2$ and that $\mathrm{DO}^-(2n,q)$ can be generated with 2^{n-1} points. Let x, y be non-orthogonal singular points in V. By our inductive hypothesis each of $U(x)$, $U(y)$, which are isomorphic to $\mathrm{DO}^-(2n,q)$, can be generated by 2^{n-1} points. By (2.2) $\mathscr{P} = \langle U(x), U(y)\rangle_\Gamma$ and therefore $\mathscr{P}$ can be generated by $2 \times 2^{n-1} = 2^n$ points.

2.2 An embedding for $\mathbf{DO}^-(2n+2,q)$. Let V, Q be as in the introduction. Let $\tilde{V} = \mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} V$. Define a scalar multiplication of $\mathbb{F}_{q^2}$ on $\tilde{V}$ as follows:

$$\alpha\left(\sum_{i=1}^{m} \beta_i \otimes z_i\right) = \sum_{i=1}^{m} (\alpha\beta_i) \otimes z_i.$$

In this way $\tilde{V}$ becomes a vector space over $\mathbb{F}_{q^2}$. We identify vectors $v \in V$ with $1 \otimes v \in \tilde{V}$. Then the basis $v_1, w_1, v_2, w_2, \ldots, v_{n+1}, w_{n+1}$ is a basis for $\tilde{V}$. We may extend Q to a quadratic form $\tilde{Q}$ over $\mathbb{F}_{q^2}$ on $\tilde{V}$ by extension of scalars as follows: for X_i, $Y_i \in \mathbb{F}_{q^2}$ set $\tilde{Q}(\sum_{i=1}^{n+1}(X_i v_i + Y_i w_i)) = \sum_{i=1}^{n} X_i Y_i + aX_{n+1}^2 + bX_{n+1}Y_{n+1} + cY_{n+1}^2$. Over $\mathbb{F}_{q^2}$ the polynomial $aX^2 + bX + c$ has two distinct roots and consequently there now exist singular subspaces of dimension $n+1$. This means that $(\tilde{V}, \tilde{Q})$ is a hyperbolic orthogonal space. Let $\tilde{\mathscr{S}}_k$ denote the collection of totally singular subspaces of $\tilde{V}$ of dimension k. Then the following is well known:

(i) The collection $\tilde{\mathscr{S}}_{n+1}$ divides into two classes $\tilde{\mathscr{S}}_{n+1}^+, \tilde{\mathscr{S}}_{n+1}^-$ where for subspaces U_1, U_2 in the same class, $\dim(U_1/(U_1 \cap U_2) = \dim(U_2/U_1 \cap U_2)$ is even,

(ii) Every element of $\tilde{\mathscr{S}}_n$ lies in one element of $\tilde{\mathscr{S}}_{n+1}^+$ and one element of $\tilde{\mathscr{S}}_{n+1}^-$,

(iii) Every element of $\tilde{\mathscr{S}}_{n-1}$ is contained in q^2+1 elements in $\tilde{\mathscr{S}}_{n+1}^+$ and $\tilde{\mathscr{S}}_{n+1}^-$ (note that it is q^2+1 since we are over the field $\mathbb{F}_{q^2}$.)

The incidence geometry $(\tilde{\mathscr{S}}_{n+1}^+, \tilde{\mathscr{S}}_{n-1})$ is a strong parapolar space commonly referred to as the "half spin geometry" and denoted by $D_{n+1,n+1}(q^2)$. It is known that this geometry has an absolutely universal embedding of dimension 2^n ([14]) which is an irreducible module for the isometry group $G(\tilde{V}, \tilde{Q}) \cong \mathrm{SO}^+(2n+2,q^2) \cong D_{n+1}(q^2)$. We remark that this module remains irreducible when restricted to $G(V, Q) \cong \mathrm{SO}^-(2n+2,q) \cong {}^2D_{n+1}(q)$. We use this embedding to construct an embedding for $\mathrm{DO}^-(2n+2,q)$ as follows.

Let $e : \tilde{\mathscr{S}}_{n+1}^+ \to \mathbb{PG}(X)$, $\dim X = 2^n$, be the absolutely universal embedding of the half spin geometry. We remark that this is an irreducible module for the isometry group of $(\tilde{V}, \tilde{Q})$ which is the orthogonal group $\mathrm{SO}^+(2n+2,q^2) \cong D_{n+1}(q^2)$. Now let $f : \mathscr{P} \to \tilde{\mathscr{S}}_{n+1}^+$ be the following map: An element p of $\mathscr{P}$ is a singular subspace of V of dimension n. The span of such a subspace in $\tilde{V}$ is still a singular subspace of dimension n: This is clearly true of $\langle v_1, v_2, \ldots, v_n\rangle$. However, $G(V, Q) \leqslant G(\tilde{V}, \tilde{Q})$

and the former is transitive on the singular n-dimensional subspaces of V. As stated in (ii) above such a subspace lies in a unique member of $\tilde{\mathcal{S}}^{+}_{n+1}$. Clearly, $f(p)$ is this unique subspace. In a similar way, a line $L \in \mathcal{L}$ corresponds to a singular subspace W of V of dimension $n-1$ and spans in $\tilde{V}$ a singular subspace of the same dimension. Note that there are q^2+1 elements of $\mathcal{P}$ which are incident with W and therefore the map induced by f from L to $\{p' \in \tilde{\mathcal{S}}^{+}_{n+1} : p' \supset W\}$ is bijective. It now follows that if we set $\alpha = e \circ f$ and $X_1 = \langle \alpha(p) : p \in \mathcal{P} \rangle$ then $\alpha : \mathcal{P} \to \mathbb{PG}(X_1)$ is an embedding. However, as noted above the isometry group $G(V,Q) \cong \mathrm{SO}^-(2n+2,q)$ of (V,Q) acts irreducibly on the space X. Since X_1 is an invariant subspace it follows that $X_1 = X$. We have therefore proved:

(2.4) Theorem. *The dual polar space* $\mathrm{DO}^-(2n+2,q)$ *has an embedding* α *into a projective space* $\mathbb{PG}(X)$ *with* $\dim X = 2^n$.

Combining this with (2.3) we may now prove:

(2.5) Theorem. *The embedding* α *of* $\mathrm{DO}^-(2n+2,q)$ *into* $\mathbb{PG}(X)$ *with* $\dim X = 2^n$ *is absolutely universal.*

Proof. By (2.3) $\mathrm{DO}^-(2n+2,q)$ can be generated by 2^n points; thus any embedding into $\mathbb{PG}(2^n-1,q)$ is relatively universal. Now the quads of $\mathrm{DO}^-(2n+2,q)$ are the unitary quadrangles $\mathrm{U}(4,q^2)$ ($H_3(q^2)$ in the notation of Thas). These quadrangles have an absolutely universal embedding by ([13]). It follows by a result of Kasikova and Shult ([9]) that any embedding of $\mathrm{DO}^-(2n+2,q)$ into $\mathbb{PG}(2^n-1,q)$ is absolutely universal.

3 A generating set for $\mathrm{DU}(2n+1,q^2)$

Now let V be a $(2n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$ and h a non-degenerate hermitian form. Let $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*)$ be the unitary polar space of isotropic points and (totally) isotropic lines in V and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the associated dual polar space. Recall that we may identify $\mathcal{P}$ with the maximal (totally) isotropic subspaces of V. For a vector w we set $w^{\perp_h} = \{w \in V : h(w,v) = 0\}$ and for a subspace W of V,

$$W^{\perp_h} = \bigcap_{w \in W} w^{\perp_h}.$$

As in the orthogonal case, for W a (totally) isotropic subspace of V we set $U(W) = \{p \in \mathcal{P} : W \subset p\}$. This is a convex subspace of Γ. Now the quotient space $\overline{W} = W^{\perp_h}/W$ can be made into a unitary space of dimension $(2n+1-2d)$ where $\dim W = d$ by defining $\overline{h}(u+W, v+W) = h(u,v)$ for vectors $u, v \in W^{\perp_h}$. Moreover there is a one-to-one correspondence between the elements of $U(W)$ and the maximal singular subspaces of $\overline{W}$. This correspondence is an isomorphism of geometries and in this way we see that $U(W)$ is isomorphic to $\mathrm{DU}(2n+1-2d,q^2)$ where $\dim W = d$. In particular, for an isotropic point w of V, $U(w)$ is isomorphic to $\mathrm{DU}(2n-1,q^2)$.

Our main result, that $\mathrm{DU}(2n+1,q)$ can be generated by 2^n points, will be an immediate consequence of the following lemma:

(3.1) Lemma. *Let $x, y \in \mathscr{P}^*$, that is isotropic points of V and assume that $h(x,y) \neq 0$. Then $\langle U(x), U(y)\rangle_\Gamma = \mathscr{P}$.*

Proof. We proceed by induction on $n \geqslant 2$. Suppose first that $n = 2$. In this case $U(x)$, $U(y)$ are two opposite lines in the generalized quadrangle $\mathrm{DU}(5,q^2)$ which has parameters (q^3,q^2) and therefore by (2.1) it follows that $\langle U(x), U(y)\rangle_\Gamma = \mathscr{P}$.

Now assume that the result holds for $n = k \geqslant 2$. We must show that it holds for $n = k+1$. So assume that (V,h) is a unitary space of dimension $2(k+1)+1 = 2k+3$ with $k \geqslant 2$ and let x, y be isotropic points of V, $(x,y) \neq 0$. We must show that $\langle U(x), U(y)\rangle_\Gamma = \mathscr{P}$. It suffices to show for z an arbitrary isotropic point in V that $U(z) \subset \langle U(x), U(y)\rangle_\Gamma$.

First suppose that $z \in \mathbb{PG}\langle x,y\rangle$ that is, z is an isotropic point on the hyperbolic line of V spanned by x and y. Denote the set of isotropic points in $\langle x,y\rangle$ by γ. Now for any element $p \in U(x)$, $p^\perp \cap U(z)$ is a unique point for each $z \neq x$, $z \in \gamma$ and the set of all these points is a line. It therefore follows that $\bigcup_{z\in\gamma} U(z) \subset \langle U(x), U(y)\rangle_\Gamma$.

Let us now assume that $z \notin \gamma$ and $z \in x^{\perp_h} \cap y^{\perp_h}$. As previously noted, $U(z)$ is isomorphic to $\mathrm{DU}(2k+1,q^2)$. The subspaces $U(\langle z,x\rangle)$ and $U(\langle z,y\rangle)$ satisfy the hypotheses of the lemma and therefore by our inductive hypothesis

$$U(z) = \langle U(\langle z,x\rangle), U(\langle z,y\rangle)\rangle_\Gamma \subset \langle U(x), U(y)\rangle_\Gamma.$$

We have therefore shown that for every $z \in x^{\perp_h} \cap y^{\perp_h}$, $U(z) \subset \langle U(x), U(y)\rangle_\Gamma$.

We can now proceed as in the elliptic case. Suppose that $z_1, z_2 \in x^{\perp_h} \cap y^{\perp_h}$ are isotropic points of V and $(z_1,z_2) \neq 0$ and $z \in z_1^{\perp_h} \cap z_2^{\perp_h}$ is a singular point. Then by the above argument

$$U(z) \subset \langle U(z_1), U(z_2)\rangle_\Gamma$$

and, in turn, it follows that

$$U(z) \subset \langle U(x), U(y)\rangle_\Gamma.$$

Assume now that z is any isotropic point, $z \notin \gamma$. Then $z^{\perp_h} \cap x^{\perp_h} \cap y^{\perp_h}$ is a hyperplane of $x^{\perp_h} \cap y^{\perp_h}$. $x^{\perp_h} \cap y^{\perp_h}$ has rank $k \geqslant 2$ and consequently there must be isotropic points $z_1, z_2 \in z^{\perp_h} \cap x^{\perp_h} \cap y^{\perp_h}$, $(z_1,z_2) \neq 0$. It now follows from the above argument that $U(z) \subset \langle U(x), U(y)\rangle_\Gamma$ and the proof is complete.

We can now prove our main theorem:

(3.2) Theorem. *For $n \geqslant 2$, $\mathrm{DU}(2n+1,q)$ can be generated by 2^n points.*

Proof. The proof is exactly like the elliptic case.

It is well known that the generalized quadrangle $\mathrm{DU}(5, q^2)$ is not embeddable. Since these are quads of the dual polar space $\mathrm{DU}(2n+1, q^2)$ it follows that in general $\mathrm{DU}(2n+1, q^2)$ is not embeddable.

4 Frames for $\mathrm{DO}^-(2n+2, q)$ and $\mathrm{DU}(2n+1, q^2)$

(4.1) Theorem. *Let* $\Gamma = (\mathscr{P}, \mathscr{L})$ *be a dual polar space of type* $\mathrm{DU}(2n+1, q^2)$ *or* $\mathrm{DO}^-(2n+2, q)$ *and let* $\Delta = (\mathscr{P}, \sim)$ *be its point-collinearity graph. Let* $\mathscr{H}$ *be the class of all subgraphs of* Δ *which are isomorphic to the graph of the n-hypercube and which are isometrically embedded in* Δ. *Then the following hold*:

1. *The point-vertices of any graph* $H \in \mathscr{H}$ *generate the ambient dual polar space as a geometry.*
2. *The automorphism group of the geometry acts transitively on* $\mathscr{H}$.

Proof. Select two vertices at distance $(n-1)$ in the subgraph H. Since H is isometrically embedded, there are points at distance $(n-1)$ in Δ. Then the convex subspace hull of these two points is a dual polar space $U(p_1)$ for some polar point. Since $U(p_1)$ is a convex subspace of point-diameter $n-1$ it must contain the union of all the geodesics of H connecting the two points and no further vertices of H. Thus, $H_1 := H \cap U(p_1)$ is an $(n-1)$-hypercube isometrically embedded in $U(p_1)$. Now let

$$\mathrm{opp} : H \to H,$$

be the "opposite" mapping on H, mapping each vertex to its antipodal vertex at distance n. Then

$$H_2 := H \backslash H_1 = H_1^{\mathrm{opp}}$$

is also an isometrically embedded $(n-1)$-hypercube whose convex subspace hull is $U(p_2)$ for some polar point p_2. We claim that $U(p_1) \cap U(p_2) = \emptyset$, that is, $\{p_1, p_2\}$ is a 2-coclique in the polar space.

Otherwise, we have $U(p_1) \cap U(p_2) = U(L)$ where L is a polar line. Since each $U(p_i)$ has point-diameter $(n-1)$ for each vertex $x \in H \backslash U(p_2)$, we have x^{opp} is in $H \backslash U(p_1)$, and vice versa, so there is a bijection

$$H \backslash U(p_1) \to H \backslash U(p_2).$$

It follows that $H \cap U(L)$ is invariant under the opposite map and that forces $H \cap U(L) = \emptyset$. Thus $H_i \subset H \backslash U(p_{3-i})$, for $i = 1, 2$. But this forces the absurdity that H is not connected, since no vertex of $U(p_1) \backslash U(p_2)$ can be collinear with a vertex of $U(p_2) \backslash U(p_1)$ when these two subspaces intersect non-trivially.

Thus we have $U(p_1) \cap U(p_2) = \emptyset$. If $n = 3$ then H_i is a square, and as we have seen H_i generates $U(p_2)$ as a geometry by (2.1). Otherwise we can maintain this assertion by induction. Thus the subspace generated by H contains both $U(p_1)$ and $U(p_2)$ and it is shown by lemmas (2.2), (3.1) that these two generate Γ.

It remains to prove the transitivity of the classical groups $\mathrm{U}(2n+1,q^2)$ or $\mathrm{O}^-(2n+2,q)$ on $\mathscr{H}$. We take $H = H_1 \cup H_2$ with $H_i \subset U(p_i)$, $i = 1,2$ as in the first part of the proof. Now let K be any other subgraph in $\mathscr{H}$. Then by taking two vertices at distance $(n-1)$ in K, and forming their convex subspace hull, we may partition K into two $(n-1)$-hypercubes K_1 and K_2, living in opposite subspaces $U(q_1)$ and $U(q_2)$, respectively, as we did for H in the first part of this proof. Now there is an element in the relevant classical group taking (q_1,q_2) to (p_1,p_2) by Witt's theorem, so we may assume from here on that $q_i = p_i$ for $i = 1,2$. Since p_1,p_2 are non-orthogonal, the stabilizer in G of (p_1,p_2) is a classical group of the same type of rank one less. So by induction it contains an element g taking the isometrically embedded hypercube K_1 to H_1 in the subspace $U(p_1)$. Now the mapping $f : U(p_1) \to U(p_2)$ which takes each point of $U(p_1)$ to the unique point of $U(p_2)$ collinear with it is an isomorphism of point-line geometries commuting with the action of g on both sides. One sees from the hypercube graph that $K_2 = f(K_1)$ and that $H_2 = f(H_1)$. Thus we also have $K_2^g = H_2$, and so $K^g = H$ and the transitivity is proved.

5 A survey of embeddings and generation of dual polar spaces

In this section we survey what is known to us regarding the embeddings and generation of the dual polar spaces. We exclude from this survey dual polar spaces for the hyperbolic orthogonal spaces since the lines of these geometries have just two points.

5.1 The symplectic dual polar spaces, $\mathbf{DSp}(2n,q)$. We begin with the situation where the underlying field has more than two elements. Cooperstein ([3]) has completely determined this situation:

(5.1) Theorem. *Assume* $q > 2$. *Then the following hold*:

(1) *The dual polar space* $\mathrm{DSp}(2n,q)$ *has generating rank* $\binom{2n}{n} - \binom{2n}{n-2}$.

(2) $\mathrm{DSp}(2n,q)$ *has an absolutely universal embedding and its dimension is equal to its generating rank.*

When a geometry $\Gamma = (\mathscr{P},\mathscr{L})$ has three points on a line then there is a standard construction for an absolutely universal embedding: let V be the space over $\mathbb{F}_2$ with $\mathscr{P}$ as basis and let W be the subspace spanned by all $x+y+z$ where $\{x,y,z\} = L \in \mathscr{L}$. Then V/W with the map $x \to x + W$ is the absolutely universal embedding. Since $\mathrm{DSp}(2n,2)$ has an obvious embedding, the so-called spin embedding, it has a universal embedding. The dimension of this embedding has been the subject of much investigation and A. Brouwer of Technical University Eindhoven made the following conjecture:

(5.2) Conjecture. *The dimension of the universal embedding of* $\mathrm{DSp}(2n,2)$ *is* $\dfrac{(2^n+1)(2^{n-1}+1)}{3}$.

Brouwer, in unpublished work, demonstrated the truth of this conjecture for $n \leqslant 4$. In ([8]), Cooperstein and Shult proved that $\mathrm{DSp}(6,2)$ can be generated by 15 points. In ([4]), Cooperstein constructed generating sets with 57 points for $\mathrm{DSp}(8,2)$ and 187 points for $\mathrm{DSp}(10,2)$, the latter proving Brouwer's conjecture for $n = 5$. Cooperstein's methods were used to construct a generating set of 716 points for $\mathrm{DSp}(12,2)$ while Bardoe and A. A. Ivanov were able to show computationally that Brouwer's conjecture holds in this case and the case for $n = 7$ as well ([1]). It is not known whether the generation can be improved to 715. Recently, Brouwer's conjecture has been settled by Paul Li ([10]) making use of ideas developed by P. McClurg ([11]):

(5.3) Theorem. *The universal embedding for* $\mathrm{DSp}(2n,2)$ *has dimension* $\dfrac{(2^n+1)(2^{n-1}+1)}{3}$.

5.2 The unitary dual polar spaces, $\mathrm{DSU}(2n,q)$. As with the previous discussion we begin with the situation where the underlying field has more than two elements. The situation here was completely settled by Cooperstein ([5]):

(5.4) Theorem. *Assume $q > 2$. Then the following hold*:

(1) *The dual polar space* $\mathrm{DSU}(2n,q^2)$ *can be generated by* $\dbinom{2n}{n}$ *points.*

(2) $\mathrm{DSU}(2n,q^2)$ *has an absolutely universal embedding of dimension* $\dbinom{2n}{n}$.

When $q = 2$ our knowledge is not very complete. We know that there are embeddings and since lines have three points there is a universal embedding. A. A. Ivanov has made the following

(5.5) Conjecture. *The universal embedding for* $\mathrm{DSU}(2n,4)$ *has dimension* $\dfrac{4^n+2}{3}$.

When $n = 2$, $\mathrm{DU}(4,4)$ is the classical generalized quadrangle $\mathrm{O}^-(6,2)$ given by an elliptic quadric in 6 dimensions and therefore has a 6 dimensional embedding. The four points of a circuit generate a grid. Any additional point generates a $(2,2)$ subquadrangle and any further point generates the entire generalized quadrangle. Consequently the 6-dimensional embedding is universal.

The situation for $n = 3$ is also entirely known with the embedding determined by Yoshiara, ([15]), and the generation by Cooperstein, ([6]):

(5.6) Theorem. (1) *The universal embedding of* $\mathrm{DU}(6,4)$ *has dimension* 22.
(2) $\mathrm{DU}(6,4)$ *can be generated by* 22 *points.*

5.3 The orthogonal dual polar spaces, $\mathrm{DO}(2n+1,q)$, $q > 2$. We restrict ourselves to the case q is odd since $\mathrm{DSO}(2n+1,2^n)$ is isomorphic to $\mathrm{DSp}(2n,2^n)$ and this case has been previously discussed. The situation for this geometry is also entirely known. Independently several groups (Blok–Brouwer [2], Cooperstein–Shult [7], and Ronan–

Smith [12]) have found generating sets for this geometry, while the universal embedding was determined by Wells ([14]):

(5.7) Theorem. *Let* $q > 2$. *Then the following hold*:

(1) *The orthogonal dual polar space* $\mathrm{DSO}(2n+1, q)$ *can be generated by* 2^n *points.*

(2) $\mathrm{DSO}(2n+1, q)$ *has an absolutely universal embedding and its dimension is* 2^n.

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