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On connected divisors

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Abstract. A vanishing theorem for numerically connected divisors, first given by Bombieri for surfaces, is established in any dimension. A definition of *k-connected divisors* is proposed, then such divisors on threefolds are studied.

1 Introduction

Before the coming of $\mathbb{Q}$-divisors and of the Kawamata–Viehweg theorem, Bombieri [4] noted a vanishing theorem, to whose effect $h^1(\mathscr{I}_D) = 0$, for any numerically connected divisor D on a smooth surface S, with $D^2 > 0$.

The notion of numerically connected divisor, due to Franchetta [6], is an algebraic analogue of topological connectedness—it basically reduces to the latter when D has no multiple components. It should be remarked that the vanishing theorem cited above had already been stated by Franchetta [7], albeit not in the language of cohomology.

In the case of surfaces, it makes sense to strengthen the notion of numerical connectedness into that of k-connectedness, introduced by Bombieri [4], k being a measure of how connected the divisor D is.

A few years later, van de Ven [18] proved that every very ample divisor on a surface is 2-connected, with only two exceptions.

Nowadays the Kawamata–Viehweg theorem gives much stronger vanishings, but they come at a price: the divisor D must be nef and big; also, the proof requires the full force of $\mathbb{Q}$-divisor techniques (see e.g. [15]).

In the present paper, we generalize both Bombieri's and van de Ven's theorems by using a more down-to-earth approach. Indeed we prove that, for a numerically connected divisor D on a smooth n-dimensional variety X, $h^1(\mathscr{I}_D) = 0$, provided that $D^n > 0$ and $h^0(D) \geqslant 3$. Subsequently, we introduce the notion of k-connected divi-

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sors for higher dimensional varieties, which reduce to Bombieri's in the case of surfaces. Equipped with this definition, we prove that every very ample divisor on a smooth threefold is 3-connected, but for a finite number of exceptions, which are completely described.

In some more detail, the paper is organized as follows.

In the second section we prove that, for any numerically connected divisor D on a smooth projective n-dimensional variety $X \subset \mathbb{P}^N$, $h^1(\mathcal{I}_D) = 0$, if $D^n > 0$ and $h^0(D) \geqslant 3$. This is a consequence of the following fact: if $h^0(D) \geqslant 3$ and $|D|$ is not compounded with a pencil, then $h^1(\mathcal{I}_D) = h^1(D, \mathcal{O}_D) - 1$. The idea of the proof is essentially that the curves $C = E \cap \theta$, where $E \in |D|$ and $\theta \in G(N - n + 2, N)$, are generically reduced and irreducible, so the images of the Albanese groups $\mathrm{Alb}(\tilde{C})$ of their normalizations $\tilde{C}$ into $\mathrm{Alb}(X)$ are a continuous family of subtori, hence they are indeed a constant subgroup K of $\mathrm{Alb}(X)$; since the curves C sweep out a Zariski open subset of X, this fact forces K to be the whole of $\mathrm{Alb}(X)$, which in turn implies our statement.

The third section is devoted to the study of connected divisors on threefolds: we give a complete description of all very ample divisors on threefolds which are not 3-connected. Also in this case the proof is quite direct. We first give a uniform bound on the degree of threefolds admitting a very ample divisor which is not 3-connected. Then we plunge ourselves in the botany of algebraic varieties of low degree: by using Ionescu's classifications of such threefolds [12] and [13], we make a short list of the possible candidates for this kind of varieties, then we analyze them one by one, in order to find actual instances of such behavior. The analysis is essentially based on the study of their Picard groups, to find a divisor with a "wrong" (i.e. not 3-connected) decomposition.

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2 Generalization of a theorem of Bombieri's

The goal of this paragraph is to generalize a result of Bombieri (and Franchetta), [4], Section 3, Theorems A and B.

In what follows X is an n-dimensional smooth projective variety, $n \geqslant 3$, and $D \subset X$ is an effective divisor.

Theorem 2.1. *If $h^0(D) \geqslant 3$ and $|D|$ is not compounded with a pencil, then $h^1(X, \mathcal{I}_D) = h^0(D, \mathcal{O}_D) - 1$.*

We need the following elementary

Lemma 2.2. *Let $C \subset X$ be a reduced irreducible curve, then the image of the natural map $j_* : H^1(C, \mathcal{O}_C)^* \to \mathrm{Alb}(X)$ is a closed subgroup.*

Proof. The natural map is defined as follows: $C \hookrightarrow X$ induces $H^1(X, \mathcal{O}_X) \to H^1(C, \mathcal{O}_C)$, hence, by duality and conjugation, $\overline{H^1(C, \mathcal{O}_C)^*} \to \overline{H^1(X, \mathcal{O}_X)^*}$; since X is Kähler, the Hodge theorem gives the identification $\overline{H^1(X, \mathcal{O}_X)^*} = H^{n-1,n}(X)$; the composition with the projection $H^{n-1,n}(X) \to \frac{H^{n-1,n}(X)}{H_1(X,\mathbb{Z})} = \mathrm{Alb}(X)$ gives the map $j_* : \overline{H^1(C, \mathcal{O}_C)^*} \to \mathrm{Alb}(X)$. It is now clear that $\mathrm{Im}(j_*)$ is a subgroup; we must prove that it is closed. Let $\tilde{C} \to C$ be the normalization of C; by reasoning as before, there is a map $f_* : H^{0,1}(\tilde{C}) \to \overline{H^1(C, \mathcal{O}_C)^*}$, thus we obtain a diagram

$$\begin{tikzcd}
 & & H^{0,1}(\tilde{C}) \arrow[dll] \arrow[d] \arrow[drr, "f_*"] & & \\
\mathrm{Alb}(\tilde{C}) \arrow[drr, "f_*"'] & & \overline{H^1(X, \mathcal{O}_X)^*} \arrow[rr] \arrow[d, "j_*"] & & H^{n-1,n}(X) \arrow[dll] \\
 & & \mathrm{Alb}(X) & &
\end{tikzcd}$$

which is commutative because of the functoriality of Alb. It follows that $\mathrm{Im}(j_*) = f_*(\mathrm{Alb}(\tilde{C}))$, which is closed because $f_* : \mathrm{Alb}(\tilde{C}) \to \mathrm{Alb}(X)$ is a continuous map of compact spaces. □

We also need the following well-known results.

Theorem 2.3 (Chow's theorem). *Any continuous family of closed subgroups of a torus is constant.*

Proof. [14], Theorem II.5. □

Theorem 2.4 (Second Bertini theorem). *Let $|E|$ be a complete linear system without fixed components on a smooth complete variety Y; if $\dim \imath_{|E|}(Y) \geqslant 2$ (i.e. $|E|$ is not compounded with a pencil) then every divisor of $|E|$ is connected and the generic one is irreducible.*

Proof. [11], Theorem 7.9. □

Proof of Theorem 2.1. Suppose that $X \subset \mathbb{P}^N(\mathbb{C})$ and $\dim X = n$.

Step 1. From $0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ we get $0 \to H^0(X, \mathcal{I}_D) \to H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D) \to H^1(X, I_D) \to \ker\{H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)\}$; since $H^0(X, \mathcal{I}_D) = H^0(X, -D) = 0$ because D is effective, and $H^0(X, \mathcal{O}_X) = \mathbb{C}$, it is enough to prove that $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$ is injective; in turn, if $C \subset D$ is a (reduced irreducible) curve, from

$$\begin{tikzcd} H^1(X,\mathcal{O}_X) \arrow[r] \arrow[dr] & H^1(D,\mathcal{O}_D) \arrow[d] \\ & H^1(C,\mathcal{O}_C) \end{tikzcd}$$

it follows that it suffices to prove that $H^1(X,\mathcal{O}_X) \to H^1(C,\mathcal{O}_C)$ is injective.

Step 2. Let $B := |D| \times G(N-n+2,N)$, and let $V \subseteq B$ be the subset of elements $(E,\theta) \in B$ such that $E \cap \theta$ is an integral curve. If $|D|$ has no fixed components from the second Bertini theorem it follows that V is a Zariski open subset of B. If not, we can substitute $|D|$ with its moving part $|D'|$, and the proof works in any case. For any $b = (E,\theta) \in B$, write $C_b := E \cap \theta$, then $\mathcal{F} = \{j_*(\overline{H^1(C_b,\mathcal{O}_{C_b})^*})\}_{b\in V}$ is an algebraic family of closed subgroups of $\mathrm{Alb}(X)$, hence, by Chow's theorem, $j_*(\overline{H^1(C_b,\mathcal{O}_{C_b})^*})$ is a constant closed subgroup, call it K, of $\mathrm{Alb}(X)$.

Step 3. Set $\Sigma_p := \{b \in B \mid p \in C_b\}$, for all $p \in X$. Clearly, $\Sigma_p \cap \Sigma_q = |D|(p,q) \times (p,q)^*$, where $|D|(p,q) = \{E \in |D| \mid p,q \in E\}$ and $(p,q)^* = \{\theta \in G(N-n+2,N) \mid p,q \in \theta\}$. Since $h^0(D) \geqslant 3$ and $N-n+2 \geqslant 3$, we see that $|D|(p,q) \times (p,q)^*$ has positive dimension. An integral curve $\Gamma \subset \Delta \subset X$, Γ passing through p and q, with $\Delta \in |D|$, exists if and only if $\Sigma_p \cap \Sigma_q \nsubseteq B - V$, and the latter is a closed condition, hence $W := \{(p,q) \mid \Sigma_p \cap \Sigma_q \cap V \neq \emptyset\}$ is a Zariski open subset of $X \times X$, i.e. for all $(p,q) \in W$, there exists a curve in the family $\{C_b\}_{b\in V}$ connecting them.

Step 4. Let $\omega : X \to \mathrm{Alb}(X)$ be the Albanese map (with respect to a fixed base point), and let $\Omega : (p,q) \in X \times X \to \omega(p) - \omega(q) \in \mathrm{Alb}(X)$; it is well-known that $\mathrm{Im}(\Omega)$ generates $\mathrm{Alb}(X)$, as a group, moreover, Ω is a closed map, because it is continuous between compact spaces. Now, for all $(p,q) \in W$, $\Omega(p,q) = \omega(p) - \omega(q) \in j_*(\overline{H^1(C_b,\mathcal{O}_{C_b})^*}) = K$; it follows that $\Omega(W) \subseteq K$, so $\Omega(X \times X) = \Omega(\overline{W}) \subseteq K$, because Ω is a closed map and K is a closed subgroup. But $\Omega(X \times X)$ generates $\mathrm{Alb}(X)$, hence $K = \mathrm{Alb}(X)$.

Conclusion. We have found an integral curve (indeed a family of them) $C \subset X$ such that the map $j_* : \overline{H^1(C,\mathcal{O}_C)^*} \to \mathrm{Alb}(X)$ is surjective; since $\dim \mathrm{Alb}(X) = \dim H^{n-1,n}(X)$, also the map $\overline{H^1(C,\mathcal{O}_C)^*} \to H^{n-1,n}(X)$ is surjective, hence $H^1(X,\mathcal{O}_X) \to H^1(C,\mathcal{O}_C)$ is injective and the theorem is proved. □

We can now extend Bombieri's theorem to n-dimensional varieties.

Definition 2.1. Let D be an effective divisor of a n-dimensional variety X, $n \geqslant 2$. D is called *numerically connected* if there exists an ample divisor H such that, for any decomposition $D = D_1 + D_2$ with D_1 and D_2 effective, $H^{n-2}D_1D_2 > 0$.

Lemma 2.5. *If D is a numerically connected divisor on a smooth variety X, then* $h^0(D,\mathcal{O}_D) = 1$.

Proof. [17], Lemma 3. □

Corollary 2.6. *Let X be a smooth n-dimensional variety and let D be a numerically connected divisor of X such that $h^0(X, D) \geqslant 3$ and $D^n > 0$, then $h^1(X, -D) = 0$.*

Proof. As $D^n > 0$, $|D|$ is not compounded with a pencil. By the previous theorem and lemma we have then $h^1(X, -D) = h^0(D, \mathcal{O}_D) - 1 = 0$. $\square$

Remark 2.1. Among the hypotheses of the previous corollary, only the numerical connectedness refers specifically to the divisor D, as opposed to the linear series $|D|$. But D is not numerically connected only if it is not topologically connected or has multiple components; furthermore, D is a moving divisor, i.e. $h^0(D) > 1$, thus, by the second Bertini theorem, the generic element of $|D|$ is irreducible and reduced outside the fixed component, if any; the upshot is that Corollary 2.6 holds if we replace numerical connectedness with the hypothesis that $|D|$ has no fixed component.

Now, a divisor D such that $h^0(D) \geqslant 3$, $D^n > 0$ and $|D|$ has no fixed component is very close to being nef and big, so in this case Corollary 2.6 is a weak form of the Kawamata–Viehweg theorem, but its proof avoids the use of $\mathbb{Q}$-divisors.

Note also that the hypothesis $D^n > 0$ is used only to say that $|D|$ is not compounded of a pencil, so if we substitute it with the latter, Corollary 2.6 is not a consequence of Kawamata–Viehweg theorem any longer. This last form is basically the one stated by Franchetta [7] in the case of surfaces.

3 Connected divisors on threefolds

In [18], in order to study the spannedness of adjoint divisors on a surface with the help of the Bombieri–Franchetta theorem, the author shows that every very ample divisor on a surface is 2-connected, with some exceptions (see [18] theorem I), according to the following definition of Bombieri.

Definition 3.1. An effective divisor D on a smooth surface S is *k-connected* if for any decomposition $D = D_1 + D_2$ with D_1 and D_2 effective, $D_1 D_2 \geqslant k$.

In this paragraph we want to generalize van de Ven's theorem to threefolds. In order to do so we need a good definition of k-connectedness for effective divisors on higher dimensional manifolds, for which we propose the following.

Definition 3.2. Let X be a n-dimensional smooth variety and let D be an effective divisor on X. We say that D is *k-connected* if for any decomposition $D = D_1 + D_2$ with D_1 and D_2 effective, we have $D^{n-2} D_1 D_2 \geqslant k$.

Remark 3.1. (i) Of course, in the case of a surface, the previous definition is in agreement with Bombieri's; it is unfortunate though that, in dimension greater than 2, a k-connected divisor is not in general numerically connected, unless it is ample.

(ii) When X is a 3-fold, by the previous definition it is obvious that, if a very ample divisor D is not k-connected, then there exists a generic hyperplane section $S \in |D|$ and a very ample divisor $D_{|S}$ which is not k-connected. Unfortunately we cannot use the results on this topic contained in [1] to study the k-connectedness on X because, as we shall see, it involves 3-folds of low degree which are outside of the range considered in [1].

We fix now some notation that we use throughout this section.

Notation.

(X, D)	(3-dimensional smooth variety; very ample divisor)
$\mathbb{P}^n$	the ambient space of X;
$d = D^3$	the degree of X;
$g = g(X)$	the sectional genus of X;
S	the generic hyperplane section of X;
$\mathrm{Pic}(X)$	the Picard group of the variety X;
$\mathrm{Num}(X)$	the additive group of divisors of X modulo numerical equivalence;
$\equiv$	numerical equivalence of divisors;
$\mathbb{P}(\mathscr{E})$	projectivization of the vector bundle $\mathscr{E}$ over a variety B;
T	its tautological bundle;
π	the natural projection of $\mathbb{P}(\mathscr{E})$ onto B.

Now we can proceed to generalize van de Ven's theorem, and first of all we prove some lemmata.

Lemma 3.1. *Let $X = \mathbb{P}(\mathscr{E})$ over a smooth curve C* ($\mathrm{rank}(\mathscr{E}) = 3$). *Assume that X is embedded in $\mathbb{P}^n$ as a scroll by the very ample tautological divisor $D = T$, then T is never* 2*-connected.*

Proof. Let F be the numerical class of a fibre. Since any fibre is embedded as a two dimensional linear space, there is always a hyperplane in $\mathbb{P}^n$ containing it. For any $P \in C$, the elements of $|D - F_P|$ correspond to the hyperplanes containing the fibre F_P, hence $D - F_P$ is effective and we have the effective decomposition $T = D_1 + D_2$ with $D_1 \equiv T - F$ and $D_2 \equiv F$ in $\mathrm{Num}(X)$. Now we can compute $TD_1D_2 = T(T - F)F = T^2F = 1$. □

Remark 3.2. (i) It is easy to see that if X is a 3-dimensional scroll over a curve C there are only two possible effective decompositions $D = T = D_1 + D_2$ such that $TD_1D_2 \leqslant 2$, namely $D_1 \equiv T - F$ and $D_2 \equiv F$, or $D_1 \equiv T - 2F$ and $D_2 \equiv 2F$; in the first case $DD_1D_2 = 1$, in the second case $DD_1D_2 = 2$.

(ii) Note that Lemma 3.1 holds for any rank $r \geqslant 2$. Indeed the decomposition $T = D_1 + D_2$, with $D_1 \equiv T - F$ and $D_2 \equiv F$, is still possible for any rank and $T^{r-2}(T - F)F = T^{r-1}F = 1$.

If X is a quadric fibration over a smooth curve C, then there exists a rank 4 vector bundle $\mathscr{E}$ over C such that X is a divisor in $W := \mathbb{P}(\mathscr{E})$; moreover D is the restriction to X of the tautological divisor T of W, $\mathrm{Num}(W)$ is generated by T and the class F of a fibre, and $X \equiv 2T + bF$ in $\mathrm{Num}(W)$ for a suitable integer b (see [9], p. 135). For such varieties we can prove the following lemma.

Lemma 3.2. *Let X be a quadric fibration over a smooth curve C. Then D is never 3-connected.*

If $\mathrm{Num}(X) \simeq \mathbb{Z} \oplus \mathbb{Z} = \langle T_{|X}, F_{|X} \rangle$, then any effective decomposition $D = D_1 + D_2$ with $DD_1D_2 = 2$ is of type $D_1 \equiv T_{|X} - F_{|X}$, $D_2 \equiv F_{|X}$.

$\mathrm{Num}(X) \simeq \mathbb{Z} \oplus \mathbb{Z} = \langle T_{|X}, F_{|X} \rangle$ when C is rational and X has at least one singular fibre.

Proof. For any $P \in C$ the sections of the divisor $T_{|X} - F_{P|X}$ correspond to the hyperplanes of $\mathbb{P}^n$ containing F_P, so the divisor is always effective and if we choose $D_1 \equiv T_{|X} - F_{|X}$, $D_2 \equiv F_{|X}T$ then $DD_1D_2 = T_{|X}(T_{|X} - F_{|X})F_{|X} = T(T - F)F(2T + bF) = 2$.

If $\mathrm{Num}(X) \simeq \mathbb{Z} \oplus \mathbb{Z} = \langle T_{|X}, F_{|X} \rangle$, any effective decomposition $T_{|X} = D = D_1 + D_2$ is of type $D_1 \equiv \alpha T_{|X} + \beta F_{|X} \equiv (\alpha T + \beta F)_{|X}$ and $D_2 \equiv \gamma T_{|X} + \delta F_{|X} \equiv (\gamma T + \delta F)_{|X}$ in $\mathrm{Num}(X)$ with $\alpha + \gamma = 1$, $\beta + \delta = 0$, $\alpha \geqslant 0$, $\gamma \geqslant 0$. Hence we can assume $D_1 \equiv T_{|X} - hF_{|X}$, $D_2 \equiv hF_{|X}$ with $h \geqslant 1$, and we can compute $DD_1D_2 = T_X(T_{|X} - hF_{|X})hF_{|X} = T(T - hF)hF(2T + bF) = 2h$ so that $DD_1D_2 = 2$ implies $h = 1$.

From [12] (see Lemma 3.8 and Proposition 0.6) we know that $\mathrm{Pic}(X) = \mathrm{Num}(X) = \langle T_{|X}, F_{|X} \rangle$ when C is rational and the fibration has at least one singular fibre. □

Lemma 3.3. *If $d \leqslant 4$, then the divisor D is 2-connected, unless X is a scroll over a curve and D its tautological divisor.*

Proof. To prove the lemma it suffices to consider only linearly normal 3-dimensional varieties such that $d \leqslant 4$. Looking at the well-known list of such varieties contained in [12], one sees that X is a hypersurface, a complete intersection or a scroll over a curve. When X is either a hypersurface in $\mathbb{P}^r$ with $r \geqslant 4$ or a complete intersection with $\dim(X) \geqslant 3$, we have that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by the hyperplane section D. In these cases there are no effective decompositions $D = D_1 + D_2$ as D generates $\mathrm{Pic}(X)$. If X is a scroll we can use Lemma 3.1. □

Remark 3.3. Note that the previous lemma is still true when $\dim(X) \geqslant 4$. Indeed [12] shows that, when $\dim(X) \geqslant 4$ and $\deg(X) \leqslant 4$, X is a hypersurface, a complete intersection or a scroll over a curve. In the first cases one can argue as above, in the last case one can use Remark 3.2 (ii).

Lemma 3.4. *Assume that there exists an effective divisor H such that $HD^2 = 1$ and $H^2D = 0$, then $X = \mathbb{P}(\mathscr{E})$ for a suitable rank 3 vector bundle over a smooth curve C, D is the tautological divisor T and H is numerically equivalent to a fibre.*

Proof. Obviously the couple (X, D) is neither $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ nor $(\boldsymbol{Q}_3, \mathcal{O}_{\boldsymbol{Q}_3}(1))$, hence, by Theorems (11.2) and (11.7) of [9], to prove that $(X, D) = (\mathbb{P}(\mathscr{E}), T)$, it is enough to show that $K_X + 2D$ is not nef.

Let S be a smooth element of $|D|$; it suffices to show that $(K_X + 2D)_{|S} = K_S + D_{|S}$ is not nef. By assumptions $(H_{|S})^2 = 0$ and $H_{|S}D_{|S} = 1$ so that by Proposition 1 of [18] S is a (blown-up) ruled surface and $H_{|S}$ is a fibre of a (blown-up) ruling of S. It is well known that, in this situation, $K_S \equiv -2\sigma^* C_0 + \alpha\sigma^* f + E_1 + \cdots + E_k$, $D_{|S} \equiv \sigma^* C_0 + \beta\sigma^* f + \gamma_1 E_1 + \cdots + \gamma_k E_k$, $H_{|S} \equiv \sigma^* f$ for suitable integers α, β, γ_i where σ is the blowing up, E_i the exceptional divisors, and C_0 and f generate the numerical equivalence group of the minimal model of S. So we get that $K_S + D_{|S}$ is not nef as $(K_S + D_{|S})H_{|S} = -1$.

Now we have only to show that H is numerically equivalent to a fibre. Since $\mathrm{Num}(X) = \langle T, F\rangle$, we have $H \equiv aT + bF$ for suitable integers a, b. The hypotheses imply that $aT^3 + b = 1$ and $a^2T^3 + 2ab = 0$, as $T^2F = 1$. Hence $a + ab = 0$. If $b = -1$, we have $T^3 = 1$ or $T^3 = 2$, which is not possible, as (X, T) is not $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ nor $(\boldsymbol{Q}_3, \mathcal{O}_{\boldsymbol{Q}_3}(1))$. Therefore $a = 0$, $b = 1$. $\square$

Proposition 3.5. *Suppose that* (i) $d \geqslant 5$, (ii) *there exists an effective divisor H such that $HD^2 = 2$, $H^2D = 0$ and* (iii) *for a generic element $S \in |D|$, either $H_{|S}$ is a smooth conic, or $H_{|S} = h_1 + h_2$ is a singular reduced conic with $h_1^2 = h_2^2 = -1$. Then one of the following happens*:

a) *$X = \mathbb{P}(\mathscr{E})$ for a suitable rank 3 vector bundle over a smooth curve C, D is the tautological divisor T and H is numerically equivalent to two fibres*;

b) *X is a quadric fibration over a smooth curve C, and H is a fibre.*

Proof. As in the previous proof if $K_X + 2D$ is not nef we get that $(X, D) = (\mathbb{P}(\mathscr{E}), T)$. In this case we have only to show that H is numerically equivalent to two fibres. $\mathrm{Num}(X) = \langle T, F\rangle$, so that $H \equiv aT + bF$ for a, b suitable integers. By assumptions we get: $aT^3 + b = 2$ and $a^2T^3 + 2ab = 0$ as $T^2F = 1$. Hence $2a + ab = 0$. If $b = -2$ we have $aT^3 = 4$ hence $d \leqslant 4$ which is not possible. Therefore $a = 0$, $b = 2$.

If $K_X + 2D = 0$, then X is a Del Pezzo 3-fold and $5 \leqslant d \leqslant 8$ (see [9], pp. 45 and 72), moreover: if $d = 5$, X is the intersection of $G(1,4)$ in $\mathbb{P}^9$ with 3 general hyperplanes; if $d = 6$, X is either the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(T_{\mathbb{P}^2})$, embedded by its tautological divisor; if $d = 7$, X is the blow-up σ of $\mathbb{P}^3$ at one point, $D = 2\sigma^*L - E$; ($\langle L\rangle = \mathrm{Pic}(\mathbb{P}^3)$, E the exceptional divisor); if $d = 8$, $(X, D) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. Now it is easy to see that an effective divisor H satisfying the assumptions does not exist in every case but only for $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which is a quadric fibration, and if H is a fibre, i.e. we get case b). From now on we can assume that $K_X + 2D \neq 0$.

If $K_X + 2D$ is nef, it is also effective and spanned, by Corollary 9.2.3 of [3], and we can consider the adjunction morphism $\Phi := \Phi_{|K_X+2D|}$.

Now let S be a smooth element of $|D|$, then $(K_X + 2D)_{|S} = K_S + D_{|S}$, so the restriction $\Phi_{|S}$ is the adjunction morphism for S. By assumptions $(H_{|S})^2 = 0$ and if $H_{|S}$ is a smooth conic, by Proposition 1 of [18], S is a (blown-up) ruled surface and $H_{|S}$ is a fibre of a (blown-up) ruling of S. If $H_{|S} = h_1 + h_2$ is the union of two (-1) lines intersecting at one point, let $\varphi : S \to S'$ be the contraction of h_1. S' is a smooth surface and $\varphi(h_2)$ is a smooth rational curve on S' such that $(\varphi(h_2))^2 = 0$. By Proposition 1 of [18], S' is a (blown-up) ruled surface and $\varphi(h_2)$ is a fibre of a (blown-up) ruling of S'. Then S is a blown-up ruled surface, at one point at least, and $H_{|S}$ is a fibre of a blown-up ruling of S.

It is well known that, in this situation, $K_S \equiv -2\sigma^* C_0 + \alpha\sigma^* f + E_1 + \cdots + E_k$, $D_{|S} \equiv 2\sigma^* C_0 + \beta\sigma^* f + \gamma_1 E_1 + \cdots + \gamma_k E_k$, $H_{|S} \equiv \sigma^* f$ for suitable integers α, β, γ_i where σ is the blowing up, E_i the exceptional divisors, and C_0 and f generate the numerical equivalence group of the minimal model of S. So we get that $(K_S + D_{|S})\sigma^* f = 0$. Now $K_S + D_{|S} \neq 0$ as $K_X + 2D \neq 0$, hence $\Phi_{|S}$ is a fibration, so $\dim(\operatorname{Im}\Phi) = 1$ and by Theorem 11.2.4 of [3] we get case b). $\square$

Now we can prove the following theorem:

Theorem 3.6. *Let D be a very ample divisor on a smooth 3-dimensional variety X. Then D is 2-connected unless X is a scroll over a curve and D is the tautological divisor.*

Proof. Let $D = D_1 + D_2$ be an effective decomposition of D. Put $a = DD_1^2$, $b = DD_1D_2$, $c = DD_2^2$ so that $d = \deg(X) = D^3 = a + 2b + c > 0$. If X is not 2-connected, then $b \leqslant 1$. Note that $(D_1 + D_2)D_1D = a + b > 0$ and $(D_1 + D_2)D_2D = b + c > 0$ therefore $a \geqslant 0$ and $c \geqslant 0$. We can assume that $a > 0$ or $c > 0$ otherwise $\deg(X) \leqslant 2$ and we can use Lemma 3.3. Say $a > 0$ and let S be a smooth element of $|D|$ and let us consider $D_{1|S}$ and $D_{2|S}$. As $\left(D_2 - \frac{b}{a}D_1\right)D_1D = 0$ in $H^2(X, \mathbb{Q})$, we have that $\left(D_{2|S} - \frac{b}{a}D_{1|S}\right)D_{1|S} = 0$ in $H^2(S, \mathbb{Q})$. By looking at the proof of theorem I in [18], we have to consider only two cases:

1) $(D_{2|S})^2 = c = (D_{1|S})^2 = a = 1$, then $d \leqslant 4$ and we can use Lemma 3.3;

2) $(D_{2|S})^2 = c = 0$ and $D_{2|S}D_{1|S} = b = 1$, hence $D_2^2 D = 0$, and $D_2D^2 = D_2(D_1 + D_2)D = 1$, so we can apply Lemmata 3.4 and 3.1 with $H = D_2$. $\square$

Now we prove the main theorem of this section.

Theorem 3.7. *Let X be a 3-dimensional variety. Let D be a very ample divisor of X, and let $D = D_1 + D_2$ be an effective decomposition of D. Then D is 3-connected unless:*

i) *X is a scroll over a smooth curve, $D = T$ is the tautological divisor, $D_1 \equiv T - F$, $D_2 \equiv F$ (F is the numerical class of a fibre), see Theorem 3.6;*

ii) *X is a scroll over a smooth curve, $D = T$ is the tautological divisor, $D_1 \equiv T - 2F$ is effective, $D_2 \equiv 2F$ (see Remark 3.2);*

iii) *X is a quadric fibration over a smooth curve, $D = T_{|X}$, $D_1 \equiv T_{|X} - F_{|X}$, $D_2 \equiv F_{|X}$ where $\langle T, F\rangle = \operatorname{Num}(W)$ and X is a divisor in $W = \mathbb{P}(\mathscr{E})$ (see Lemma 3.2);*

iv) *X is the blowing up at one point of another smooth 3-fold X', $D = \sigma^*\Delta - E$, $D_1 = \sigma^*\Delta - 2E$ is effective, $D_2 = E$, where σ is the blow up, E is the exceptional divisor and Δ is a suitable divisor of X'.*

v) *(X, D) is one of the exceptional cases considered below:*
   – *$X = \mathbb{P}^3$, $D = 2L$ where $\operatorname{Pic}(\mathbb{P}^3) = \langle L\rangle$, $D_1 = D_2 = L$;*

– $X = \mathbb{P}(T_{\mathbb{P}^2}), D = T$ *the tautological divisor*, $D_1 = T - \pi^* l$, $D_2 = \pi^* l$ *where* $\langle l \rangle = \mathrm{Pic}(\mathbb{P}^2)$;
– X *is the blowing up of* $\mathbb{P}^3$ *at one point*, $D = 2\sigma^* L - E$, $D_1 = \sigma^* L$, $D_2 = \sigma^* L - E$, *where* σ *is the blowing up*, E *the exceptional divisor*, $\langle L \rangle = \mathrm{Pic}(\mathbb{P}^3)$.

Proof. Put $a = DD_1^2$, $b = DD_1 D_2$, $c = DD_2^2$, $d = \deg(X) = D^3 = a + 2b + c > 0$ as before. By Theorem 3.6 D is 2-connected unless we are in case i), so we can assume that $b = 2$ and we have $a + 2 > 0$ and $2 + c > 0$. Moreover $a > 0$ or $c > 0$, otherwise $\deg(X) \leqslant 4$ and then we can use Theorem 3.6 and we are done. In any case we can assume $d \geqslant 5$.

Let S be a smooth element of $|D|$. As in the proof of Theorem 3.6, we can assume that $a > 0$ and we get two cases by looking at the proof of theorem I of [18]:

1) $D_{2|S} = \frac{2}{a} D_{1|S}$ in $H^2(S, \mathbb{Q})$.

2) $\left(D_{2|S} - \frac{2}{a} D_{1|S}\right)^2 < 0$.

Case 1): $c = \frac{4}{a}$, hence $c > 0$ so $(a, c) = (1, 4), (2, 2)$ or $(4, 1)$. By the symmetric definition of a and c we have to consider only the first two cases.

Case 1a): $(a, c) = (1, 4)$, hence $d = 9$ and $D_{2|S} \equiv 2D_{1|S}$, so that $D_{|S} \equiv 3D_{1|S}$. Looking at the arithmetic genus of $D_{1|S}$, we have $2p_a(D_{1|S}) - 2 = (K_S + D_{1|S})D_{1|S} = K_S D_{1|S} + 1$. On the other hand, $2g(S) - 2 = (K_S + D_{|S})D_{|S} = 3K_S D_{1|S} + 9 = 6p_a(D_{1|S})$. As $\deg(D_{1|S}) = 3$ and $D_{1|S}$ is a (pure) one-dimensional scheme without embedded components, $p_a(D_{1|S}) \leqslant 1$, so that $g(S) = g(X)$ is 1 or 4. If $g(X) = 1$, X is a Del Pezzo 3-fold (we are assuming that X is not a scroll over a curve, otherwise we are in case i) by Theorem 12.3 of [9]), but there are no such 3-folds with $d = 9$. Hence $g(X) = 4$. By looking at the list of linearly normal varieties of degree 9 contained in [5] we have to check the following:

1a.1) X is the Segre embedding of $\mathbb{P}^1 \times Y$ in $\mathbb{P}^7$ where Y is the cubic surface in $\mathbb{P}^3$, i.e. the blowing up σ of $\mathbb{P}^2$ at 6 points in general position. We can consider X as $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is the rank 2 vector bundle $O_Y(1) \oplus O_Y(1)$ over Y. D is the tautological divisor T and $\mathrm{Pic}(X) = \langle T, \pi^*\sigma^* l, \pi^* E_1, \ldots, \pi^* E_6 \rangle$ where $\mathrm{Pic}(\mathbb{P}^2) = \langle l \rangle$ and $E_1, \ldots, E_6$ are the exceptional divisors. Recall that $L := O_Y(1) = 3\sigma^* l - E_1 - \cdots - E_6$. As usual the only possible effective decomposition is $D = D_1 + D_2$ with $D_1 = T - \pi^*\Delta$ and $D_2 = \pi^*\Delta$ for some effective divisor Δ of Y. By considering the extension: $0 \to L \to \mathscr{E} \to L \to 0$ we get that D_1 can be effective only if $L - \Delta = 3\sigma^* l - E_1 - \cdots - E_6 - \Delta$ is effective. As $c_1(\mathscr{E}) = 2L$ we have $T^2 = \pi^*(2L)T - \pi^*[c_2(\mathscr{E})]$. Now let us compute $2 = DD_1D_2 = T(T - \pi^*\Delta)\pi^*\Delta = 2L\Delta - \Delta^2 = (L - \Delta)\Delta + L\Delta$. Let us put $\Delta = a\sigma^* l + b_1 E_1 + \cdots + b_6 E_6$ for suitable integers b_i and a with $3 \geqslant a \geqslant 0$, then we have $0 < L\Delta = 3a + \sum b_i$ and $0 \leqslant (L - \Delta)\Delta = (3 - a)a + \sum b_i(b_i + 1)$. Hence $L\Delta = 1$ or $L\Delta = 2$. If $L\Delta = 1$ then Δ is a line on Y and it is well known (see e.g. [10], p. 402) that $\Delta = E_i$ or $\Delta = \sigma^* l - E_i - E_j$ or $\Delta = 2\sigma^* l - E_1 - \cdots - \check{E}_i - \cdots - E_6$, but in any case $(L - \Delta)\Delta \neq 1$. If $L\Delta = 2$ and $a = 0, 1$ then it is easy to see that $(L - \Delta)\Delta = 0$ is not possible; if $a = 2$ then $\sum b_i^2 = 2$ hence $b_1^2 = b_2^2 = 1$, $b_i^2 = 0$, $i \geqslant 3$, but in this case $L - \Delta$ cannot be effective; if $a = 3$ then $\sum b_i^2 = 7$ with $b_i \leqslant -1$ to have $L - \Delta$ effective, and this is not possible. So we have no suitable decompositions.

1a.2) X in $\mathbb{P}^7$ is a quadric fibration over $\mathbb{P}^1$. Unfortunately we do not know whether there always exist singular fibres, so we cannot use Lemma 3.2. Let S be a generic hyperplane section of X; then S is the blowing up at 11 simple points of a rational ruled surface $\boldsymbol{F}_e$ with $0 \leqslant e \leqslant 4$ (see [5]). Let C_0 and f be the generators of $\mathrm{Num}(\boldsymbol{F}_e)$, let $\sigma : S \to \boldsymbol{F}_e$ be the blowing up and let $E_1, \dots, E_{11}$ be the exceptional divisors so that $\mathrm{Num}(S) = \langle \sigma^* C_0, \sigma^* f, E_1, \dots, E_{11} \rangle$. By the Lefschetz theorem on hyperplane sections we have that $\mathrm{Pic}(X)$ injects into $\mathrm{Pic}(S)$ (recall that X is regular). We know that $D_{|S} \equiv 2\sigma^* C_0 + (e+7)\sigma^* f - \sum E_i$, (see [5]), if we assume that there exists an effective decomposition such that $D = D_1 + D_2$, $DD_1D_2 = 2$, hence we get an effective decomposition $D_{|S} = D_{1|S} + D_{2|S}$, $D_{1|S}D_{2|S} = 2$. Let us put $D_{1|S} \equiv a\sigma^* C_0 + b\sigma^* f + \sum c_i E_i$ and $D_{2|S} \equiv a'\sigma^* C_0 + b'\sigma^* f + \sum c_i' E_i$ with $a + a' = 2$, $b + b' = e + 7$, $c_i + c_i' = 1$, $a \geqslant 0, a' \geqslant 0, b \geqslant ae, b' \geqslant a'e, 2 = -aa'e + ab' + ba' - \sum c_i c_i'$. If $a = a' = 1$ we have $2 = 7 - \sum c_i c_i'$, as $c_i c_i' \leqslant 0$ for any i it is not possible. If $a = 2$, $a' = 0$ we have $2 = 2b' - \sum c_i c_i'$. As $c_i c_i' \leqslant 0$ for any i it must be $b' = 1$, $c_i c_i' = 0$ for any i, $D_{2|S} \equiv \sigma^* f + \sum c_i' E_i$ or $b' = 0$ and $\sum c_i c_i' = -2$ $D_{2|S} \equiv \sum c_i' E_i$. In the first case, by intersecting $D_{2|S}$ with $D_{|S}$ it is easy to see that $D_{2|S} \equiv \sigma^* f - E_i$ for some i or $D_{2|S} \equiv \sigma^* f$. In the second case $D_{2|S} \equiv E_i$ for some i. In both cases $c \leqslant 0$ in contradiction with our assumptions for Case 1): this variety will be considered in Case 2).

1a.3) X is $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^7$, D is the tautological divisor T, where $\mathscr{E}$ is a rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Pic}(X) = \langle T, \pi^* H_1, \pi^* H_2 \rangle$ where $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle H_1, H_2 \rangle$ and $c_1(\mathscr{E}) = 3H_1 + 3H_2$ (see [5]). As usual the only possible effective decomposition is $D = D_1 + D_2$ with $D_1 = T - a\pi^* H_1 - b\pi^* H_2$ and $D_2 = a\pi^* H_1 + b\pi^* H_2$, a, b non-negative integers. Let us compute: $2 = DD_1D_2 = T(T - a\pi^* H_1 - b\pi^* H_2) \cdot (a\pi^* H_1 + b\pi^* H_2) = (3H_1 + 3H_2)(aH_1 + bH_2) - (aH_1 + bH_2)^2 = 3a + 3b - 2ab$. It would be $3(a+b) = 2(ab+1)$, which implies $(a,b) = (4,2)$ or $(2,4)$. In both cases an easy computation shows that $D_1 T^2 = -c_2(\mathscr{E}) = -9$, hence D_1 is not effective, so we have no suitable decompositions.

Case 1b): $(a,c) = (2,2)$ hence $d = 8$. It is more useful to consider the list of all linearly normal degree 8 varieties contained in [13]. We can exclude hypersurfaces and complete intersections for which we have nothing to prove. We can also exclude varieties which obviously give rise to i) ... iii). Note that the generic hyperplane section of $\mathbb{P}^1 \times \boldsymbol{Q}_3$ in $\mathbb{P}^9$ (where $\boldsymbol{Q}_3$ is the smooth 3-dimensional hyperquadric) is a quadric fibration and by Theorem 3.4 of [12] we can use Lemma 3.2 and we get iii). Note also that the complete intersection of $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7$ and a smooth generic hyperquadric is a quadric fibration over $\mathbb{P}^1$ and, by direct calculation, it is easy to see that X has 8 singular fibres, so we can use Lemma 3.2 and get iii).

We have to check the following other varieties:

1b.1) X is $\mathbb{P}^3$, $D = 2L$ where $\mathrm{Pic}(\mathbb{P}^3) = \langle L \rangle$. It is easy to see that the only effective decomposition is $D = L + L$. Obviously $DD_1D_2 = 2$ and we get an exception.

1b.2) X is the double covering of Z, a generic hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$ embedded in $\mathbb{P}^7$, $D = f^*(H_{1|Z} + H_{2|Z})$ where f is the covering and $Z = H_1 + H_2$ is the hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^3) = \langle H_1, H_2 \rangle$. X is also a quadric fibration by Theorem 4.2 of [12], its fibres are double coverings, branched over conics, of the planes which are fibres of the natural projection $Z \to \mathbb{P}^1$. Such conics are the intersections of the fibres of Z with a $(0,2)$ divisor of $\mathbb{P}^1 \times \mathbb{P}^2$ so that there are 6 singular conics among them. In fact, by looking at the proof of Theorems 4.2 and 4.3 of [12], we know that the fibres of X are linearly equivalent to $f^*(H_{1|Z}) = K_X + 2D$, moreover $K_X = f^*(K_Z) + R$, where R is the ramification divisor of f, and $K_Z = (-H_1 - 3H_2)_{|Z}$ by adjunction theory. Hence the branching divisor is $f_*R = f_*f^*(H_{2|Z}) = 2H_{2|Z}$, i.e. the intersection of Z with a $(0,2)$ divisor of $\mathbb{P}^1 \times \mathbb{P}^2$. Now, by direct calculation, it is easy to see that there are the singular conics.

Any double covering of $\mathbb{P}^2$ branched over a singular conic is a singular quadric, so we can apply Lemma 3.2 and we get iii).

1b.3) X is $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^7$, D is the tautological divisor T, where $\mathscr{E}$ is a rank 2 vector bundle over $\mathbb{P}^2$, for which there exists an exact sequence: $0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{E} \to I_Y(4) \to 0$ and Y is the scheme of 8 distinct points, not belonging to any line or conic (see Theorem 4.1 of [12]). $\mathrm{Pic}(X) = \langle T, \pi^*l \rangle$ where l is the generator of $\mathrm{Pic}(\mathbb{P}^2)$ and $c_1(\mathscr{E}) = 4l$. As usual the only possible decomposition is $D = D_1 + D_2$ with $D_1 = T - h\pi^*l$ and $D_2 = h\pi^*l$ for some positive integer h. By looking at the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2}(-h) \to \mathscr{E}(-h) \to I_Y(4-h) \to 0$ we have that $T - h\pi^*l$ is not effective if $h \geqslant 4$, or $h = 3, 2$ as the points are in general position. The only possibility is $h = 1$ and $h^0(\mathbb{P}^2, \mathscr{E}(-1)) = 2$. Let us compute $DD_1D_2 = T(T - \pi^*l)\pi^*l = T^2\pi^*l - 1 = \pi^*[c_1(\mathscr{E})]T\pi^*l - 1 = 3$.

1b.4) X is $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^6$, D is the tautological divisor T, where $\mathscr{E}$ is a rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ for which there exists an exact sequence: $0 \to \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathscr{E} \to I_Y(3l_1 + 3l_2) \to 0$ where Y is the scheme of 10 points and $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle l_1, l_2 \rangle$. $\mathrm{Pic}(X) = \langle T, \pi^*l_1, \pi^*l_2 \rangle$ and $c_1(\mathscr{E}) = 3l_1 + 3l_2$. As usual the only possible decomposition is $D = D_1 + D_2$ with $D_1 = T - \alpha\pi^*l_1 - \beta\pi^*l_2$ and $D_2 = \alpha\pi^*l_1 + \beta\pi^*l_2$, $\alpha \geqslant 0$, $\beta \geqslant 0$. By looking at the exact sequences $0 \to \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-\alpha l_1 - \beta l_2) \to \mathscr{E}(-\alpha l_1 - \beta l_2) \to \mathscr{I}_Y[(3-\alpha)l_1 + (3-\beta)l_2] \to 0$ and $0 \to \mathscr{I}_Y[(3-\alpha)l_1 + (3-\beta)l_2] \to \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}[(3-\alpha)l_1 + (3-\beta)l_2] \to \mathscr{O}_Y[(3-\alpha)l_1 + (3-\beta)l_2]_{|Y} \to 0$ we get that D_1 is effective only if $\alpha \leqslant 3$ and $\beta \leqslant 3$.

Let us compute $DD_1D_2 = T(T - \alpha\pi^*l_1 - \beta\pi^*l_2)(\alpha\pi^*l_1 + \beta\pi^*l_2) = T(\alpha T\pi^*l_1 + \beta T\pi^*l_2 - 2\alpha\beta F) = [\pi^*(3l_1 + 3l_2)](\alpha\pi^*l_1 + \beta\pi^*l_2) - 2\alpha\beta = 3(\alpha + \beta) - 2\alpha\beta$. It is easy to see that there are no suitable values of α and β such that $DD_1D_2 = 2$.

1b.5) X in $\mathbb{P}^5$ is a regular fibration over $\mathbb{P}^1$ in complete intersections of type $(2,2)$ and the generic hyperplane section $S \in |D|$ is a smooth minimal elliptic surface of Kodaira dimension 1. $g(S) = 7$ and the elliptic fibration over $\mathbb{P}^1$ is given by $|K_S|$. Note that the fibration over $\mathbb{P}^1$ is the rational map Φ associated to $|K_X + D|$ (see also [2]) and K_X is not nef because $(K_{X|S})\Gamma = (K_S - D_{|S})\Gamma < 0$ for any $\Gamma \in |K_S|$. Φ is the Mori contraction of the extremal ray $[R]$, see [16], where R is a suitable rational curve contained in a fibre of X. In this case we have an exact sequence $0 \to \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(X) \to \mathbb{Z}$, hence $\mathrm{Pic}(X) = \langle D, F \rangle$ and any effective decomposition $D = D_1 + D_2$ is such that $D_1 = D - hF$, $D_2 = hF$ for $h \geqslant 1$. Let us compute $DD_1D_2 = D(D - hF)hF = hD^2(D + K_X) = 4h \geqslant 3$.

Case 2): $c < \frac{4}{a}$. Since $c \geqslant -1$ we consider:

Case 2a): $a > 0$, $c = -1$. D_2 is an effective divisor which is a plane in X as $D^2D_2 = 1$, moreover $D_{2|S}$ is a line in S such that $(D_{2|S})^2 = -1$ and by looking at the exact sequence $0 \to \mathscr{O}_X(D_2 - D) \to \mathscr{O}_X(D_2) \to \mathscr{O}_S(D_{2|S}) \to 0$ we get that $h^0(X, D_2) \leqslant h^0(S, D_{2|S}) = 1$, hence $h^0(X, D_2) = 1$.

Therefore $D_2 = E$ is the exceptional divisor of some blow-up, i.e. there exists a smooth 3-fold X' and a point P on X' such that X is the blow-up of X' at P. Let us call σ this blow-up. $\mathrm{Pic}(X) = \langle \sigma^* \mathrm{Pic}(X'), E \rangle$, $D_2 = E$, $D_1 = \sigma^*\Delta + kE$ for some divisor Δ of X' and for some integer k. Let us compute: $2 = DD_1D_2 = (\sigma^*\Delta + (k+1)E) \cdot (\sigma^*\Delta + kE)E = k(k+1)$, hence $k = 1$ or $k = -2$. If $k = 1$, $D = \sigma^*\Delta + 2E$ and $D_{|E}$ would be not very ample which is not possible. If $k = -2$ we get iv).

Case 2b) $a > 0$, $c = 0$. D_2 is an effective divisor whose degree is 2. If D_2 is an irreducible quadric we can apply Proposition 3.5 to $H = D_2$ by using generic hyperplane sections and we get case ii) or iii). If D_2 is an nonreduced quadric, $D_2 = 2P$ and we can apply Lemma 3.4 to $H = P$ and we get case i). If D_2 is the union of two planes P_1 and P_2 disjoint or intersecting at one point we can proceed as follows: let l_1 and l_2 the respective generators of $\mathrm{Num}(P_1)$ and $\mathrm{Num}(P_2)$, we have $D_{|P_1} = l_1$ $D_{|P_2} = l_2$ $D_{1\,|\,P_1} = ml_1$ $D_{1\,|\,P_2} = nl_2$ with m, n non-negative integers (by the effectiveness of D_1) such that $m + n = 2$. Then we can write $D = (D_1 + P_1) + P_2$ and we can compute $D(D_1 + P_1)P_2 = DD_1P_2 = D_{|P_2}D_{1\,|\,P_2} = l_2nl_2 = n$. If $n \leqslant 1$ then D is not 2-connected and we can use Theorem 3.6 to get case i). Hence $n = 2$ and $m = 0$, but in this case we can write $D = (D_1 + P_2) + P_1$ and we can compute: $D(D_1 + P_2)P_1 = DD_1P_1 = D_{|P_1}D_{1\,|\,P_1} = l_1ml_1 = m = 0$, thus D is not 2-connected and we can use Theorem 3.6 in any case.

So we have only one possibility: D_2 is the union of P_1 and P_2 intersecting along a line. By using the above notation now we have that $D(D_1 + P_1)P_2 = n + 1$, $D(D_1 + P_2)P_1 = m + 1$: we can use Theorem 3.6 unless $m = n = 1$. If $m = n = 1$ we have $P_1^2D = D_{|P_1}P_{1\,|\,P_1} = l_1(D - D_1 - P_2)_{|P_1} = l_1(l_1 - l_1 - l_1) = -1$ and $P_2^2D = -1$ in the same way. Therefore a smooth element $S \in |D|$ cuts D_2 along a singular reduced conic $h_1 + h_2$ such that $(h_i)^2 = -1$. By Proposition 3.5 we get ii) or iii).

When $a > 0$, $c > 0$ and $c < \frac{4}{a}$, we have that $6 \leqslant d \leqslant 8$. We have considered all degree 8 varieties in 1b), so we have the following:

Case 2c) $d = 6$. Let us look at the list of all linearly normal degree 6 varieties contained in [12]. As before we can exclude hypersurfaces, complete intersections and varieties which obviously give rise to i) ... iii). We check the following cases:

2c.1) X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Note that X is also a quadric fibration, but all its fibres are smooth. Let $\mathrm{Pic}(X)$ be generated by H_1, H_2, H_3 with $D = H_1 + H_2 + H_3$. It is easy to see that the only possible effective decomposition is $D = D_1 + D_2$ with $D_1 = H_1 + H_2$, $D_2 = H_3$ so that $DD_1D_2 = (H_1 + H_2 + H_3)(H_1 + H_2)H_3 = 2$. By considering the natural projection onto the first factor we see that this case is iii) in spite of the fact that we cannot use Lemma 3.2.

2c.2) X is $\mathbb{P}(T_{\mathbb{P}^2})$ and D is the tautological divisor T. Let l be the generator of $\mathrm{Pic}(\mathbb{P}^2)$ so that $\mathrm{Pic}(X) = \langle T, \pi^*l \rangle$. Let f be the numerical class of a fibre. Now if $D = D_1 + D_2$ as usual, it is easy to see that it must be $D_1 = T - h\pi^*l$ and $D_2 = h\pi^*l$ with $h > 0$. The Euler sequence for $\mathbb{P}^2$, twisted by $\mathcal{O}_{\mathbb{P}^2}(-h)$ yields: $0 \to \mathcal{O}_{\mathbb{P}^2}(-h) \to \mathcal{O}_{\mathbb{P}^2}(1-h)^{\oplus 3} \to T_{\mathbb{P}^2}(-h) \to 0$, so that the only possibility is $h = 1$. We compute: $DD_1D_2 = T(T - \pi^*l)\pi^*l = T^2\pi^*l - Tf = [\pi^*c_1(T_{\mathbb{P}^2})]T\pi^*l - 1 = 3Tf - 1 = 2$ and we get one exceptional case.

2c.3) X is the double covering of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a $(2, 2)$ divisor. D is

$f^*(H_1 + H_2)$ where $f : X \to \mathbb{P}^1 \times \mathbb{P}^2$ is the covering and H_1, H_2 are the generators of $\mathrm{Pic}(X) \simeq \mathrm{Num}(X)$. X is a quadric fibration too by Theorem 3.4 and Corollary 3.3 of [12], and its fibres are double coverings of the planes in $|H_1|$, branched over the conics which are the intersections of the planes with the $(2,2)$ divisor. Among these ones there are surely 6 singular conics (see the proof of Theorem 3.4 of [12]) so that the corresponding quadrics are singular. Now Lemma 3.2 implies iii).

2c.4) X is the Bordiga scroll over $\mathbb{P}^2$, i.e. $X = \mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a rank 2 vector bundle defined by the following extension: $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_Y(4) \to 0$ where Y is the scheme of 10 distinct points in $\mathbb{P}^2$ in general position (see Theorem 4.1 of [12]). D is the tautological divisor T. Let l be the generator of $\mathrm{Pic}(\mathbb{P}^2)$ so that $\mathrm{Pic}(X) = \langle T, \pi^* l\rangle$. Let f be the numerical class of a fibre. It is easy to see that any possible effective decomposition $D = D_1 + D_2$ implies $D_1 = T - h\pi^* l$, $D_2 = h\pi^* l$ with $h > 0$. By combining the exact sequences $0 \to \mathcal{O}_{\mathbb{P}^2}(-h) \to \mathscr{E}(-h) \to \mathscr{I}_Y(4-h) \to 0$ and $0 \to \mathscr{I}_Y(4-h) \to \mathcal{O}_{\mathbb{P}^2}(4-h) \to \mathcal{O}_Y(4-h) \to 0$ we get that D_1 cannot be effective if $h \geqslant 4$. As the 10 points do not belong to any line, conic or cubic, $h^0(\mathbb{P}^2, \mathscr{I}_Y(4-h)) = 0$ for $h = 1, 2, 3$ and we have no suitable decompositions.

Case 2d): $d = 7$. Let us look at the list of all linearly normal degree 7 varieties contained in [12]. As before we can exclude hypersurfaces, complete intersections and varieties which obviously give rise to i) ... iii). We check the following cases:

2d.1) X is the blowing up of $\mathbb{P}^3$ at one point, $D = 2\sigma^* L - E$ where σ is the blowing up, E is the exceptional divisor, $\mathrm{Pic}(\mathbb{P}^3) = \langle L\rangle$, $\mathrm{Pic}(X) = \langle \sigma^* L, E\rangle$. It is easy to see that there are only two possible decompositions: $D_1 = 2\sigma^* L - 2E$, $D_2 = E$ and $D_1 = \sigma^* L - E$, $D_2 = \sigma^* L$. In both cases $DD_1D_2 = 2$. The first one belongs to iv), the second one is an exceptional case.

2d.2) X is the blowing up of $\mathbb{P}^3$ along a smooth curve C which is a complete intersection of type $(2,2)$ or, equivalently, X is a divisor of type $(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^3$ (see Theorem 3.4 of [12]). Hence X is a quadric fibration over $\mathbb{P}^1$ with singular fibres and by Lemma 3.2 we get iii).

2d.3) X is $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^6$, D is the tautological divisor T, where $\mathscr{E}$ is a rank 2 vector bundle over $\mathbb{P}^2$ for which there exists an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_Y(4) \to 0$ and Y is the scheme of 9 distinct points not belonging to any line or conic. We can proceed as in Case 1b.3) and 2c.4).

2d.4) X is $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^5$, D is the tautological divisor T, where $\mathscr{E}$ is a rank 2 vector bundle over Z, the cubic surface in $\mathbb{P}^3$, for which there exists an exact sequence: $0 \to \mathcal{O}_Z \to \mathscr{E} \to \mathscr{I}_Y(2) \to 0$ and Y is a scheme of 5 points. Z is the blowing up of $\mathbb{P}^2$ at 6 distinct points in general position. Let σ be the blowing up, l the generator of $\mathrm{Pic}(\mathbb{P}^2)$ and E_i the exceptional divisors, then $\mathrm{Pic}(Z) = \langle \sigma^* l, E_1, \dots, E_6\rangle$, $\mathrm{Pic}(X) = \langle T, \pi^*\sigma^* l, \pi^* E_i\rangle$. Let $H = 3\sigma^* l - E_1 - \cdots - E_6$ be the hyperplane section of Z, $c_1(\mathscr{E}) = 2H$. As usual the only possible decomposition is $T = D_1 + D_2$ with $D_1 = T - \pi^*\Delta$ and $D_2 = \pi^*\Delta$ for some effective divisor Δ of Z such that $\Delta = \alpha\sigma^* l - \beta_1 E_1 - \cdots - \beta_6 E_6$, $\alpha, \beta_i \in \mathbb{Z}$. Let us compute $DD_1D_2 = T(T - \pi^*\Delta)(\pi^*\Delta) = 2H\Delta - \Delta^2 = \alpha(6-\alpha) + \sum \beta_i(\beta_i - 2) = 2$. Note that $2H - \Delta = (6-\alpha)\sigma^* l + \sum(\beta_i - 2)E_i$ has to be effective, by looking at $0 \to \mathcal{O}_Z(-\Delta) \to \mathscr{E}(-\Delta) \to \mathscr{I}_Y(2H - \Delta) \to 0$, and at $0 \to \mathscr{I}_Y(2H-\Delta) \to \mathcal{O}_Z(2H-\Delta) \to \mathcal{O}_Y(2H-\Delta) \to 0$, because we are assuming that $h^0(X, D_1) = h^0(X, T - \pi^*\Delta) = h^0(Z, \mathscr{E}(-\Delta)) \neq 0$. Hence $0 \leqslant \alpha \leqslant 6$. If $\alpha = 6$ we have

$\beta_i \geqslant 2$ for any i as $2H - \Delta$ is effective, so that it is not possible $DD_1D_2 = 2$. If $\alpha = 5$ or $\alpha = 1$ to get $DD_1D_2 = 2$ we must have $\beta_1 = \beta_2 = \beta_3 = 1$ and $\beta_4 = \beta_5 = \beta_6 = 0$ or 2, but in these cases $2H - \Delta$ or Δ would not be effective. If $\alpha = 4$ or $\alpha = 2$ to get $DD_1D_2 = 2$ we must have $\beta_i = 1$ for any i, but also in this case $2H - \Delta$ or Δ would not be effective. If $\alpha = 3$ it is not possible that $DD_1D_2 = 2$. If $\alpha = 0$ we have $\beta_i \leqslant 0$ for any i as Δ is effective, so that it is not possible that $DD_1D_2 = 2$. So in fact there are no suitable effective decompositions for T.

2d.5) X is the blowing up at a point P of Y, a smooth complete intersection of type $(2,2,2)$ in $\mathbb{P}^6$. $D = \sigma^*H - E$ where σ is the blow-up, E is the exceptional divisor, H is the class of a hyperplane section of Y, $\mathrm{Pic}(Y) = \langle H \rangle$, $\mathrm{Pic}(X) = \langle \sigma^*H, E \rangle$. By imposing $DD_1D_2 = 2$ it is easy to see that the only possible effective decomposition is $D_1 = \sigma^*H - 2E$, $D_2 = E$. D_1 is effective by the existence of tangent hyperplanes at P to Y and we get case iv).

2d.6) X in $\mathbb{P}^5$ is a regular fibration over $\mathbb{P}^1$ in a cubic surface and the generic hyperplane section $S \in |D|$ is a smooth minimal elliptic surface of Kodaira dimension 1. $g(S) = 6$ and the elliptic fibration over $\mathbb{P}^1$ is given by $|K_S|$. As in Case 1b.5) the fibration over $\mathbb{P}^1$ is the rational map Φ associated to $|K_X + D|$ (see also [2]) and K_X is not nef because $(K_{X|S})\Gamma = (K_S - D_{|S})\Gamma < 0$ for any $\Gamma \in |K_S|$. Φ is the Mori contraction of the extremal ray $[R]$, see [16], where R is a suitable rational curve on some fibre of X. In this case we have an exact sequence $0 \to \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(X) \to \mathbb{Z}$, hence $\mathrm{Pic}(X) = \langle D, F \rangle$ and any effective decomposition $D = D_1 + D_2$ is such that $D_1 = D - hF$, $D_2 = hF$ for $h \geqslant 1$. Let us compute $DD_1D_2 = D(D - hF)hF = hD^2(D + K_X) = 3h \geqslant 3$. □

Remark 3.4. One can conjecture that, in general, if X is a smooth n-dimensional variety and D is a very ample divisor on X, then D is n-connected but for a finite list of exceptions. The natural next step in such an investigation is $n = 4$. In this case, as suggested by the referee, it should be not too difficult to prove the conjecture, at least in high degree, using theorem C of [1].

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