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# Higher dimensional polarized varieties with non-integral nefvalue

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**Abstract.** Let $X$ be an $n$-dimensional normal projective variety with terminal, Gorenstein, $\mathbb{Q}$-factorial singularities. Let $L$ be an ample line bundle on $X$. Let $\tau$ be the nefvalue of $(X, L)$. Then we classify $(X, L)$, describing the structure of the nefvalue morphism of $(X, L)$, when $\tau$ satisfies $n - k < \tau < n - k + 1$ and $n \geqslant 2k - 3$, $k \geqslant 4$. In the smooth case, we discuss the case $n = 2k - 4$, $k \geqslant 5$, as well.

**Key words.** Complex polarized $n$-fold, ample line bundle, nefvalue, nefvalue morphism, Gorenstein, terminal, $\mathbb{Q}$-factorial singularities, adjunction theory, special varieties.

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## Introduction

Let $X$ be an $n$-dimensional projective variety with terminal, Gorenstein, $\mathbb{Q}$-factorial singularities and let $L$ be an ample line bundle on $X$. If the canonical bundle $K_X$ is not nef, the Kawamata rationality theorem and the Kawamata–Shokurov basepoint free theorem imply that there is a fraction $\tau = u/v$, with $u$, $v$ positive coprime integers, and a morphism $\phi : X \to W$ with connected fibers onto a normal projective variety $W$ such that $vK_X + uL \approx \phi^* H$ for an ample line bundle $H$ on $W$ and $u \leqslant \max_{w \in W}\{\dim \phi^{-1}(w)\} + 1$. We call $\tau$ the *nefvalue* and $\phi$ the *nefvalue morphism* of $(X, L)$ respectively.

Thus $\tau \leqslant n + 1$ and by the Kobayashi–Ochiai theorem $\tau = n + 1$ if and only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

It is a natural question to classify polarized pairs $(X, L)$ in terms of the numerical values of $\tau$ and the structure of the morphism $\phi$. The range $n - 3 \leqslant \tau < n + 1$ has been extensively studied by several authors. We refer to [4, Chapter 7] for the case $n - 3 < \tau < n + 1$ with $n \geqslant 5$, to [7] for the $n = 4$ case, to [11], [12] for the case $\tau = n - 3$, and to [1] for a refinement in a more general context when $\phi$ is birational with $\tau = n - 1, n - 2$. Recently, the case where $\tau$ is not integer satisfying the condition $n - 4 < \tau < n - 3$, with $n \geqslant 5$ (as well as the case when $\tau$ satisfies $n - 3 < \tau < n - 2$), has been studied in [13].

In this paper we consider the more general situation when $\tau = u/v$ is not integer and satisfies $n-k < \tau < n-k+1$, with $n \geqslant 2k-3$, $k \geqslant 4$, which includes the results of [13]. If $X$ is smooth, we study also the case $n = 2k-4$, $k \geqslant 5$. Following [3], we use a new polarization $A$ on $X$ such that the nefvalue of $(X, A)$ is $u$. Whenever $n \geqslant 2k-3$ we fall in the range up to the second reduction in the adjunction theoretic sense, i.e., $u \geqslant n-2$. If $n = 2k-4$, then $u = n-3$ and we need the third adjunction results [11], as well as the classification [2] of some codimension 2 small contractions which occur.

## 1 Background material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties $V$ (i.e., irreducible and reduced projective schemes), and we follow the usual notation in algebraic geometry. We denote by $\approx$ (respectively $\sim$) the linear (respectively numerical) equivalence of line bundles.

The book [4] is a good reference for standard results and notation of adjunction theory. We also refer to [8] for some facts from Mori theory we use.

The paper is based on the following special case of a major theorem of Kawamata [8].

**Theorem 1.1** (Kawamata rationality theorem). *Let $V$ be a normal projective variety of dimension $n$ with terminal Gorenstein singularities. Let $\pi : V \to Y$ be a projective morphism onto a variety $Y$. Let $L$ be a $\pi$-ample Cartier divisor of $V$. If $K_V$ is not $\pi$-nef then*

$$\tau = \min\{t \in \mathbb{R} \mid K_V + tL \text{ is } \pi\text{-nef}\}$$

*is a rational number. Furthermore expressing $\tau = u/v$ with $u$, $v$ coprime positive integers, we have $u \leqslant b+1$ where $b = \max_{y \in Y}\{\dim_{\mathbb{C}(y)} \pi^{-1}(y)\}$.*

**Definition 1.2.** Let $V$ be a normal variety of dimension $n$ with terminal Gorenstein singularities. Let $\pi : V \to Y$ be a projective morphism onto a variety $Y$. Let $\mathscr{L}$ be a $\pi$-ample Cartier divisor of $V$. Assume that $K_V$ is not $\pi$-nef. Let $\tau$ be the positive rational number given by the Kawamata rationality theorem (1.1).

We say that the rational number $\tau$ is the *$\pi$-nefvalue* of $(V, \mathscr{L})$. If $Y$ is a point, $\tau$ is called the *nefvalue* of $(V, \mathscr{L})$. Note also that, if $Y$ is a point, then $K_V + \tau\mathscr{L}$ is nef and hence by Theorem 1.1 we have that $\tau = u/v$ for two coprime positive integers, $u$ and $v$. Thus by the Kawamata–Shokurov basepoint free theorem we know that $|m(vK_V + u\mathscr{L})|$ is basepoint free for all $m \gg 0$. Therefore for such $m$, $|m(K_V + \tau\mathscr{L})|$ defines a morphism $f : V \to \mathbb{P}_{\mathbb{C}}$. Let $f = s \circ \phi$ be the Remmert–Stein factorization of $f$ where $\phi : V \to W$ is a morphism with connected fibers onto a normal projective variety, $W$, and $s : W \to \mathbb{P}_{\mathbb{C}}$ is a finite-to-one morphism. By [4, (1.1.3)] we know that the morphism, $\phi$, is the same for any $m > 0$ such that $|m(vK_V + u\mathscr{L})|$ is basepoint free, and thus only depends on $(V, \mathscr{L})$. Note that, by [4, (1.1.3)], $s$ is an embedding for $m \gg 0$ and therefore $f = \phi$ for $m \gg 0$. We call $\phi : V \to W$ the *nefvalue morphism* of $(V, \mathscr{L})$. We also know by [4, (1.1.3)] that there is an ample line bundle $H$ on $W$ such that $vK_V + u\mathscr{L} \cong \phi^* H$.

**Remark 1.3.** Let $V$ be as in Theorem 1.1 and $\mathcal{L}$ an ample line bundle on $V$. Let $\tau$ be the nefvalue of $(V, \mathcal{L})$ and $\phi$ the nefvalue morphism of $(V, \mathcal{L})$. Then $\mathcal{L}$ is $\phi$-ample and

$$\tau = \min\{\tau \in \mathbb{R} \mid K_V + t\mathcal{L} \text{ is nef}\} = \min\{t \in \mathbb{R} \mid K_V + t\mathcal{L} \text{ is } \phi\text{-nef}\}.$$

That is $\tau$ coincides with the $\phi$-nefvalue of $(V, \mathcal{L})$.

**Lemma 1.4** ([4, (1.5.5)]). *Let $(V, \mathcal{L})$ be as in Theorem* 1.1. *A real number $\tau$ is the nefvalue of $(V, \mathcal{L})$ if and only if $K_V + \tau\mathcal{L}$ is nef but not ample.*

Let us recall a few results from adjunction theory.

**Lemma 1.5** ([4, (3.3.2)]). *Let $\mathcal{L}$ be a nef and big line bundle on a normal projective variety, $V$, of dimension $n$ with only terminal Gorenstein singularities. Then if $t(aK_V + b\mathcal{L}) \approx \mathcal{O}_V$ for some integers $a > 0$, $b > 0$, $t > 0$ one has $aK_V + b\mathcal{L} \approx \mathcal{O}_V$, and $b/a \leqslant n+1$. If $a$, $b$ are coprime, there exists a nef and big line bundle $M$ on $V$ such that $K_V \approx -bM$, $\mathcal{L} \approx aM$. If $\mathcal{L}$ is ample, then so is $M$.*

**1.6 Special varieties.** Let $V$ be a normal Gorenstein variety of dimension $n$, and let $L$ be an *ample* line bundle on $V$. We say that $V$ is a *Gorenstein–Fano variety* (or simply that $V$ is *Fano*) if $-K_V$ is ample. We say that $(V, L)$ is a *Del Pezzo variety* (respectively a *Mukai variety*) if $K_V \approx -(n-1)L$ (respectively $K_V \approx -(n-2)L$).

We also say that $(V, L)$ is a *scroll* (respectively a *quadric fibration*; respectively a *Del Pezzo fibration*; respectively a *Mukai fibration*) over a normal variety $Y$ of dimension $m$ if there exists a surjective morphism with connected fibers $p : V \to Y$, such that $K_V + (n-m+1)L \approx p^*\mathcal{L}$ (respectively $K_V + (n-m)L \approx p^*\mathcal{L}$; respectively $K_V + (n-m-1)L \approx p^*\mathcal{L}$; respectively $K_V + (n-m-2)L \approx p^*\mathcal{L}$) for some *ample* line bundle $\mathcal{L}$ on $Y$.

We say that a normal Gorenstein $n$-dimensional variety $V$ is a *Fano variety of index i*, if $i$ is the largest positive integer such that $K_V \approx -iH$ for some ample line bundle $H$ on $V$. Note that $i \leqslant n+1$ (see Lemma 1.5 below) and $n - i + 1$ is referred to as the *co-index* of $V$.

We refer to Fujita [5] and [6] for classification results on Del Pezzo varieties. Note that Del Pezzo manifolds are completely described by Fujita [5, I, §8]. We refer to Mukai [9] and [10] for results on Mukai varieties.

We also refer e.g. to [4, (3.1.6)] for a generalized version of Kobayashi–Ochiai theorem (characterizing projective spaces and quadrics) which we systematically use in the sequel.

The following useful fact was noted in [13, (1.1)]. It is an easy consequence of the Kawamata rationality theorem (1.1), and the assumption that $\tau$ is not integer.

**Lemma 1.7** (Zhao). *Let $V$ be an $n$-dimensional normal projective variety with Gorenstein, terminal, $\mathbb{Q}$-factorial singularities. Let $\mathcal{L}$ be an ample line bundle on $V$. Let $\tau$ be the nefvalue of $(V, \mathcal{L})$. By the Kawamata rationality theorem, $\tau = u/v$, with $u$, $v$*

*positive coprime integers. Assume $n - k < \tau < n - k + 1$ for positive $k < n$. Then $2 \leqslant v \leqslant \frac{n}{n-k}$ and $\tau = n - k + \frac{i}{v}$ for some positive integer $i < v$ and $i$, $v$ are coprime.*

Finally, let us recall for reader's convenience the main results from [3].

**Lemma 1.8** ([3, (1.1), (1.2)]). *Let $X$ be a normal projective variety with terminal Gorenstein singularities. Let $L$ be an ample line bundle on $X$. Let $\varphi : X \to W$ be a surjective morphism onto a normal variety $W$. Assume that $\varphi$ has at least one positive dimensional fiber and that $vK_X + uL \approx \varphi^* H$, for some ample line bundle $H$ on $W$ and coprime integers $u$, $v$.*

1. *There exist positive integers $a$, $b$ such that $av - bu = 1$;*
2. *Let $A := bK_X + aL$. Then $A$ is ample, $K_X + uA \approx \varphi^*(aH)$ and $u$ is the nefvalue of $(X, A)$.*

**Theorem 1.9** ([3, (1.4)]). *Let $X$ be a projective variety of dimension $n$ with Gorenstein rational singularities. Assume $K_X$ not nef. Let $L$ be an ample line bundle on $X$. Let $\tau = u/v$ be the nefvalue of $(X, L)$, $u$, $v$ coprime positive integers. Let $\phi : X \to W$ be the nefvalue morphism of $(X, L)$. Let $A := bK_X + aL$ be an ample line bundle on $X$ given by Lemma* 1.8.

1. *Assume that $u = \max_{w \in W}\{\dim \phi^{-1}(w)\} + 1$. Then $(X, A)$ is a scroll over $W$ under $\phi$. If $X$ is smooth, or more generally if $\mathrm{cod}_X \mathrm{Sing}(X) > \dim W$, then $(X, A)$ is in fact a $\mathbb{P}^{u-1}$-bundle over $W$ under $\phi$. Furthermore $\phi$ is a fiber type contraction of an extremal ray.*
2. *Assume that $u = \max_{w \in W}\{\dim \phi^{-1}(w)\}$. If $\phi$ is not birational, then either*
   (a) *$(X, A)$ is a scroll over $W$ under $\phi$; or*
   (b) *$(X, A)$ is a quadric fibration over $W$ under $\phi$, and all fibers are equidimensional.*

   *If $\phi$ is birational, $X$ is smooth, and $u \geqslant (n+1)/2$, then*
   (c) *$\phi$ is the simultaneous contraction of a finite number of extremal rays and is an isomorphism outside of $\phi^{-1}(\mathscr{B})$ where $\mathscr{B}$ is an algebraic subset of $W$ which is the disjoint union of irreducible components of dimension $n - u - 1$. Let $B$ be an irreducible component of $\mathscr{B}$ and let $E = \phi^{-1}(B)$. The general fiber, $\Delta$, of the restriction, $\phi_E$ of $\phi$ to $E$ is a linear $\mathbb{P}^u$, $(\Delta, A_\Delta) \cong (\mathbb{P}^u, \mathcal{O}_{\mathbb{P}^u}(1))$, $\mathcal{N}_{E/X|\Delta} \cong \mathcal{O}_{\mathbb{P}^u}(-1)$ and $W$ is factorial with terminal singularities.*

Note that if $X$ has terminal singularities, then $X$ has rational singularities and it is a general fact that $\mathrm{cod}_X \mathrm{Sing}(X) \geqslant 3$, so that the above condition $\mathrm{cod}_X \mathrm{Sing}(X) > \dim W$ is always true if $\dim W \leqslant 2$.

## 2 The case of dimension $n \geqslant 2k - 3$

The following theorem includes the results of [13], which correspond to the cases $k = 3, 4$.

**Theorem 2.1.** *Let $X$ be a normal projective variety of dimension $n \geqslant 2k-3$, $k \geqslant 4$, with terminal, Gorenstein, $\mathbb{Q}$-factorial singularities. Let $L$ be an ample line bundle on $X$. Let $\tau$ be the nefvalue of $(X, L)$ and let $\phi : X \to W$ be the nefvalue morphism of $(X, L)$. Assume $n-k < \tau < n-k+1$. Then $(X, L)$ is described as follows:*

1. $n = 2k$, $\tau = \frac{n+1}{2}$, $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$;

2. $n = 2k-1$, $\tau = \frac{n}{2}$, $A := K_X + kL$ *is ample and either*:
   (a) $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(2))$, $\mathcal{Q}$ *a hyperquadric in* $\mathbb{P}^{n+1}$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is a* $\mathbb{P}^{n-1}$*-bundle over a smooth curve, and* $\phi$ *is a fiber type contraction of an extremal ray*;

3. $n = 2k-2$, $\tau = \frac{n-1}{2}$, $A := K_X + (k-1)L$ *is ample and either*:
   (a) $(X, A)$ *is a Del Pezzo variety*, $L \approx 2A$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is a quadric fibration over a smooth curve and all fibers are equidimensional*, *or*
   (c) $(X, A)$, $\phi : X \to W$, *is a scroll over a normal surface*; *or*
   (d) $(X, A)$, $\phi : X \to W$, *is a* $\mathbb{P}^{n-2}$*-bundle over a normal surface*; *furthermore* $\phi$ *is a fiber type contraction of an extremal ray*; *or*
   (e) $\phi : X \to W$ *is the simultaneous contraction to distinct smooth points of disjoint divisors* $E_i \cong \mathbb{P}^{n-1}$ *such that* $E_i \subset \mathrm{Reg}(X)$, $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ *and* $A_{E_i} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for* $i = 1, \ldots, t$. *Furthermore* $A_W := (\phi_* A)^{**}$ *and* $K_W + (n-1)A_W$ *are ample and* $K_X + (n-1)A \approx \phi^*(K_W + (n-1)A_W)$;

4. $n = 2k-3$, $\tau = \frac{n-2}{2}$, $A := K_X + (k-2)L$ *is ample and either*:
   (a) $(X, A)$ *is a Mukai variety*, $L \approx 2A$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is a Del Pezzo fibration over a smooth curve*; *or*
   (c) $(X, A)$, $\phi : X \to W$, *is a quadric fibration over a normal surface*; *or*
   (d) $(X, A)$, $\phi : X \to W$, *is a scroll over a normal threefold*; *or*
   (e) $\phi : X \to W$ *is the simultaneous contraction of a finite number of extremal rays and is an isomorphism outside of* $\phi^{-1}(Z)$, *where* $Z$ *is an algebraic subset of* $W$ *such that* $\dim Z \leqslant 1$. *Moreover* $\phi$ *is the blowing up of* $W$ *along* $Z$ *and the following cases can occur*:
      i. *The* 1*-dimensional component* $Z_1$ *of* $Z$ *is the disjoint union of locally complete intersection curves and it is contained in the regular set of* $W$; *or*
      ii. *If* $z$ *is a* 0*-dimensional component of* $Z$, *then* $\phi^{-1}(z)$ *is an irreducible reduced divisor and either* $(E, A_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ *with* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$, *or* $(E, A_E) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ *a* (*possibly singular*) *hyperquadric in* $\mathbb{P}^n$, *with* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathcal{Q}}(-1)$;

5. $n = 6$, $\tau = \frac{7}{3}$, $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(3))$;

6. $n = 9$, $\tau = \frac{10}{3}$, $(X, L) \cong (\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(3))$;

7. $n = 7$, $\tau = \frac{8}{3}$, $(X, L) \cong (\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(3))$;

8. $n = 7$, $\tau = \frac{7}{3}$ *and either*:

(a) $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(3))$, $\mathcal{Q}$ *hyperquadric in* $\mathbb{P}^8$; *or*
(b) $A := 2K_X + 5L$ *is ample*, $(X, A)$, $\phi : X \to W$, *is a* $\mathbb{P}^6$*-bundle over a smooth curve*; *moreover* $\phi$ *is a fiber type contraction of an extremal ray*;

9. $n = 5$, $\tau = \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{6}{5}$ *and* $(X, L)$ *is described as in* [13, (1.2), (iv)].

*Proof.* Throughout the proof we use over and over all the results from §1 without always explicitly referring to them. Let $\tau = \frac{u}{v}$, where $v \geqslant 2$ since $\tau$ is not integer. By Lemma 1.8 there exist positive integers $a$, $b$ such that $av - bu = 1$ and the line bundle $A := bK_X + aL$ is ample. Thus

$$K_X + uA = a(vK_X + uL) \tag{1}$$

and hence $K_X + uA \approx \phi^*(\mathscr{H})$ for some ample line bundle $\mathscr{H}$ on $W$ and $u$ is the nef-value of $(X, A)$.

We put $m(\phi) := \max_{w \in W}\{\dim \phi^{-1}(w)\}$ and, if $\phi$ is not birational, we denote by $f(\phi)$ the dimension of the general fiber $F$. Note that in this case $K_F + uA_F \approx \mathcal{O}_F$ and hence

$$u \leqslant f(\phi) + 1 \leqslant m(\phi) + 1 \leqslant n + 1. \tag{2}$$

Let us first consider the case $v = 2$. Then, by Lemma 1.7,

$$\tau = n - k + \frac{1}{2} = \frac{2n - 2k + 1}{2}$$

and hence, recalling the assumption on $n$, one has $n + 1 \geqslant u = 2n - 2k + 1 \geqslant n - 2$.

If $u = n + 1$, then $n = 2k$, $A = K_X + (k+1)L$, $(X, A) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and we are in Case 1.

If $u = n$, or $n = 2k - 1$, we have $\tau = \frac{n}{2}$ and $A = K_X + kL$. Then $K_X + nA = k(2K_X + nL)$ by (1). Since $K_X + nA$ nef and big implies $K_X + nA$ ample by [4, (7.2.3)], we conclude that $\phi$ is not birational. Hence we have $u = n \leqslant m(\phi) + 1 \leqslant n + 1$. Then either $m(\phi) = n$ and $\phi$ contracts $X$ to a point, or $u = n = m(\phi) + 1$. In the first case $2K_X + nL \approx \mathcal{O}_X$, so that $-K_X \approx nM$, $L \approx 2M$ for some ample line bundle $M$ on $X$ (and hence $A \approx (-n + 2k)M = M$), and therefore $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(2))$ as in Case 2 (a). In the latter case, by Theorem 1.9, $(X, A)$ is a $\mathbb{P}^{n-1}$-bundle over $W$ as in Case 2 (b).

If $u = n - 1$, or $n = 2k - 2$, then $\tau = \frac{n-1}{2}$ and $A = K_X + (k-1)L$. If $\phi$ is not birational, we have $u = n - 1 \leqslant m(\phi) + 1$, and therefore $n - 2 \leqslant m(\phi) \leqslant n$. If $m(\phi) = n$, then $\phi$ contracts $X$ to a point, and hence $2K_X + (n-1)L \approx \mathcal{O}_X$. Thus, since $n - 1$ is odd, there exists an ample line bundle $M$ on $X$ such that $K_X \approx -(n-1)M$, $L \approx 2M$ (and hence $A \approx (1 - n + 2(k-1))M = M$) and therefore $(X, A)$ is a Del Pezzo variety as in Case 3 (a). Let $m(\phi) = n - 1$. Thus (2) yields $n - 1 \leqslant f(\phi) + 1 \leqslant n$ and hence either $f(\phi) = n - 1$ or $f(\phi) = n - 2$. Since $u = m(\phi)$, and recalling that $K_X + (n-1)A \approx \phi^*(\mathscr{H})$, we conclude from Theorem 1.9 that $(X, A)$, $\phi : X \to W$, is either a quadric fibration over a smooth curve, or a scroll over a normal surface as in Cases 3 (b), 3 (c). If $m(\phi) = n - 2$, Inequality (2) gives $u = n - 1 = m(\phi) + 1$, $f(\phi) = n - 2$ and $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^{n-2}$-bundle as in Case 3 (d).

If $\phi$ is birational, since $u = n - 1$ is the nefvalue of $(X, A)$, the structure theorem [4, (7.3.2)] applies to give Case 3 (e).

Next, assume $u = n - 2$, or $n = 2k - 3$. Then $\tau = \frac{n-2}{2}$ and $A = K_X + (k - 2)L$. Assume $\phi$ is not birational. We have $u = n - 2 \leqslant m(\phi) + 1$, so that $n - 3 \leqslant m(\phi) \leqslant n$. If $m(\phi) = n$, then $\phi$ contracts $X$ to a point, and hence $2K_X + (n - 2)L \approx \mathcal{O}_X$. Thus, since $n - 2$ is odd, there exists an ample line bundle $M$ on $X$ such that $K_X \approx -(n - 2)M$, $L \approx 2M$ (so that $A \approx (2 - n + 2(k - 2))M = M$) and therefore $(X, A)$ is a Mukai variety as in Case 4 (a). Let $m(\phi) = n - 1$. Then (2) yields $n - 2 \leqslant f(\phi) + 1 \leqslant n$ and hence $n - 3 \leqslant f(\phi) \leqslant n - 1$. Let $f(\phi) = n - 1$ (respectively $f(\phi) = n - 2$; respectively $f(\phi) = n - 3$). Thus, since $K_X + (n - 2)A \approx \phi^*(\mathscr{H})$, we see that $(X, A)$, $\phi : X \to W$, is a Del Pezzo fibration over $W$ as in Case 4 (b) (respectively $(X, A)$, $\phi : X \to W$, is a quadric fibration over $W$ as in Case 4 (c); respectively $(X, A)$, $\phi : X \to W$, is a scroll over $W$ as in Case 4 (d)). Assume now $m(\phi) = n - 2$. Then $n - 2 \leqslant f(\phi) + 1 \leqslant n - 1$, and hence either $f(\phi) = n - 2$, or $f(\phi) = n - 3$. Since $u = m(\phi)$, we conclude from Theorem 1.9 that $(X, A)$, $\phi : X \to W$, is either a quadric fibration over a normal surface (and all fibers are equidimensional in this case) as in 4 (c), or a scroll over a normal threefold as in 4 (d). Finally, let $m(\phi) = n - 3$. Then we find $f(\phi) = n - 3$ and, since $u = m(\phi) + 1$, $(X, A)$, $\phi : X \to W$, is again a scroll over a normal threefold as in Case 4 (d) (and in fact a linear $\mathbb{P}^{n-3}$-bundle if $X$ is smooth by Theorem 1.9).

If $\phi$ is birational, since $u = n - 2$ is the nefvalue of $(X, A)$, the structure theorem [1, Theorem 3] (see also [4, (7.5.3)] in the smooth case) applies to give Case 4 (e).

From now on, we may assume $v \geqslant 3$. Lemma 1.7 yields the inequality

$$3 \leqslant v \leqslant \frac{n}{n-k}. \tag{3}$$

If $n \geqslant 2k - 1$, we find $3k \geqslant 2n \geqslant 2(2k - 1)$, or $k \leqslant 2$, contradicting our assumption on $k$.

Let $n = 2k - 2$. Then $3k \geqslant 2n \geqslant 4k - 4$, or $k \leqslant 4$. Hence $k = 4$, $n = 6$ and $v = 3$. Therefore Lemma 1.7 yields $\tau = 2 + \frac{i}{3}$, with $i = 1, 2$. If $i = 2$ one has $\tau = \frac{8}{3}$, $u = 8$, which contradicts the bound $u \leqslant 7$ from the Kawamata rationality theorem (1.1). Thus $i = 1$, $\tau = \frac{7}{3}$, and hence $u = 7 = m(\phi) + 1$. Then $m(\phi) = 6$, so that $\phi$ contracts $X$ to a point. In this case $3K_X + 7L \approx \mathcal{O}_X$, and we are in Case 5.

Assume now $n = 2k - 3$. Inequality (3) gives now $n \leqslant 9$, so that $n = 9, 7, 5$ by parity.

Let $n = 9$. Then $k = 6$ and $v = 3$. Therefore $\tau = 3 + \frac{i}{3}$ with $i = 1, 2$. If $i = 2$, then $\tau = \frac{11}{3}$, $u = 11$, contradicting the bound $u \leqslant 10$ from Theorem 1.1. Thus $i = 1$, $\tau = \frac{10}{3}$ and hence $u = 10 = m(\phi) + 1$, so that $m(\phi) = 9$ and $\phi$ contracts $X$ to a point. In this case $3K_X + 10L \approx \mathcal{O}_X$, and we are in Case 6.

Let $n = 7$. Then $k = 5$ and again $v = 3$ by (3). Therefore $\tau = 2 + \frac{i}{3}$ with $i = 1, 2$. If $i = 2$ we have $\tau = \frac{8}{3}$, and $u = 8 = m(\phi) + 1$. Thus $m(\phi) = 7$, so that $\phi$ contracts $X$

to a point. In this case $3K_X + 8L \approx \mathcal{O}_X$, and we are in Case 7. If $i = 1$, then $\tau = \frac{7}{3}$ and $u = 7 \leqslant m(\phi) + 1 \leqslant 8$, so that either $u = m(\phi) = 7$, or $u = 7 = m(\phi) + 1$. If $u = m(\phi) = 7$, $\phi$ contracts $X$ to a point and therefore $3K_X + 7L \approx \mathcal{O}_X$, so we are in Case 8 (a). Let $u = 7 = m(\phi) + 1$. Note that $A = 2K_X + 5L$ in this case. If $\phi$ is not birational, Theorem 1.9 applies to say that $(X, A)$ is a $\mathbb{P}^6$-bundle over $W$ under $\phi$ as in Case 8 (b). We claim that $\phi$ is not birational. Indeed, otherwise, we conclude from Lemma 1.8 that $K_X + 7A$ (=$5(3K_X + 7L)$) is nef and big and not ample. Since $n = 7$, this contradicts [4, (7.2.3)].

Let $n = 5$. Then $k = 4$ and $3 \leqslant v \leqslant 5$ by (3). The relations $\tau = \frac{u}{v} = 1 + \frac{i}{v}$, $(i, v) = 1$, $i < v$, and $u \leqslant n + 1 = 6$ yield for $\tau$ the values $\frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{3}{2}, \frac{6}{5}$. If $\tau = \frac{3}{2}$ we are in the previous Case 4 of the statement. The remaining cases are described in [13, (1.2), (iv)], to which we refer for details. □

**Remark 2.2.** Note that, if $X$ is smooth, in the scroll Cases 3 (c) and 4 (d) of Theorem 2.1, $\phi$ is a contraction of an extremal ray by [4, (14.1.1)]. Furthermore, if $A$ is very ample, then $\phi$ is a linear $\mathbb{P}^{n-\dim(W)}$-bundle by [4, (14.1.3)].

## 3 The case of dimension $n = 2k - 4$

In this section we deal with the case of a manifold of dimension $n = 2k - 4$. The smoothness assumption is needed to use the Ionescu–Wiśniewski inequality (see e.g. [4, (6.3.6)]).

**Theorem 3.1.** *Let $X$ be a smooth projective variety of dimension $n = 2k - 4$, $k \geqslant 5$. Let $L$ be an ample line bundle on $X$. Let $\tau$ be the nefvalue of $(X, L)$ and let $\phi : X \to W$ be the nefvalue morphism of $(X, L)$. Assume $n - k < \tau < n - k + 1$. Then $(X, L)$ is described as follows:*

1. $\tau = \frac{n-3}{2}$, *$A := K_X + (k-3)L$ is ample, and either:*
   (a) *$(X, A)$ is a Fano variety of co-index 4, i.e., $K_X \approx -(n-3)A$, $L \approx 2A$; or*
   (b) *$(X, A)$, $\phi : X \to W$, is a Mukai fibration over a smooth curve; or*
   (c) *$(X, A)$, $\phi : X \to W$, is a Del Pezzo fibration over a normal surface; or*
   (d) *$(X, A)$, $\phi : X \to W$, is a quadric fibration over a normal threefold; or*
   (e) *$(X, A)$, $\phi : X \to W$, is a scroll over a normal fourfold; or*
   (f) *$(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^{n-4}$-bundle over a normal fourfold; furthermore $\phi$ is a fiber type contraction of an extremal ray; or*
   (g) *$n \geqslant 8$, $\phi$ is the simultaneous contraction of a finite number of extremal rays and is an isomorphism outside of $\phi^{-1}(\mathcal{B})$ where $\mathcal{B}$ is an algebraic subset of $W$ which is the disjoint union of irreducible components of dimension 2. Let $B$ be an irreducible component of $\mathcal{B}$ and let $E = \phi^{-1}(B)$. The general fiber, $\Delta$, of the restriction, $\phi_E$ of $\phi$ to $E$ is a linear $\mathbb{P}^{n-3}$, $(\Delta, A_\Delta) \cong (\mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1))$, $\mathcal{N}_{E/X|\Delta} \cong \mathcal{O}_{\mathbb{P}^{n-3}}(-1)$ and $W$ is factorial with terminal singularities; or*
   (h) *$n = 6$. Let $R$ be an extremal ray subordinated to $\phi$, i.e., $(K_X + 3A) \cdot R = 0$. Let $E$ be an irreducible component of the exceptional locus of the contraction $\rho : X \to Y$ of $R$. Let $\Delta$ be any irreducible component of any fiber of the*

*restriction, $\rho_E$, of $\rho$ to $E$. Thus $\rho$ is a birational third adjoint contraction with supporting divisor $K_X + 3A$, and either*:

i. *$\rho$ is of divisorial type, $E$ is a prime divisor and $E$, $\Delta$ are described as in* [11, Theorem 1.3]; *or*

ii. $E = \Delta$, $E \cong \mathbb{P}^4$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}(-1)$.

2. $n = 12$, $\tau = \frac{13}{3}$, $(X, L) \cong (\mathbb{P}^{12}, \mathcal{O}_{\mathbb{P}^{12}}(3))$;

3. $n = 10$ *and either*:
   (a) $\tau = \frac{11}{3}$, $(X, L) \cong (\mathbb{P}^{10}, \mathcal{O}_{\mathbb{P}^{10}}(3))$; *or*
   (b) $\tau = \frac{10}{3}$, $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(3))$, *$\mathcal{Q}$ a hyperquadric in* $\mathbb{P}^{11}$; *or*
   (c) $\tau = \frac{10}{3}$, *$A := 2K_X + 7L$ is ample, $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^9$-bundle over a smooth curve, and $\phi$ is a fiber type contraction of an extremal ray*;

4. $n = 8$, $\tau = \frac{7}{3}$, *$A := 2K_X + 5L$ is ample and either*:
   (a) *$(X, A)$ is a Del Pezzo variety, $L \approx 3A$*; *or*
   (b) *$(X, A)$, $\phi : X \to W$, is a quadric fibration over a nonsingular curve, and all fibers are equidimensional*; *or*
   (c) *$(X, A)$, $\phi : X \to W$, is a scroll over a normal surface*; *or*
   (d) *$(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^6$-bundle over a normal surface, and $\phi$ is a fiber type contraction of an extremal ray*; *or*
   (e) *$\phi : X \to W$ is the simultaneous contraction to distinct smooth points of disjoint divisors $E_i \cong \mathbb{P}^7$ such that $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}^7}(-1)$ and $A_{E_i} \cong \mathcal{O}_{\mathbb{P}^7}(1)$ for $i = 1, \ldots, t$. Furthermore $A_W := (\phi_* A)^{**}$ and $K_W + 7A_W$ are ample and $K_X + 7A \approx \phi^*(K_W + 7A_W)$*;

5. $n = 8$, $\tau = \frac{8}{3}$, *$A := K_X + 3L$ is ample and either*:
   (a) $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(3))$, *$\mathcal{Q}$ a hyperquadric in* $\mathbb{P}^9$; *or*
   (b) *$(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^7$-bundle over a nonsingular curve, and $\phi$ is a fiber type contraction of an extremal ray*;

6. $n = 8$, $\tau = \frac{9}{4}$, $(X, L) \cong (\mathbb{P}^8, \mathcal{O}_{\mathbb{P}^8}(4))$;

7. $n = 6$, $\tau = \frac{4}{3}$, *$A := 2K_X + 3L$ is ample and either*:
   (a) *$(X, A)$ is a Mukai variety, $L \approx 3A$*; *or*
   (b) *$(X, A)$, $\phi : X \to W$, is a Del Pezzo fibration over a smooth curve*; *or*
   (c) *$(X, A)$, $\phi : X \to W$, is a quadric fibration over a normal surface*; *or*
   (d) *$(X, A)$, $\phi : X \to W$, is a scroll over a normal threefold*; *or*
   (e) *$(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^3$-bundle over a normal threefold, and $\phi$ is the contraction of an extremal ray*; *or*
   (f) *$\phi$ is the simultaneous contraction of a finite number of extremal rays and is an isomorphism outside of $\phi^{-1}(\mathcal{B})$ where $\mathcal{B}$ is an algebraic subset of $W$ which is the disjoint union of irreducible components of dimension* 1. *Let $B$ be an irreducible component of $\mathcal{B}$ and let $E = \phi^{-1}(B)$. The general fiber, $\Delta$, of the restriction, $\phi_E$, of $\phi$ to $E$ is a linear $\mathbb{P}^4$, $(\Delta, A_\Delta) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$, $\mathcal{N}_{E/X\,|\,\Delta} \cong \mathcal{O}_{\mathbb{P}^4}(-1)$ and $W$ is factorial with terminal singularities*;

8. $n = 6$, $\tau = \frac{5}{3}$, $A := K_X + 2L$ *is ample and either*:
   (a) $(X, A)$ *is a Del Pezzo variety*, $L \approx 3A$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is a quadric fibration over a smooth curve, and all fibers are equidimensional*; *or*
   (c) $(X, A)$, $\phi : X \to W$, *is a scroll over a normal surface*; *or*
   (d) $(X, A)$, $\phi : X \to W$, *is a* $\mathbb{P}^4$*-bundle over a normal surface, and* $\phi$ *is the contraction of an extremal ray*; *or*
   (e) $\phi$ *is the simultaneous contraction of a finite number of extremal rays and is an isomorphism outside of* $\phi^{-1}(\mathscr{B})$ *where* $\mathscr{B}$ *is the union of a finite set of points. For each point* $b \in \mathscr{B}$ *let* $E = \phi^{-1}(b)$. *Then* $(E, A_E) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^5}(-1)$ *and* $W$ *is factorial with terminal singularities*;
9. $n = 6$, $\tau = \frac{5}{4}$, $A := 3K_X + 4L$ *is ample and either*:
   (a) $(X, A)$ *is a Del Pezzo variety*, $L \approx 4A$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is as in one of cases* 8 (b), 8 (c), 8 (d), 8 (e) *respectively*;
10. $n = 6$, $\tau = \frac{7}{4}$, $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(4))$;
11. $n = 6$, $\tau = \frac{6}{5}$, $A := 4K_X + 5L$ *is ample and either*:
   (a) $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(5))$, $\mathcal{Q}$ *a hyperquadric in* $\mathbb{P}^7$; *or*
   (b) $(X, A)$, $\phi : X \to W$, *is a* $\mathbb{P}^5$*-bundle over a nonsingular curve*; *furthermore* $\phi$ *is a contraction of an extremal ray*;
12. $n = 6$, $\tau = \frac{7}{5}$, $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(5))$;
13. $n = 6$, $\tau = \frac{7}{6}$, $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(6))$.

*Proof.* Throughout the proof we use over and over all the results from §1 without always explicitly referring to them. Let $\tau = \frac{u}{v}$, where $v \geqslant 2$ since $\tau$ is not integer. By Lemma 1.8 there exist positive integers $a$, $b$ such that $av - bu = 1$ and the line bundle $A := bK_X + aL$ is ample. Thus $K_X + uA = a(vK_X + uL)$ and hence $K_X + uA \approx \phi^*(\mathscr{H})$ for some ample line bundle $\mathscr{H}$ on $W$ and $u$ is the nefvalue of $(X, A)$.

We put $m(\phi) := \max_{w \in W}\{\dim \phi^{-1}(w)\}$ and, if $\phi$ is not birational, we denote by $f(\phi)$ the dimension of the general fiber $F$. Note that in this case $K_F + uA_F \approx \mathcal{O}_F$ and hence Inequality (2) holds true.

*Step* I: Let us first consider case $v = 2$. From Lemma 1.7 we have

$$\tau = n - k + \frac{1}{2} = \frac{2k - 7}{2} = \frac{n - 3}{2}.$$

Therefore $u = n - 3$ and hence $A = K_X + (k - 3)L$.

If $\phi$ is not birational, then the same arguments as in the proof of Theorem 2.1 lead to Cases (a) to (f) in 1.

Thus we can assume $\phi$ birational. If $n \geqslant 8$, we are in the range $u \geqslant \frac{n+1}{2}$ and therefore we are in Case 1 (f) by using Theorem 1.9, (c).

Then we can assume $n = 6$. Hence $u = 3$ is the nefvalue of $(X, A)$ and $K_X + 3A \approx \phi^*(\mathscr{H})$. Let $R$ be an extremal ray subordinated to $K_X + 3A$ (i.e., $(K_X + 3A) \cdot R = 0$) and let $E$ be an irreducible component of the exceptional locus of the contraction $\rho =$

$\mathrm{cont}_R : X \to Y$ of $R$. Let $\Delta$ be any irreducible component of any fiber of the restriction, $\rho_E$, of $\rho$ to $E$. Then, since $X$ is smooth, the Ionescu–Wiśniewski inequality (see e.g. [4, (6.3.6)]) yields $\dim E + \dim \Delta \geqslant \dim X + \ell(R) - 1$, where $\ell(R)$ denotes the length of $R$. In our case $\ell(R) = 3$ (cf. [4, (4.2.15)]), so that the above inequality gives

$$\dim E + \dim \Delta \geqslant 8. \tag{4}$$

Thus $2 \dim E \geqslant 8$, or $\dim E \geqslant 4$. Note that since $K_X + 3A$ is the supporting divisor of $\rho$, $\rho$ is a 6-dimensional third reduction in the sense of [11]. If $\dim E = 5$, i.e., if $\rho$ is of divisorial type, then $E$, $\Delta$ are completely described in [11, Theorem 1.3]. We are in Case 1 (h), i. If $\dim E = 4$, Inequality (4) yields $\dim \Delta \geqslant 4$, which implies $\Delta = E$ and hence $\rho$ contracts $E$ to a point. Thus [2, (5.8.1)] applies to give Case 1 (h), ii.

Thus from now on we can assume $v \geqslant 3$. Inequality (3) gives for $n$ the possible values $n = 12, 10, 8, 6$.

If $n = 12, 10$, the same arguments as in the proof of Theorem 2.1 (cases $n = 9, 7$) easily lead to Cases 2, 3.

*Step* II: The case $n = 8$. We have $k = 6$ and (3) yields $v = 3, 4$. We deal first with the case $v = 3$. From Lemma 1.7 either $\tau = \frac{7}{3}$, $u = 7$ or $\tau = \frac{8}{3}$, $u = 8$.

Let $\tau = \frac{7}{3}$, so that $A = 2K_X + 5L$. Assume $\phi$ is not birational. We have $7 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$ from Inequality (2) and hence $6 \leqslant m(\phi) \leqslant 8$. If $m(\phi) = 8$, $\phi$ contracts $X$ to a point so that $3K_X + 7L \approx \mathcal{O}_X$. It follows that $K_X \approx -7A$, $L \approx 3A$, and we are in Case 4 (a). If $u = m(\phi) = 7$, one has $6 \leqslant f(\phi) \leqslant 7$. Then, recalling that $K_X + 7A \approx \phi^*(\mathscr{H})$, $(X, A)$, $\phi : X \to W$, is a quadric fibration over a nonsingular curve as in Case 4 (b) if $f(\phi) = 7$; and $(X, A)$, $\phi : X \to W$, is a scroll over a normal surface as in Case 4 (c) if $f(\phi) = 6$. If $m(\phi) = 6$, then $u = m(\phi) + 1$ and $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^6$-bundle over a normal surface as in Case 4 (d). Whenever $\phi$ is birational, since $u = 7 = n - 1$ is the nefvalue of $(X, A)$, we are in Case 4 (e) by using [4, (7.3.2)].

Let $\tau = \frac{8}{3}$, so that $A = K_X + 3L$. If $\phi$ is not birational, we have $8 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$ from Inequality (2) and hence $7 \leqslant m(\phi) \leqslant 8$. If $m(\phi) = 8$, $\phi$ contracts $X$ to a point, so that $3K_X + 8L \approx \mathcal{O}_X$ and we are in Case 5 (a). If $m(\phi) = 7$, then $u = m(\phi) + 1$ and $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^7$-bundle over a smooth curve as in Case 5 (b).

We claim that $\phi$ is not birational. Indeed, if it was, then $K_X + 8A = K_X + 8(K_X + 3L) = 3(3K_X + 8L)$ would be nef and big and not ample; since $n = 8$ this is not possible by [4, (7.2.3)].

Let $v = 4$. From Lemma 1.7 either $\tau = \frac{9}{4}$, $u = 9$, or $\tau = \frac{11}{4}$, $u = 11$. The second case contradicts the bound $u \leqslant 9$ from the Kawamata rationality theorem. Therefore $\tau = \frac{9}{4}$. Then $u = m(\phi) + 1 = 9$, that is $m(\phi) = 8$ and $\phi : X \to W$ contracts $X$ to a point. Hence $4K_X + 9L \approx \mathcal{O}_X$ and we are in Case 6.

*Step* III: The case $n = 6$. We have $k = 5$ and (3) yields $3 \leqslant v \leqslant 6$.

Let $v = 3$. From Lemma 1.7 either $\tau = \frac{4}{3}$, $u = 4$ or $\tau = \frac{5}{3}$, $u = 5$. Consider first the case $\tau = \frac{4}{3}$. Then $A = 2K_X + 3L$. Assume $\phi$ is not birational. Then Inequality (2) yields $4 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$, so that $3 \leqslant m(\phi) \leqslant 6$. If $m(\phi) = 6$, $\phi$ contracts $X$ to a point, and therefore $3K_X + 4L \approx \mathcal{O}_X$; it follows that $K_X \approx -4A$, $L \approx 3A$ and we are in Case 7 (a). If $m(\phi) = 5$, then $3 \leqslant f(\phi) \leqslant 5$. If $f(\phi) = 5$ (respectively

$f(\phi) = 4$; respectively $f(\phi) = 3$), recalling that $K_X + 4L \approx \phi^*(\mathscr{H})$, we see that $(X, A)$, $\phi : X \to W$, is a Del Pezzo fibration over a smooth curve as in Case 7 (b) (respectively a quadric fibration over a normal surface as in Case 7 (c); respectively a scroll over a normal threefold as in Case 7 (d)). If $u = m(\phi) = 4$, then $3 \leqslant f(\phi) \leqslant 4$ and $(X, A)$, $\phi : X \to W$, is either a quadric fibration over a normal surface if $f(\phi) = 4$ (and all fibers are equidimensional since $u = m(\phi)$), or a scroll over a normal threefold if $f(\phi) = 3$; we fall again in Cases 7 (c), 7 (d). If $m(\phi) = 3$, then $u = m(\phi) + 1$ and $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^3$-bundle over a normal threefold as in Case 7 (e). Whenever $\phi$ is birational Theorem 1.9, (c) applies to give Case 7 (f).

Let $\tau = \frac{5}{3}$, and hence $A = K_X + 2L$. If $\phi$ is not birational, Inequality (2) gives $5 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$. Thus $4 \leqslant m(\phi) \leqslant 6$ and exactly the same argument as in the case $\tau = \frac{4}{3}$, shows that we are in one of Cases 8 (a), 8 (b), 8 (c), 8 (d) (note that in Case 8 (c) all fibers are equidimensional since $u = m(\phi)$). If $\phi$ is birational we are in Case 8 (e) by using again Theorem 1.9, (c).

Let $v = 4$. From Lemma 1.7 either $\tau = \frac{5}{4}$, $u = 5$, or $\tau = \frac{7}{4}$, $u = 7$. Let $\tau = \frac{5}{4}$, so that $A = 3K_X + 4L$. If $\phi$ is not birational, we have $5 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$, so that $4 \leqslant m(\phi) \leqslant 6$. If $m(\phi) = 6$, $\phi$ contracts $X$ to a point and hence $4K_X + 5L \approx \mathcal{O}_X$. Thus $K_X \approx -5A$, $L \approx 4A$ and $(X, A)$ is a Del Pezzo variety as in Case 9 (a). If $u = m(\phi) = 5$, we have $4 \leqslant f(\phi) \leqslant 5$. Therefore, since $K_X + 5A \approx \phi^*(\mathscr{H})$, we see that $(X, A)$, $\phi : X \to W$, is a quadric fibration over a smooth curve, and all fibers are equidimensional, if $f(\phi) = 5$; and $(X, A)$, $\phi : X \to W$, is a scroll over a normal surface if $f(\phi) = 4$; we find the first two cases of 9 (b). If $m(\phi) = 4$, then $u = m(\phi) + 1$ and $(X, A)$, $\phi : X \to W$, is a $\mathbb{P}^4$-bundle over a normal surface as in the third case of 9 (b). If $\phi$ is birational, Theorem 1.9 applies again and we are in the last case of 9 (b).

Let $\tau = \frac{7}{4}$. Since $u = m(\phi) + 1 = 7$ we have $m(\phi) = 6$, that is $\phi$ contracts $X$ to a point, and therefore $4K_X + 7L \approx \mathcal{O}_X$; we are in Case 10 (a).

Next, let us assume $v = 5$. Lemma 1.7 yields $\tau = 1 + \frac{i}{5}$ with $i = 1, 2, 3, 4$. Hence we find for $\tau$ the possible numerical values $\frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}$. Clearly the last two cases cannot occur since they contradict the bound $u \leqslant 7$ from the Kawamata rationality theorem.

Let $\tau = \frac{6}{5}$, and hence $A = 4K_X + 5L$. Note that $\phi$ is not birational. Indeed, if it was, $K_X + 6A$ $(=5(5K_X + 6L))$ would be nef and not ample, contradicting [4, (7.2.3)]. Thus $\phi$ is a fibration satisfying $6 \leqslant f(\phi) + 1 \leqslant m(\phi) + 1$, and hence $5 \leqslant m(\phi) \leqslant 6$. If $m(\phi) = 6$, $\phi$ contracts $X$ to a point, so that $5K_X + 6L \approx \mathcal{O}_X$ and we find Case 11 (a). If $m(\phi) = 5$, we are in Case 11 (b) since $u = m(\phi) + 1$.

Finally, let $\tau = \frac{7}{5}$. Since $u = m(\phi) + 1 = 7$ we have $m(\phi) = 6$ and therefore $5K_X + 7L \approx \mathcal{O}_X$; we are in Case 12.

If $v = 6$, then $\tau = 1 + \frac{i}{6}$, $i = 1, 5$, by Lemma 1.7. The case $i = 5$ is excluded by the usual bound $u \leqslant 7$. Therefore $\tau = \frac{7}{6}$ and hence $6K_X + 7L \approx \mathcal{O}_X$; we are in Case 13. □

**Remark 3.2.** Note that in the scroll Cases 1 (e) (with $n \geqslant 7$), 4 (c), 7 (d), 8 (c) of Theorem 3.1, $\phi$ is a contraction of an extremal ray by [4, (14.1.1)]. Furthermore, if $A$ is very ample, in Cases 4 (c), 7 (d), 8 (c), $\phi$ is a linear $\mathbb{P}^{n-\dim(W)}$-bundle by [4, (14.1.3)].

Note also that in the quadric fibration Cases 1 (d) (with $n \geqslant 7$), 4 (b), 7 (c), 8 (b), $\phi$ is a contraction of an extremal ray by [4, (14.2.1)].

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