Adv. Geom. **3** (2003), 361–363

Advances in Geometry

An effective Bertini theorem over finite fields

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(Communicated by G. Korchmáros)

Abstract. Let p be a prime, q a power of p and K the algebraic closure of the finite field $\mathrm{GF}(p)$. Let $X \subset \mathbf{P}^N(K)$ be an irreducible variety. Set $n := \dim(X)$ and $d := \deg(X)$. Here we prove that if $q \geqslant d(d-1)^n$ there is a hyperplane H of $\mathbf{P}^N(K)$ defined over $\mathrm{GF}(q)$ and transversal to X.

2000 Mathematics Subject Classification. 14N05

1 Introduction

Fix a prime integer p and call K the algebraic closure of the finite field $\mathrm{GF}(p)$. Fix a finite set $\{H_i\}_{i \in I}$ of hyperplanes of $\mathbf{P}^N(K)$. Let $X \subset \mathbf{P}^N(K)$ be an irreducible variety. A hyperplane H of $\mathbf{P}^N(K)$ is said to be transversal to X if $H \notin \check{X}$, where $\check{\mathbf{P}}^N(K)$ is the dual projective space and $\check{X} \subset \check{\mathbf{P}}^N(K)$ is the dual variety of X. If X is smooth, $H \notin \check{X}$ if and only if H does not contain any embedded tangent space of X. If X is singular and $H \notin \check{X}$, then for every $P \in X_{\mathrm{reg}}$ the hyperplane H does not contain any embedded tangent space to X at P. A classical and very weak form of Bertini's theorem ([5], Parts 2) and 3) of Theorem 6.3, or [3], Theorem II.8.18) says that a general hyperplane H of $\mathbf{P}^N(K)$ is transversal to X. Here "general" means "in a non-empty Zariski open subset U of $\check{\mathbf{P}}^N(K)$". The non-empty open set U depends on X. When can one be sure that there is $i \in I$ such that H_i is transversal to X? We are interested in the case in which $\{H_i\}_{i \in I}$ is the set of all hyperplanes of $\mathbf{P}^N(K)$ defined over $\mathrm{GF}(q)$, q a power of p, i.e. the set of all hyperplanes of $\mathbf{P}^N(K)$ spanned by a subset of $\mathrm{PG}(N,q)$. In this note we prove the following result.

Theorem 1. *Let $X \subset \mathbf{P}^N(K)$ be an irreducible variety. Set $n := \dim(X)$ and $d := \deg(X)$. Assume $q \geqslant d(d-1)^n$. Then there exists a hyperplane H of $\mathbf{P}^N(K)$ defined over $\mathrm{GF}(q)$ and transversal to X.*

We stress that in Theorem 1 we do not require that X is defined over $\mathrm{GF}(q)$. Even if X is defined over $\mathrm{GF}(q)$ and smooth, Theorem 1 gives the existence of a hyper-

* The author was partially supported by MURST and GNSAGA of INdAM (Italy).

plane H defined over $\mathrm{GF}(q)$ and transversal to X at each of its K-points, not just at each of its $\mathrm{GF}(q)$-points.

2 Proof of Theorem 1

Lemma 1. *Let* $X \subset \mathbf{P}^N(K)$ *be an irreducible variety. Set* $n := \dim(X)$ *and* $d := \deg(X)$. *Then* $\deg(\check{X}) \leqslant d(d-1)^n$.

Proof. The dual variety of X is a hypersurface if and only if the general contact locus of X is zero-dimensional ([6], p. 174). First assume $\dim(\check{X}) = N-1$ and $n = N-1$. Fix a general line $D \subset \check{\mathbf{P}}^N(K)$. We have $\deg(\check{X}) = \mathrm{card}(D \cap \check{X})$. D is induced by the pencil of all hyperplanes through a codimension two linear subspace V of $\mathbf{P}^N(K)$. Fix homogeneous coordinates $x_0, \dots, x_{n+1}$ of $\mathbf{P}^N(K)$ such that $V = \{x_n = x_{n+1} = 0\}$ and let $f(x_0, \dots, x_{n+1})$ be a degree d homogeneous equation of X. By the generality of D the varieties X and V are transversal. Every $H \in \check{X} \cap D$ corresponds to a solution of the system

$$f = 0, \quad \partial f/\partial x_i = 0, \quad 0 \leqslant i \leqslant n-1 \tag{1}$$

We claim that $\deg(\check{X}) \leqslant d(d-1)^n$. The claim is true if the system (1) has only finitely many solutions by Bezout theorem. However, the system (1) has seldom finitely many solutions and never if $\dim(\mathrm{Sing}(X)) > 0$, because every singular point of X is a solution of the system (1). However, by the generality of D we need only to compute the number of all hyperplanes in the pencil associated to V and tangent to X at some smooth point of X. Since the general contact locus of X is zero-dimensional, each point of $D \cap \check{X}$ corresponds to a connected component of the set of all solutions of the system (1). Hence we conclude by [2], Example 8.4.6. Now we assume $\dim(\check{X}) = N-1$ and $n \leqslant N-2$. Take a general linear subspace W of $\mathbf{P}^N(K)$ with $\dim(W) = N-n-2$. By the generality of W we have $W \cap X = \emptyset$. Let $\pi : \mathbf{P}^N(K) \backslash W \to \mathbf{P}^{n+1}(K)$ be the linear projection from W. Set $Y := \pi(X)$. By the generality of W the morphism $\pi \mid X : X \to Y$ is birational and $\deg(Y) = \deg(X)$. A line D' of $\check{\mathbf{P}}^N(K)$ corresponds to the pencil of all hyperplanes containing a codimension two linear subspace V' of $\mathbf{P}^N(K)$. By the generality of W to compute $\deg(\check{X})$ we may use a line D' corresponding to a codimension two linear subspace V' containing W. For such V' the closure of $\pi(V' \backslash W)$ in $\mathbf{P}^{n+1}(K)$ is a codimension two linear subspace, V, of $\mathbf{P}^{n+1}(K)$. If V' is general with the only restriction that $W \subset V'$, then V is a general codimension two linear subspace of $\mathbf{P}^{n+1}(K)$. Hence $\deg(\check{X}) = \deg(\check{Y})$. In the same way computing the contact locus of X (resp. Y) with a general element of $\check{X}$ (resp. $\check{Y}$) we obtain that if $\check{X}$ is a hypersurface, then $\check{Y}$ is a hypersurface. Hence we conclude by the case $N = n+1$ and $\dim(\check{X}) = N-1$ just proved. Now assume $\dim(\check{X}) < N-1$. In particular $\dim(X) \geqslant 2$ because the dual variety of an integral curve (not a line) is always a hypersurface. We use induction on the integer $\dim(X)$. Let $M \subset \mathbf{P}^N(K)$ be a general hyperplane. Hence by Bertini's theorem over K the scheme $X \cap M$ is an integral variety with $\dim(X \cap M) = n-1$

and $\deg(X \cap M) = d$. Call M^* the point of $\check{\mathbf{P}}^N(K)$ associated to M. Since M is general, $M^* \notin \check{X}$. Let $(X \cap M)\check{} \subset \check{M}$ be the dual variety of $X \cap M$ seen as a subvariety of M. The linear projection $f : \check{\mathbf{P}}^N(K) \backslash \{M^*\} \to \check{M}$ induces a surjection of $\check{X}$ onto $(X \cap M)\check{}$ ([4], Prop. 4.7 (ii)). Since M is general and $\check{X}$ is not a hypersurface, $f \mid \check{X}$ is birational onto its image. Hence $\deg(\check{X}) = \deg((X \cap M)\check{})$. By the inductive assumption we obtain $\deg(\check{X}) = \deg((X \cap M)\check{}) \leqslant d(d-1)^{n-1}$ as claimed. □

Proof of Theorem 1. Choose homogeneous coordinates $z_0, \ldots, z_N$ of $\check{\mathbf{P}}^N(K)$. By Lemma 1 we have $\deg(\check{X}) \leqslant d(d-1)^n$. Hence there is a homogeneous polynomial $G(z_0, \ldots, z_N)$ with $\deg(G) = \deg(\check{X}) \leqslant d(d-1)^n$, $G \mid \check{X} \equiv 0$ and $G \neq 0$. It is very easy to check that there is no non-zero homogeneous polynomial of degree at most q vanishing on $\mathrm{PG}(n, q)$: with the terminology of [1], [6] and [7] any $\mathrm{PG}(n, q)$ has Property $\mathrm{FFN}(q)$, i.e. it satisfies the Finite Field Nullstellensatz of order q. Hence $\deg(\check{X})$ does not contain the dual $\mathrm{PG}(N, q)$, i.e. there is a hyperplane H defined over $\mathrm{GF}(q)$ and transversal to X. □

Remark. Take X as in the statement of Theorem 1. The proof of Theorem 1 shows that it is sufficient to take $q \geqslant \deg(\check{X})$.

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Received 1 June, 2001; revised 8 October, 2002

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