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Morphisms of projective spaces over rings

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Dedicated to Francis Buekenhout on the occasion of his 65th birthday

Abstract. The fundamental theorem of projective geometry is generalized for projective spaces over rings. Let ${}_RM$ and ${}_SN$ be modules. Provided some weak conditions are satisfied, a morphism $g : \mathscr{P}(M)\backslash E \to \mathscr{P}(N)$ between the associated projective spaces can be induced by a semilinear map $f : M \to N$. These conditions are satisfied for instance if S is a left Ore domain and if the image of g contains three independent free points. No assumptions are made on the module M, and both modules may have some torsion.

Introduction

Two different approaches to projective spaces associated to modules are usually considered. One may choose as set of points the set of all submodules generated by a unimodular element, as defined in [20], or one may choose the lattice of all submodules, as defined in [3]. In the first approach one avoids the pathology (?) of small points contained in big points. But the price to pay is important.

Following [9] it would be desirable if one had a functor from the category of modules and semilinear maps to a category of projective spaces and morphisms. But this is impossible with the first approach. Consider the ring $R := \mathbb{Z}/4\mathbb{Z}$ and the linear map $f : R^3 \to R^3$ defined by $f(x, y, z) = (x + y, x + 3y, z)$. One easily shows that f cannot induce a map $\mathscr{P}(R^3) \to \mathscr{P}(R^3)$ that preserves the incidence relation. So with this first approach we must restrict our attention to semilinear maps that preserve unimodular elements, and this is not natural.

In the present paper the projective space $\mathscr{P}(M)$ associated to a module M is defined as the set of all cyclic (i.e. one-generated) submodules. This is equivalent to the second approach. Using axioms of Faigle and Herrmann [5] we propose a definition of projective spaces based on a single operator $\vee$.

Morphisms of projective spaces are defined in the second section. It is shown that one has a functor from the category of modules and semilinear maps to the category of projective spaces and morphisms (this implies that a morphism must be a partially defined map between the point sets).

The main result of this paper is a generalization of the fundamental theorem of projective geometry. It is proved in Section 3 by following mainly the lines of the proof given in [6]. *Let* ${}_RM$ *and* ${}_SN$ *be modules and* $g : \mathscr{P}(M)\backslash E \to \mathscr{P}(N)$ *a morphism between the associated projective spaces. We suppose that the ring* S *is directly finite, and that the image of* g *contains three independent free points* B_1, B_2, B_3 *satisfying a weak condition* (C3). *Then there exists a semilinear map* $f : M \to N$ *which induces* g. *Moreover, the map* f *is unique up to multiplication with a unit.*

This condition (C3) requires that for any $C_1, C_2 \in \mathscr{P}(N)$, there exists a point B_i which is independent from all the points of the line $C_1 \vee C_2$. In Section 4 we show that this condition is satisfied provided S is a left Ore domain. In Section 5 we show that it is satisfied provided S is a right Bezout domain and B_1, B_2, B_3 generate a direct summand.

In the literature, most generalizations of the fundamental theorem deal with isomorphisms. See for instance [18], [13], [12], [4] and [15]. Several interesting results in that direction (and others) can be found in [10]. Closer to our theorem is the result of Brehm [2]. His triangle-property resembles condition (C3), but it applies to the module M, not to N. The reason is that Brehm's homomorphisms preserve disjointness. Since we do not make such assumptions, our Theorem 3.2 generalizes Theorem 1 in [2]. On the other hand, Brehm's result is very general, because homomorphisms do not preserve cyclic submodules.

For classical projective spaces (over division rings), the present version of the fundamental theorem was first proved in [8] and independently by Havlicek [11]. It generalized a former version due to Brauner [1] on *linear maps*. In the case of projective lattice geometries, these linear maps are discussed in [14]. Recently, a further generalization of the fundamental theorem for classical projective spaces appeared in [7]. It is possible that this generalization also applies to the case of projective spaces associated to modules.

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1 Projective spaces

Definition 1.1. A *projective space* is a set P of *points* together with a binary operator $\vee : P \times P \to 2^P$ which satisfies (at least) the following axioms:

(P1) $a \in b \vee a$ for all $a, b \in P$,

(P2) if $a \in b \vee c$, then $a \vee b \subseteq b \vee c$,

(P3) if $a \vee a = b \vee b$, then $a = b$,

(P4) if $a \in b \vee p$ and $p \in c \vee d$, then there exists $q \in b \vee c$ with $a \in q \vee d$,

(P5) if $a \in b \vee c$ and $a \notin b \vee b$, then there exists $d \in c \vee c$ with $a \vee b = b \vee d$.

According to axioms (P1) and (P2) one has $a \vee b = b \vee a$. The last two axioms were introduced by Faigle and Herrmann in [5] as properties (A7) and (A6).

In an equivalent way, a projective space can be defined as a partially ordered set together with a binary operator satisfying suitable axioms. The partial order associated to a projective space P is given by $a \leqslant b$ if and only if $a \in b \vee b$.

Proposition 1.2. *Let M be a (left) module over an arbitrary ring R (with 1). On the set $\mathscr{P}(M)$ of all nonzero cyclic submodules of M we define an operator $\vee$ by $A \in B \vee C$ if and only if $A \subseteq B + C$. Then $\mathscr{P}(M)$ becomes a projective space.*

Proof. We verify axiom (P5). Let $A, B, C \in \mathscr{P}(M)$ with $A \subseteq B + C$ and $A \nsubseteq B$. Say $A = Ra$, $B = Rb$ and $C = Rc$. There exist $\lambda, \mu \in R$ such that $a = \lambda b + \mu c$ and $\mu c \neq 0$. Putting $D = R\mu c$ one easily shows that $A + B = B + D$. □

Definition 1.3. A *subspace* of a projective space P is a subset $E \subseteq P$ with the property that $a, b \in E$ implies $a \vee b \subseteq E$. Trivially, the set $\mathscr{L}(P)$ of all subspaces of P is closed under arbitrary intersections and directed unions. Therefore $\mathscr{L}(P)$ is a complete algebraic lattice for the inclusion order.

Lemma 1.4. *Let P be a projective space. Then for any points $a, b \in P$ the set $a \vee b$ is the smallest subspace containing a and b (this justifies the notation). In particular, $a \vee a$ is the smallest subspace containing a.*

Proof. Let $p, q \in a \vee b$ and $r \in p \vee q$. By axiom (P4) there exists $s \in p \vee a$ such that $r \in s \vee b$. Since $p \in a \vee b$ implies $p \vee a \subseteq a \vee b$ by (P2), one gets $s \in b \vee a$. Therefore $r \in s \vee b \subseteq b \vee a$, and this shows that $a \vee b$ is a subspace. □

Lemma 1.5. *Let E, F be two subspaces of a projective space P. Then the set $G := \bigcup\{a \vee b \mid a \in E \text{ and } b \in F\}$ is also a subspace of P.*

Proof. Let $p \in p_1 \vee p_2$ where $p_1, p_2 \in G$. There exist $a_1, a_2 \in E$ and $b_1, b_2 \in F$ with $p_1 \in a_1 \vee b_1$ and $p_2 \in a_2 \vee b_2$. We now apply three times axiom (P4):

1) Since $p \in p_1 \vee p_2$ and $p_2 \in a_2 \vee b_2$, there exists $q \in p_1 \vee a_2$ with $p \in q \vee b_2$.

2) Since $q \in a_2 \vee p_1$ and $p_1 \in a_1 \vee b_1$, there exists $a \in a_2 \vee a_1$ with $q \in a \vee b_1$.

3) Since $p \in b_2 \vee q$ and $q \in b_1 \vee a$, there exists $b \in b_2 \vee b_1$ with $p \in b \vee a$.

Therefore $p \in G$, and this shows that G is a subspace. □

Proposition 1.6. *For any projective space P the lattice $\mathscr{L}(P)$ of all subspaces of P is modular.*

Proof. Let E, F, G be three subspaces of P with $E \subseteq G$. We have to show that $(E \vee F) \wedge G \subseteq E \vee (F \wedge G)$ (the other inclusion holds trivially). We may assume that E and F are not empty. This implies $E \vee F = \bigcup\{a \vee b \mid a \in E \text{ and } b \in F\}$ by the previous lemma. Let $p \in (E \vee F) \wedge G$. There exist $a \in E$ and $b \in F$ such that $p \in a \vee b$. If $p \in a \vee a$, then $p \in E \subseteq E \vee (F \wedge G)$. Otherwise, axiom (P5) implies that there exists a point $c \in b \vee b$ such that $p \vee a = a \vee c$. One thus gets $c \in (b \vee b) \cap (p \vee a) \subseteq F \wedge G$, and hence $p \in a \vee c \subseteq E \vee (F \wedge G)$. □

Proposition 1.7. *Let M be a module over a ring R. Then the lattice $\mathscr{L}(M)$ of all submodules of M is isomorphic to the lattice $\mathscr{L}(\mathscr{P}(M))$.*

Proof. For every submodule $N \subseteq M$ the set $\varphi(N) := \{A \in \mathscr{P}(M) \mid A \subseteq N\}$ is a subspace of $\mathscr{P}(M)$, and we thus get a monotone map $\varphi : \mathscr{L}(M) \to \mathscr{L}(\mathscr{P}(M))$. Its inverse is the map ψ defined by $\psi(E) = \bigcup E$ if $E \neq \varnothing$ and $\psi(\varnothing) = \{0\}$. □

2 Morphisms

Definition 2.1. Let P, Q be two projective spaces. A *morphism* from P into Q is a partially defined map $g : P \backslash E \to Q$ satisfying the following axioms:

(M1) $a, b, c \notin E$ and $a \in b \vee c$ imply $ga \in gb \vee gc$,

(M2) $a, b \notin E$, $x \in E$ and $a \in b \vee x$ imply $ga \in gb \vee gb$,

(M3) E is a subspace of P, called the *kernel* of g.

The following lemma gives an equivalent (and shorter) definition of a morphism:

Lemma 2.2. *A partially defined map $g : P \backslash E \to Q$ between projective spaces is a morphism if and only if $g^{-1}(F) \cup E$ is a subspace for every subspace $F \subseteq Q$.*

Proof. ($\Rightarrow$) Let $b, c \in g^{-1}(F) \cup E$ and $a \in b \vee c$. We show that $a \in g^{-1}(F) \cup E$ by considering the cases **1)** $b, c \in g^{-1}(F)$, **2)** $b \in g^{-1}(F)$ and $c \in E$, **3)** $b, c \in E$.
($\Leftarrow$) Choose the subspaces $F_1 = gb \vee gc$, $F_2 = gb \vee gb$ and $F_3 = \varnothing$. □

Definition 2.3. Let $g_1 : P_1 \backslash E_1 \to P_2$ and $g_2 : P_2 \backslash E_2 \to P_3$ be two morphisms of projective spaces. The *composite* $g_2 \circ g_1$ is defined as follows: its kernel is the subspace $E = g_1^{-1}(E_2) \cup E_1$ and any element $a \notin E$ is mapped to $g_2 g_1 a$. It is a morphism because one has $(g_2 \circ g_1)^{-1}(F) \cup E = g_1^{-1}(g_2^{-1}(F) \cup E_2) \cup E_1$.

Remark 2.4. Morphisms from P_1 to P_2 are in one-to-one correspondence with maps $\mathscr{L}(P_1) \to \mathscr{L}(P_2)$ preserving arbitrary joins and cyclic subspaces (where the empty subspace is considered as a cyclic one).

Definition 2.5. Let ${}_RM$ and ${}_SN$ be modules and $\sigma : R \to S$ a homomorphism of rings. A map $f : M \to N$ is called *σ-semilinear* if it is additive and if one has $f(\lambda x) = \sigma(\lambda) f(x)$ for all $x \in M$ and $\lambda \in R$.

Proposition 2.6. *Let ${}_RM$ and ${}_SN$ be modules and $f : M \to N$ a σ-semilinear map. Then the map $\mathscr{P}f : \mathscr{P}(M) \backslash \mathscr{P}(\ker f) \to \mathscr{P}(N)$ defined by $\mathscr{P}f(Rx) = Sf(x)$, where $x \notin \ker f$, is a morphism of projective spaces.*

Proof. The map $\mathscr{P}f$ is well defined, because $Rx = Ry$ implies $Sf(x) = Sf(y)$. The conditions of Definition 2.1 (or Lemma 2.2) are easily verified. □

Proposition 2.7. *If $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ are two semilinear maps between modules, then $\mathscr{P}(f_2 \circ f_1) = \mathscr{P}f_2 \circ \mathscr{P}f_1$. This means that P is a functor from the category of modules to the category of projective spaces.*

Proof. One has $\mathscr{P}(\ker(f_2 \circ f_1)) = \mathscr{P}f_1^{-1}(\mathscr{P}(\ker f_2)) \cup \mathscr{P}(\ker f_1)$. □

Definition 2.8. Let M be a module over R. We recall that an element $a \in M$ is *free* if $\lambda a = 0$ implies $\lambda = 0$. A family of n elements $a_1, \dots, a_n \in M$ is called

1) *ω-independent* if $\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$ implies $\lambda_1 a_1 = \cdots = \lambda_n a_n = 0$,

2) *linearly independent* if $\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$ implies $\lambda_1 = \cdots = \lambda_n = 0$.

One trivially shows that a family $a_1, \dots, a_n$ is linearly independent if and only if it is ω-independent and each a_i is free.

Theorem 2.9. *Let ${}_RM$ and ${}_SN$ be modules and $f, h : M \to N$ two semilinear maps satisfying $\mathscr{P}f = \mathscr{P}h$. We suppose that the image of f contains two linearly independent elements y_1, y_2 with the following condition*:

(C2) *for every non-zero $z \in N$ there exists i such that y_i, z are ω-independent.*

If S is a directly finite ring (*that is, $\lambda\mu = 1$ implies $\mu\lambda = 1$*), *then there exists a unit $\varepsilon \in S$ such that $h(c) = \varepsilon f(c)$ for every $c \in M$.*

Proof. Let $x_i \in M$ with $f(x_i) = y_i$. Since $Sf(x_1) = Sh(x_1)$, there exist $\delta, \varepsilon \in S$ such that $f(x_1) = \delta h(x_1)$ and $h(x_1) = \varepsilon f(x_1)$. So one obtains $f(x_1) = \delta\varepsilon f(x_1)$, which implies $\delta\varepsilon = 1$. Therefore ε is a unit. We want to show that $h(x) = \varepsilon f(x)$ for every $x \notin \ker f = \ker h$. We first suppose that $f(x_1), f(x)$ are ω-independent. Since $\mathscr{P}f = \mathscr{P}h$, there exist two elements $\lambda, \mu \in S$ such that $h(x) = \lambda f(x)$ and $h(x_1 + x) = \mu f(x_1 + x)$. From the equality

$$\mu f(x_1) + \mu f(x) = h(x_1 + x) = \varepsilon f(x_1) + \lambda f(x)$$

it follows that $\mu = \varepsilon$ and $\mu f(x) = \lambda f(x)$. Hence $h(x) = \varepsilon f(x)$. We now suppose that $f(x_1), f(x)$ are ω-dependent. Then $f(x_2), f(x)$ are ω-independent, and we can apply the same argument (one has $h(x_2) = \varepsilon f(x_2)$ by the first case). □

Condition (C2) clearly implies Greferath's condition (Δ). However, the other assumptions of Proposition 2.10 in [10] are stronger.

3 The fundamental theorem

Definition 3.1. Let ${}_SN$ be a module and $\mathscr{P}(N)$ the associated projective space. A point $B \in \mathscr{P}(N)$ is called *free* if $B = Sb$ for some free element $b \in N$. A family of n points $B_1, \dots, B_n \in \mathscr{P}(N)$ is called *independent* if one has

$$\langle B_i \rangle \cap \langle B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n \rangle = \varnothing$$

for every $i = 1, \ldots, n$ (where $\langle \mathscr{A} \rangle$ denotes the subspace generated by a set $\mathscr{A}$). One easily shows that a family $b_1, \ldots, b_n \in N$ is ω-independent in N if and only if $Sb_1, \ldots, Sb_n$ is independent in $\mathscr{P}(N)$.

The aim of the present section is to prove the following result:

Theorem 3.2. *Let $_RM$ and $_SN$ be modules and $g : \mathscr{P}(M)\backslash E \to \mathscr{P}(N)$ a morphism between the associated projective spaces. We suppose that the image of g contains three independent free points B_1, B_2, B_3 with the following condition*:

(C3) *for any $C_1, C_2 \in \mathscr{P}(N)$ there exists i such that $(B_i \vee B_i) \cap (C_1 \vee C_2) = \varnothing$.*

If S is a directly finite ring, then there exists a semilinear map $f : M \to N$ such that $g = \mathscr{P}f$. Moreover, the map f is unique up to multiplication with a unit.

Remarks 3.3. 1) If C_1, C_2 are independent, then condition (C3) implies that B_i, C_1, C_2 are also independent.

2) If Sy is free, then y is free. By hypothesis one has $Sy = Sz$ for some free element $z \in N$, and since S is directly finite, the element y differs from z by a unit.

Condition (C3) clearly implies condition (1) of Brehm's triangle-property [2], but not condition (2). It is possible that this assumption in Theorem 3.2 can be weakened by following Brehm's idea. However, since all points have to be chosen in the image of g, the game is not worth the candle.

Lemma 3.4. *Let $g(Rx_1)$ and $g(Rx_2)$ be two independent points, and suppose that $g(Rx_1) = Sy_1$ is free. Then there exists a unique element $y_2 \in N$ such that $g(Rx_2) = Sy_2$ and $g(R(x_1 + x_2)) = S(y_1 + y_2)$.*

Proof. Let $z_2 \in N$ with $g(Rx_2) = Sz_2$. One first remarks that $R(x_1 + x_2) \notin E$, because otherwise $Rx_1 \in Rx_2 \vee R(x_1 + x_2)$ would imply $Sy_1 \subseteq Sz_2$ by (M2), in contradiction to the hypothesis. Let $z \in N$ with $g(R(x_1 + x_2)) = Sz$. We apply three times condition (M1):

1) $R(x_1 + x_2) \in Rx_1 \vee Rx_2$ implies $z = \lambda_1 y_1 + \lambda_2 z_2$,

2) $Rx_1 \in R(x_1 + x_2) \vee Rx_2$ implies $y_1 = \mu z - \mu_2 z_2$,

3) $Rx_2 \in R(x_1 + x_2) \vee Rx_1$ implies $z_2 = \nu z - \nu_1 y_1$.

From the equality $y_1 = \mu\lambda_1 y_1 + (\mu\lambda_2 - \mu_2) z_2$ one obtains $\mu\lambda_1 = 1$ (because y_1 is free) and $\mu\lambda_2 z_2 = \mu_2 z_2$. We put $y_2 = \mu_2 z_2$. Since μ is a unit of S, one gets $g(R(x_1 + x_2)) = S\mu z = S(y_1 + y_2)$ according to condition 2). From the equality $z_2 = (\nu\lambda_1 - \nu_1) y_1 + \nu\lambda_2 z_2$ one obtains $z_2 = \nu\lambda_2 z_2$. So it follows that $\nu\lambda_1 y_2 = \nu\lambda_1 \mu_2 z_2 = \nu\lambda_1 \mu\lambda_2 z_2 = \nu\lambda_2 z_2 = z_2$. Therefore $Sy_2 = Sz_2$ and the assertion is proved. The uniqueness of y_2 is obvious. □

Lemma 3.5. *Let $g(Rx_1), g(Rx_2)$ and $g(Rx_3)$ be three independent points. If there exist $y_1, y_2, y_3 \in N$ such that*

1) $g(Rx_1) = Sy_1$ *is free*, $g(Rx_2) = Sy_2$ *and* $g(Rx_3) = Sy_3$,

2) $g(R(x_1 + x_2)) = S(y_1 + y_2)$ *and* $g(R(x_1 + x_3)) = S(y_1 + y_3)$,

then $g(R(x_1 + x_2 + x_3)) = S(y_1 + y_2 + y_3)$ *and* $g(R(x_2 + x_3)) = S(y_2 + y_3)$.

Proof. One first remarks that $g(R(x_1 + x_2)), g(Rx_3)$ are independent, because $g(R(x_1 + x_2)) \in g(Rx_1) \vee g(Rx_2)$ by (M1). Since $g(R(x_1 + x_2)) = S(y_1 + y_2)$ is free, there exists by Lemma 3.4 a unique $z_3 \in N$ such that $g(Rx_3) = Sz_3$ and $g(R(x_1 + x_2 + x_3)) = S(y_1 + y_2 + z_3)$. And by symmetry there exists a unique $z_2 \in N$ such that $g(Rx_2) = Sz_2$ and $g(R(x_1 + x_2 + x_3)) = S(y_1 + z_2 + y_3)$. So one obtains $y_2 = z_2$ and $y_3 = z_3$, which proves the first assertion.

Now one considers the points $g(R(x_1 + x_2 + x_3))$ and $g(R(x_2 + x_3))$. They are independent, because $g(R(x_2 + x_3)) \in g(Rx_2) \vee g(Rx_3)$. Moreover, the first point is free. So there exists a unique $z \in N$ such that $g(R(x_2 + x_3)) = Sz$ and $g(Rx_1) = S(y_1 + y_2 + y_3 + z)$. Obviously, this implies $z = -y_2 - y_3$, and hence $g(R(x_2 + x_3)) = S(y_2 + y_3)$, which proves the second assertion. □

By hypothesis the image of the morphism g contains three independent free points B_1, B_2, B_3. We choose $A_1, A_2, A_3 \in \mathscr{P}(M) \backslash E$ such that $B_i = g(A_i)$, and $a_1, a_2, a_3 \in M$ such that $A_i = Ra_i$.

Lemma 3.6. *There exist $b_1, b_2, b_3 \in N$ such that $g(Ra_i) = Sb_i$ for each i, and $g(R(a_i + a_j)) = S(b_i + b_j)$ for all $i \neq j$.*

Proof. Let $b_1 \in N$ with $g(Ra_1) = Sb_1$. By Lemma 3.4 there exist $b_2 \in N$ such that $g(Ra_2) = Sb_2$ and $g(R(a_1 + a_2)) = S(b_1 + b_2)$, and $b_3 \in N$ with the same properties. According to Lemma 3.5 one has $g(R(a_2 + a_3)) = S(b_2 + b_3)$. □

Definition 3.7. According to Proposition 1.7 the kernel E can be written in a unique way as $E = \mathscr{P}(M_0)$ where M_0 is a submodule of M. The map $f : M \to N$ is now defined as follows. For each element $x \in M_0$ we put $f(x) = 0$. If $x \notin M_0$, then by condition (C3) there exists i such that $g(Ra_i), g(Rx)$ are independent. We put $f(x) = y$ where $y \in N$ is the unique element satisfying $g(Rx) = Sy$ and $g(R(a_i + x)) = S(b_i + y)$ (cf. Lemma 3.4).

Lemma 3.8. *The definition does not depend on the choice of the element a_i.*

Proof. Suppose that $g(Ra_1), g(Rx)$ and $g(Ra_2), g(Rx)$ are independent pairs of points. We consider $y \in N$ with $g(Rx) = Sy$ and $g(R(a_1 + x)) = S(b_1 + y)$, and we want to show that $g(R(a_2 + x)) = S(b_2 + y)$. If $g(Ra_1), g(Ra_2), g(Rx)$ are independent, then the conclusion holds by Lemma 3.5. Otherwise, condition (C3) implies that $g(Ra_1)$, $g(Ra_3), g(Rx)$ and $g(Ra_3), g(Ra_2), g(Rx)$ are both independent triples of points. So we apply twice the preceding argument. □

Proposition 3.9. *$f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in M$.*

Proof. Put $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Obviously, we may assume that $x_1 \neq 0$ and $x_2 \notin M_0$. Three different cases will be considered.

Case 1: $Rx_1 \in E$. Choose i such that $g(Ra_i), g(Rx_2)$ are independent. Since $R(a_i + x_2) \in R(a_i + x_1 + x_2) \vee Rx_1$, one gets

$$S(b_i + y_2) = g(R(a_i + x_2)) = g(R(a_i + x_1 + x_2))$$

by (M2). Similarly, $Sy_2 = g(Rx_2) = g(R(x_1 + x_2))$. By definition of the map f this shows that $f(x_1 + x_2) = f(x_2)$.

Case 2: $Rx_1 \notin E$ and $g(Rx_1), g(Rx_2)$ are independent. By condition (C3) one can choose i such that $g(Ra_i), g(Rx_1), g(Rx_2)$ are independent. One obtains

$$g(R(a_i + x_1 + x_2)) = S(b_i + y_1 + y_2) \quad \text{and} \quad g(R(x_1 + x_2)) = S(y_1 + y_2)$$

by Lemma 3.5, and this shows that $f(x_1 + x_2) = f(x_1) + f(x_2)$.

Case 3: $Rx_1 \notin E$ and $g(Rx_1), g(Rx_2)$ are dependent. By condition (C3) there exists i such that $(Sb_i \vee Sb_i) \cap (Sy_1 \vee Sy_2) = \varnothing$. If $R(x_1 + x_2) \in E$, then $f(a_i + x_1 + x_2) = f(a_i)$ according to the first case. And if $R(x_1 + x_2) \notin E$, then $g(R(x_1 + x_2)) \in Sy_1 \vee Sy_2$ implies that the points $g(Ra_i), g(R(x_1 + x_2))$ are independent, and one thus gets $f(a_i + x_1 + x_2) = f(a_i) + f(x_1 + x_2)$ by the second case. So this equality holds in any cases. On the other hand, one obtains $f(a_i + x_1 + x_2) = f(a_i + x_1) + f(x_2) = f(a_i) + f(x_1) + f(x_2)$ by applying twice Case 2, and one deduces that $f(a_i) + f(x_1 + x_2) = f(a_i) + f(x_1) + f(x_2)$. □

Proposition 3.10. *There exists a map $\sigma : R \to S$ such that $f(\lambda x) = \sigma(\lambda) f(x)$ for all $\lambda \in R$ and $x \in M$.*

Proof. For any $\lambda \in R$ and $x \notin M_0$ we remark that there exists $\mu \in S$ such that $f(\lambda x) = \mu f(x)$. This is trivial if $\lambda x \in M_0$. And if $\lambda x \notin M_0$, then (M1) implies that $Sf(\lambda x) = g(R(\lambda x)) \subseteq g(Rx) = Sf(x)$. We now define $\sigma(\lambda)$ as the unique element of S with the property that $f(\lambda a_1) = \sigma(\lambda) f(a_1)$. We have to show that $f(\lambda x) = \sigma(\lambda) f(x)$ for all $x \notin M_0$ and $\lambda \in R$.

Case 1: $g(Ra_1), g(Rx)$ are independent. Let $\mu, \nu \in S$ with $f(\lambda x) = \mu f(x)$ and $f(\lambda(a_1 + x)) = \nu f(a_1 + x)$. From the equality

$$\sigma(\lambda) f(a_1) + \mu f(x) = f(\lambda a_1 + \lambda x) = \nu f(a_1) + \nu f(x)$$

one obtains $\sigma(\lambda) = \nu$ and $\mu f(x) = \nu f(x)$. Therefore $f(\lambda x) = \sigma(\lambda) f(x)$.

Case 2: By condition (C3) we may assume that $g(Ra_2), g(Rx)$ are independent points. Since $f(\lambda a_2) = \sigma(\lambda)f(a_2)$ according to the first case, one can apply the preceding argument. □

From the equalities $\sigma(\lambda+\mu)f(a_1) = f(\lambda a_1 + \mu a_1) = \sigma(\lambda)f(a_1) + \sigma(\mu)f(a_1)$ and $\sigma(\lambda\mu)f(a_1) = \sigma(\lambda)f(\mu a_1) = \sigma(\lambda)\sigma(\mu)f(a_1)$ one deduces that σ is a homomorphism of rings. Therefore f is a semilinear map. By definition of the map f one has $g = \mathscr{P}f$. The fact that f is unique up to multiplication by a unit follows from Theorem 2.9. So the proof of Theorem 3.2 is complete.

4 Modules over left Ore domains

Let N be a module over a directly finite ring S. We suppose given two linearly independent elements $b_1, b_2 \in N$. Then condition (C2) can be written as follows:

(C2) for any $c \in N$ there exists i such that $Sb_i \cap Sc = \{0\}$.

Now let $b_1, b_2, b_3 \in N$ be three linearly independent elements. Condition (C3) in Theorem 3.2 can be written as follows:

(C3) for any $c_1, c_2 \in N$ there exists i such that $Sb_i \cap (Sc_1 + Sc_2) = \{0\}$.

We show that these conditions are satisfied provided S is a left Ore domain. We recall that a ring S is *left Ore* if $S\lambda \cap S\mu \neq \{0\}$ for all non-zero $\lambda, \mu \in S$.

Proposition 4.1. *If S is a left Ore domain, then condition* (C2) *is satisfied.*

Proof. Assume it is not. There exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in S$ such that

$$\lambda_1 b_1 = \mu_1 c \neq 0 \quad \text{and} \quad \lambda_2 b_2 = \mu_2 c \neq 0.$$

Since S is left Ore, there exist $\alpha, \beta \in S$ such that $\alpha\mu_1 = \beta\mu_2 \neq 0$. So $\alpha\lambda_1 b_1 = \alpha\mu_1 c = \beta\mu_2 c = \beta\lambda_2 b_2$ implies $\alpha\lambda_1 = \beta\lambda_2 = 0$, a contradiction. □

Remark 4.2. Suppose that S is a domain. If condition (C2) holds for any two linearly independent elements $b_1, b_2 \in N$ and if N contains some free element x, then, conversely, S is left Ore.

Proof. Assume on the contrary that there exist non-zero elements $\lambda, \mu \in S$ with $S\lambda \cap S\mu = \{0\}$. Then λx and μx are linearly independent, but $S\lambda x \cap Sx \neq \{0\}$ and $S\mu x \cap Sx \neq \{0\}$, which yields a contradiction. □

Proposition 4.3. *If S is a left Ore domain, then condition* (C3) *is satisfied.*

Proof. Assume it is not. For each $i = 1, 2, 3$ there exist $\lambda_i, \mu_i, \nu_i \in S$ such that

$$\lambda_i b_i = \mu_i c_1 + \nu_i c_2 \neq 0.$$

We may assume that $v_1 v_2 \neq 0$ or $v_1 v_3 \neq 0$ or $v_2 v_3 \neq 0$, because otherwise the preceding proposition yields a contradiction. Suppose that $v_1 v_2 \neq 0$. There exist $\alpha_1, \alpha_2 \in S$ such that $\alpha_1 v_1 = \alpha_2 v_2 \neq 0$. So we obtain

$$\alpha_1 \lambda_1 b_1 - \alpha_2 \lambda_2 b_2 = (\alpha_1 \mu_1 - \alpha_2 \mu_2) c_1.$$

If $v_3 \neq 0$, a similar argument gives a second equality

$$\beta_1 \lambda_1 b_1 - \beta_3 \lambda_3 b_3 = (\beta_1 \mu_1 - \beta_3 \mu_3) c_1.$$

And if $v_3 = 0$, we consider the equality $\lambda_3 b_3 = \mu_3 c_1$. So in both cases we obtain two equalities $\delta_2 d_2 = \gamma_2 c_1$ and $\delta_3 d_3 = \gamma_3 c_1$, where d_2, d_3 are two linearly independent elements. By the preceding proposition this is impossible. □

Corollary 4.4. *Let ${}_R M$ and ${}_S N$ be modules and $g : \mathscr{P}(M) \backslash E \to \mathscr{P}(N)$ a morphism between the associated projective spaces. If the image of g contains three independent free points, and if the ring S is a left Ore domain, then there exists a semilinear map $f : M \to N$ such that $g = \mathscr{P}f$. Moreover, the map f is unique up to multiplication with a unit.*

Remark 4.5. If each $c \in N$ is a multiple of a free element (and if the image of g contains three independent free points), then it is enough to assume that S is a left Ore ring. This is left as an easy exercise.

5 Modules over right Bezout domains

Definition 5.1. We say that a ring S satisfies the 2-*diagonal condition* (D2) if

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \neq \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. We say that S satisfies the 3-*diagonal condition* (D3) if

$$\begin{pmatrix} \mu_1 & v_1 \\ \mu_2 & v_2 \\ \mu_3 & v_3 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \neq \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$.

Remark 5.2. If a ring satisfies condition (D2), then its only idempotents are 0 and 1. In particular, it is directly finite.

Proof. If $\lambda^2 = \lambda$, then $\begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix} (\lambda \quad 1-\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$. □

As in the preceding section, we suppose given two (or three) linearly independent elements b_1, b_2 (and b_3) in N.

Proposition 5.3. *If S satisfies condition* (D2) *and $N = Sb_1 \oplus Sb_2 \oplus N'$, then condition* (C2) *is satisfied.*

Proof. Assume it is not. There exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in S$ such that

$$\lambda_1 b_1 = \mu_1 c \neq 0 \quad \text{and} \quad \lambda_2 b_2 = \mu_2 c \neq 0.$$

Put $c = \alpha_1 b_1 + \alpha_2 b_2 + c_3$. Then $\lambda_1 b_1 = \mu_1 c$ implies $\lambda_1 = \mu_1 \alpha_1$ and $0 = \mu_1 \alpha_2$, and similarly $\lambda_2 b_2 = \mu_2 c$ implies $0 = \mu_2 \alpha_1$ and $\lambda_2 = \mu_2 \alpha_2$, in contradiction to the 2-diagonal condition. □

Proposition 5.4. *If S satisfies condition* (D3) *and $N = Sb_1 \oplus Sb_2 \oplus Sb_3 \oplus N'$, then condition* (C3) *is satisfied.*

Proof. Same argument. □

Lemma 5.5. *If S is directly finite, and if the module S^2 satisfies the following intersection condition*:

(I2) *$x, y \in S^2$ and $Sx \cap Sy \neq \{0\}$ imply $x, y \in Sz$ for some $z \in S^2$,*

then the ring S satisfies both conditions (D2) *and* (D3).

Proof. We first show that S is a domain. Let $\alpha, \beta \in S$ with $\alpha\beta = 0$ and $\alpha \neq 0$. Since $\alpha(1, \beta) = \alpha(1, 0) \neq (0, 0)$, one obtains $(1, \beta) = \gamma(\lambda, \mu)$ and $(1, 0) = \delta(\lambda, \mu)$. One has $\delta\lambda = 1$ and hence $\lambda\delta = 1$. Thus $\beta = \gamma\mu = \gamma\lambda\delta\mu = 0$, and the assertion is proved. Condition (D2) then easily follows. In order to verify condition (D3), we suppose on the contrary that

$$\begin{pmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$. From $\mu_1(\alpha_2, \alpha_3) + \nu_1(\beta_2, \beta_3) = (0, 0)$ one gets $\mu_1(\alpha_2, \alpha_3) = -\nu_1(\beta_2, \beta_3)$, and one can easily show that $\mu_1(\alpha_2, \alpha_3) \neq (0, 0)$. By hypothesis one obtains $(\alpha_2, \alpha_3) = \alpha(\gamma_2, \gamma_3)$ and $(\beta_2, \beta_3) = \beta(\gamma_2, \gamma_3)$. Then

$$\begin{pmatrix} \mu_2\alpha + \nu_2\beta \\ \mu_3\alpha + \nu_3\beta \end{pmatrix} (\gamma_2 \quad \gamma_3) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix},$$

in contradiction to condition (D2). So condition (D3) is verified. □

We show that the intersection condition (I2) is satisfied provided S is a right Bezout domain. We recall that a ring S is *right Bezout* if for any $\alpha, \beta \in S$ there exist $\gamma, \delta, \varepsilon, \lambda, \mu \in S$ such that $\alpha = \gamma\delta$, $\beta = \gamma\varepsilon$ and $\gamma = \alpha\lambda + \beta\mu$.

Proposition 5.6. *If S is a right Bezout domain, then the module S^2 satisfies the condition* (I2). *In particular, S satisfies both conditions* (D2) *and* (D3).

Proof. Suppose that $\lambda(\xi_1, \xi_2) = \mu(\eta_1, \eta_2) \neq (0, 0)$. We put $\omega_1 = \lambda\xi_1 = \mu\eta_1$ and $\omega_2 = \lambda\xi_2 = \mu\eta_2$. By hypothesis one can write $\omega_1 = \nu\zeta_1$ and $\omega_2 = \nu\zeta_2$ for some $\nu = \omega_1\alpha_1 + \omega_2\alpha_2$. Since $\lambda(\xi_1, \xi_2) = \lambda(\xi_1\alpha_1 + \xi_2\alpha_2)(\zeta_1, \zeta_2)$, one concludes that $(\xi_1, \xi_2) = (\xi_1\alpha_1 + \xi_2\alpha_2)(\zeta_1, \zeta_2) = \xi(\zeta_1, \zeta_2)$. Similarly, $(\eta_1, \eta_2) = \eta(\zeta_1, \zeta_2)$. □

Corollary 5.7. *Let ${}_RM$ and ${}_SN$ be modules and $g : \mathscr{P}(M)\backslash E \to \mathscr{P}(N)$ a morphism between the associated projective spaces. If the image of g contains three free points B_1, B_2, B_3 such that $N = B_1 \oplus B_2 \oplus B_3 \oplus N'$, and if the ring S is a right Bezout domain, then there exists a semilinear map $f : M \to N$ such that $g = \mathscr{P}f$. Moreover, the map f is unique up to multiplication with a unit.*

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