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SOME COMPLEX VECTOR SYSTEMS OF PARTIAL FUNCTIONAL EQUATIONS

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Dedicated to the memory of Prof. Ljubomir Davidov

Abstract. In this paper some types of complex vector systems of partial linear and non-linear functional equations are solved.

0. Introduction

In the first part of the paper two types of complex vector systems of linear functional equations are solved. The first type of these systems are the systems of linear complex vector functional equations in which each equation contains all of the unknown functions. The last two theorems represent the second type of systems of partial linear functional equations considered in which not all equations contain all the unknown functions.

The complex vector systems of nonlinear partial functional equations presented in the second part may be sorted in two types. The first three systems solved are considered as complex vector systems of partial quadratic functional equations with real parameters. The last three systems solved represent complex vector systems of partial functional equations of higher order without parameters.

The results presented here expand the results which were obtained in [16].

Now we will introduce the following notations:

Let $\mathcal{V}$, $\mathcal{V}'$ be finite dimensional complex vector spaces. Throughout the paper $\mathbf{Z}_r,\ r \in \mathbf{N}$ are vectors in $\mathcal{V}$. We may assume that $\mathbf{Z}_r = (z_{r1}(t), \dots, z_{rn}(t))^T$, where the components $z_{rj}(t)$ $(1 \le j \le n)$ are complex functions, and $\mathbf{O} = (0, 0, \dots, 0)^T$ is the zero vector in $\mathcal{V}$ or $\mathcal{V}'$. We also denote by $\mathcal{V}^0$ the subspace of all real vectors in $\mathcal{V}$ (thus $\mathcal{V} = \mathcal{V}^0 \oplus i\mathcal{V}^0$), and by $\mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ the space of linear mappings $\mathcal{V}^0 \to \mathcal{V}'$.

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1. Complex Vector Systems of Linear Partial Functional Equations

1.1. Systems in which each equation contains all the unknown functions.

Now we prove the following results.

Theorem 1.1. *The general solution of the system of functional equations*

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O}\,, \tag{1.1}$$

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)+g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O}\,, \tag{1.2}$$

is determined by

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2)&=F_1(\mathbf{Z}_1)-F_2(\mathbf{Z}_2)\,,\\ f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=F_2(\mathbf{Z}_1)+G_1(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\ g_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=-F_1(\mathbf{Z}_1)+G_2(\mathbf{Z}_1,\mathbf{Z}_3)-G_1(\mathbf{Z}_2,\mathbf{Z}_3)+G_2(\mathbf{Z}_2,\mathbf{Z}_3)+A\,,\\ f_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)&=-G_2(\mathbf{Z}_1,\mathbf{Z}_2)+G_3(\mathbf{Z}_3,\mathbf{Z}_4)\,,\\ g_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)&=-G_2(\mathbf{Z}_1,\mathbf{Z}_2)-G_3(\mathbf{Z}_3,\mathbf{Z}_4)-A\,, \end{aligned} \tag{1.3}$$

where $F_1,\ F_2,\ G_r\ (r=1,2,3)$ are arbitrary functions with values in $\mathcal{V}'$, and A is an arbitrary constant complex vector from $\mathcal{V}'$.

Proof. By putting $\mathbf{Z}_r=\mathcal{C}_r\ (r=3,4,5,6)$ into (1.1), we obtain

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)=F_1(\mathbf{Z}_1)-F_2(\mathbf{Z}_2), \tag{1.4}$$

where we introduced the notations

$$\begin{aligned} F_1(\mathbf{Z}_1)&=-g_1(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_4)-g_2(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_5,\mathcal{C}_6)\,,\\ F_2(\mathbf{Z}_2)&=f_1(\mathbf{Z}_2,\mathcal{C}_3,\mathcal{C}_4)+f_2(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_5,\mathcal{C}_6)\,. \end{aligned}$$

By virtue of the expression (1.4), by putting $\mathbf{Z}_r=\mathcal{C}_r\ (r=1,5,6)$ into (1.1) we get

$$f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)=F_2(\mathbf{Z}_2)+G_1(\mathbf{Z}_3,\mathbf{Z}_4)\,, \tag{1.5}$$

where

$$G_1(\mathbf{Z}_3,\mathbf{Z}_4)=-F_1(\mathcal{C}_1)-g_1(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_4)-f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathcal{C}_5,\mathcal{C}_6)-g_2(\mathcal{C}_1,\mathbf{Z}_3,\mathcal{C}_5,\mathcal{C}_6)\,.$$

If we put $\mathbf{Z}_r=\mathcal{C}_r\ (r=5,6)$ into (1.1), in view of (1.4), (1.5) we have

$$g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)=-F_1(\mathbf{Z}_1)-G_1(\mathbf{Z}_3,\mathbf{Z}_4)+G_2'(\mathbf{Z}_3,\mathbf{Z}_4)-H(\mathbf{Z}_1,\mathbf{Z}_3)\,, \tag{1.6}$$

where we introduced the notations

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) = -f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathcal{C}_5,\mathcal{C}_6), \quad H(\mathbf{Z}_1,\mathbf{Z}_3) = g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathcal{C}_5,\mathcal{C}_6).$$

By substituting (1.4), (1.5) and (1.6) into (1.1) and (1.2), the system becomes

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) - H(\mathbf{Z}_1,\mathbf{Z}_3) + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}, \tag{1.7}$$

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) - H(\mathbf{Z}_1,\mathbf{Z}_3) + g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}. \tag{1.8}$$

By putting $\mathbf{Z}_r = \mathcal{C}_1$ $(r = 1,3)$ into (1.8), after replacing $\mathbf{Z}_4$ by $\mathbf{Z}_3$ and putting $\mathbf{Z}_1 = \mathcal{C}_1$ into (1.7), the equations (1.7) and (1.8) become

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) - H(\mathcal{C}_1,\mathbf{Z}_3) + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}, \tag{1.9}$$

$$G_2'(\mathcal{C}_1,\mathbf{Z}_3) - H(\mathcal{C}_1,\mathcal{C}_1) + g_2(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathcal{C}_1,\mathcal{C}_1,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}. \tag{1.10}$$

By subtracting (1.10) from (1.9), we obtain

$$f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) = F_3(\mathbf{Z}_3) - G_2'(\mathbf{Z}_3,\mathbf{Z}_4) + G_3(\mathbf{Z}_5,\mathbf{Z}_6), \tag{1.11}$$

where

$$F_3(\mathbf{Z}_3) = G_2'(\mathcal{C}_1,\mathbf{Z}_3) - H(\mathcal{C}_1,\mathcal{C}_1) + H(\mathcal{C}_1,\mathbf{Z}_3),$$
$$G_3(\mathbf{Z}_5,\mathbf{Z}_6) = f_2(\mathcal{C}_1,\mathcal{C}_1,\mathbf{Z}_5,\mathbf{Z}_6).$$

From (1.7) and (1.11) we obtain

$$g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = H(\mathbf{Z}_1,\mathbf{Z}_3) - F_3(\mathbf{Z}_3) - G_3(\mathbf{Z}_5,\mathbf{Z}_6). \tag{1.12}$$

We substitute the functions f_2 and g_2 determined by (1.11) and (1.12) (with $\mathbf{Z}_3,\mathbf{Z}_4$ replaced by $\mathbf{Z}_1,\mathbf{Z}_3$ and *vice versa*) into the equation (1.8) and obtain

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) + H(\mathbf{Z}_3,\mathbf{Z}_4) - F_3(\mathbf{Z}_4) = G_2'(\mathbf{Z}_1,\mathbf{Z}_3) + H(\mathbf{Z}_1,\mathbf{Z}_3) - F_3(\mathbf{Z}_1). \tag{1.13}$$

It is clear that both sides of this equality are a function just of $\mathbf{Z}_3$, say, $F_4(\mathbf{Z}_3)$. Then we have

$$G_2'(\mathbf{Z}_3,\mathbf{Z}_4) + H(\mathbf{Z}_3,\mathbf{Z}_4) = F_3(\mathbf{Z}_4) + F_4(\mathbf{Z}_3). \tag{1.14}$$

If in (1.14) we replace $\mathbf{Z}_3,\mathbf{Z}_4$ by $\mathbf{Z}_1,\mathbf{Z}_3$, we obtain

$$G_2'(\mathbf{Z}_1,\mathbf{Z}_3) + H(\mathbf{Z}_1,\mathbf{Z}_3) = F_3(\mathbf{Z}_3) + F_4(\mathbf{Z}_1).$$

The last two equalities, together with (1.13), yield

$$F_3(\mathbf{Z}_1) - F_4(\mathbf{Z}_1) = F_3(\mathbf{Z}_3) - F_4(\mathbf{Z}_3). \tag{1.15}$$

It is clear that both sides of this equality are equal to a constant complex vector A, and

$$F_4(\mathbf{Z}_3) = F_3(\mathbf{Z}_3) - A.$$

On the basis of this, the equation (1.14) takes on the form

$$H(\mathbf{Z}_3,\mathbf{Z}_4) = -G_2(\mathbf{Z}_3,\mathbf{Z}_4) + F_3(\mathbf{Z}_4) - A, \tag{1.16}$$

where we introduced a new function

$$G_2(\mathbf{Z}_3,\mathbf{Z}_4) = G_2'(\mathbf{Z}_3,\mathbf{Z}_4) - F_3(\mathbf{Z}_3). \tag{1.17}$$

From (1.16), (1.15), (1.12), (1.11), (1.6), (1.5) and (1.4) there follows the result (1.3). □

This theorem generalizes the result given in [6].

Theorem 1.2. *The general solution of the system of functional equations*

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O}\,, \tag{1.18}$$

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O} \tag{1.19}$$

is determined by

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2)&=F_1(\mathbf{Z}_1)-F_2(\mathbf{Z}_2)\,,\\ f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=F_2(\mathbf{Z}_1)+F_3(\mathbf{Z}_2)+G_2(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\ g_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=F_2(\mathbf{Z}_1)-G_1(\mathbf{Z}_2,\mathbf{Z}_3)-G_2(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\ f_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)&=G_1(\mathbf{Z}_1,\mathbf{Z}_2)-H(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)\,,\\ g_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)&=-F_1(\mathbf{Z}_1)-F_2(\mathbf{Z}_1)-F_3(\mathbf{Z}_2)+H(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)\,, \end{aligned} \tag{1.20}$$

where F_r $(r=1,2,3)$, G_j $(j=1,2)$ and H are arbitrary functions with values in $\mathcal{V}'$.

Proof. If we put $\mathbf{Z}_r=\mathcal{C}_r$ $(r=3,4,5,6)$ into (1.18), we obtain

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)=F_1(\mathbf{Z}_1)-F_2(\mathbf{Z}_2)\,, \tag{1.21}$$

where

$$\begin{aligned} F_1(\mathbf{Z}_1)&=-g_1(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_4)-g_2(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_5,\mathcal{C}_6)\,,\\ F_2(\mathbf{Z}_2)&=f_1(\mathbf{Z}_2,\mathcal{C}_3,\mathcal{C}_4)+f_2(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_5,\mathcal{C}_6)\,. \end{aligned}$$

By putting $\mathbf{Z}_r=\mathcal{C}_r$ $(r=2,4)$ into (1.18), by virtue of the expression (1.21) we get

$$g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=-F_1(\mathbf{Z}_1)-G(\mathbf{Z}_1,\mathbf{Z}_3)+H(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\,, \tag{1.22}$$

where

$$\begin{aligned} H(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)&=-f_1(\mathcal{C}_2,\mathbf{Z}_3,\mathcal{C}_4)-f_2(\mathbf{Z}_3,\mathcal{C}_4,\mathbf{Z}_5,\mathbf{Z}_6)+F_2(\mathcal{C}_2)\,,\\ G(\mathbf{Z}_1,\mathbf{Z}_3)&=-g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathcal{C}_4)\,. \end{aligned}$$

By virtue of the expressions (1.21) and (1.22), if we put $\mathbf{Z}_r=\mathcal{C}_r$ $(r=1,2)$ into (1.18), we get

$$f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)=-H(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)+G_1(\mathbf{Z}_3,\mathbf{Z}_4)\,, \tag{1.23}$$

where we have put

$$G_1(\mathbf{Z}_3,\mathbf{Z}_4)=-f_1(\mathcal{C}_2,\mathbf{Z}_3,\mathbf{Z}_4)-g_1(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_4)+G(\mathcal{C}_1,\mathbf{Z}_3)+F_2(\mathcal{C}_2)\,.$$

In view of the expressions (1.21), (1.22) and (1.23), the system of functional equations (1.18) and (1.19) becomes

$$f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)-F_2(\mathbf{Z}_2)-G(\mathbf{Z}_1,\mathbf{Z}_3)+G_1(\mathbf{Z}_3,\mathbf{Z}_4)=\mathbf{O}\,, \tag{1.24}$$

$$g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)-F_2(\mathbf{Z}_2)-G(\mathbf{Z}_1,\mathbf{Z}_3)+G_1(\mathbf{Z}_3,\mathbf{Z}_4)=\mathbf{O}\,. \tag{1.25}$$

By putting $\mathbf{Z}_2 = \mathcal{C}_2$ into (1.24), after exchanging the roles of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in (1.25), subtracting (1.24) from (1.25), we get

$$f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = F_2(\mathbf{Z}_1) + G(\mathbf{Z}_2,\mathbf{Z}_3) - G(\mathbf{Z}_1,\mathbf{Z}_3) + f_1(\mathcal{C}_2,\mathbf{Z}_3,\mathbf{Z}_4) - F_2(\mathcal{C}_2)\,,$$

or if we put $\mathbf{Z}_1 = \mathcal{C}_2$, we obtain

$$f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = G(\mathbf{Z}_2,\mathbf{Z}_3) + G_2(\mathbf{Z}_3,\mathbf{Z}_4)\,, \tag{1.26}$$

where

$$G_2(\mathbf{Z}_3,\mathbf{Z}_4) = f_1(\mathcal{C}_2,\mathbf{Z}_3,\mathbf{Z}_4) - G(\mathcal{C}_2,\mathbf{Z}_3)\,.$$

From (1.25) and (1.26) it follows that

$$g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = F_2(\mathbf{Z}_2) - G_1(\mathbf{Z}_3,\mathbf{Z}_4) - G_2(\mathbf{Z}_3,\mathbf{Z}_4)\,. \tag{1.27}$$

We substitute the functions f_1 and g_1 determined by (1.26) and (1.27) into the equation (1.24) and find

$$G(\mathbf{Z}_2,\mathbf{Z}_3) - F_2(\mathbf{Z}_2) = G(\mathbf{Z}_1,\mathbf{Z}_3) - F_2(\mathbf{Z}_1)\,.$$

It is clear that both sides of this equality are equal to a function $F_3(\mathbf{Z}_3)$ (independent both of $\mathbf{Z}_1$ and $\mathbf{Z}_2$), thus

$$G(\mathbf{Z}_2,\mathbf{Z}_3) = F_2(\mathbf{Z}_2) + F_3(\mathbf{Z}_3)\,. \tag{1.28}$$

Thus, on the basis of the equations (1.28), (1.27), (1.26), (1.23), (1.22) and (1.21), we obtain (1.20). □

Theorem 1.3. *The general solution of the system of functional equations*

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) &+ f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) \\ &+ f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}\,, \end{aligned} \tag{1.29}$$

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) &+ g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + f_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) \\ &+ g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O} \end{aligned} \tag{1.30}$$

is determined by

$$
\begin{aligned}
f_0(\mathbf{Z}_1,\mathbf{Z}_2) &= F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_2)\,,\\
f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= F_2(\mathbf{Z}_1) + G_1(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\
g_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= F_2(\mathbf{Z}_1) + F(\mathbf{Z}_2) + F_3(\mathbf{Z}_3) - G_1(\mathbf{Z}_2,\mathbf{Z}_3) - A\,,\\
f_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) &= F_3(\mathbf{Z}_1) - F(\mathbf{Z}_1) - F_3(\mathbf{Z}_2) + G_2(\mathbf{Z}_3,\mathbf{Z}_4) + A\,,\\
g_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) &= F_3(\mathbf{Z}_1) - F(\mathbf{Z}_1) - F_3(\mathbf{Z}_2) - G_2(\mathbf{Z}_3,\mathbf{Z}_4)\,,
\end{aligned}
\tag{1.31}
$$

where F_r $(r=1,2,3)$, G_j $(j=1,2)$ and H are arbitrary functions with values in $\mathcal{V}'$, $F=F_1+F_2+F_3$ and A is an arbitrary constant complex vector from $\mathcal{V}'$.

Proof. By putting $\mathbf{Z}_r=\mathcal{C}_r$ $(r=3,4,5,6)$ into (1.29), we obtain

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2) = F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_2)\,, \tag{1.32}$$

where

$$
\begin{aligned}
F_1(\mathbf{Z}_1) &= -g_1(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_4) - g_2(\mathbf{Z}_1,\mathcal{C}_3,\mathcal{C}_5,\mathcal{C}_6)\,,\\
F_2(\mathbf{Z}_2) &= f_1(\mathbf{Z}_2,\mathcal{C}_3,\mathcal{C}_4) + f_2(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_5,\mathcal{C}_6)\,.
\end{aligned}
$$

If we put $\mathbf{Z}_r=\mathcal{C}_r$ $(r=1,5,6)$ into (1.29), in view of the expression (1.32) we get

$$f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = F_2(\mathbf{Z}_2) + G_1(\mathbf{Z}_3,\mathbf{Z}_4)\,, \tag{1.33}$$

where

$$G_1(\mathbf{Z}_3,\mathbf{Z}_4) = -F_1(\mathcal{C}_1) - g_1(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_4) - f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathcal{C}_5,\mathcal{C}_6) - g_2(\mathcal{C}_1,\mathbf{Z}_3,\mathcal{C}_5,\mathcal{C}_6)\,.$$

By putting $\mathbf{Z}_r=\mathcal{C}_r$ $(r=1,5,6)$ into (1.30), by virtue of the expression (1.32) we obtain

$$g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = F_2(\mathbf{Z}_2) + G(\mathbf{Z}_3,\mathbf{Z}_4)\,, \tag{1.34}$$

where we have put

$$G(\mathbf{Z}_3,\mathbf{Z}_4) = -F_1(\mathcal{C}_1) - f_1(\mathcal{C}_1,\mathbf{Z}_3,\mathbf{Z}_4) - g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathcal{C}_5,\mathcal{C}_6) - f_2(\mathcal{C}_1,\mathbf{Z}_3,\mathcal{C}_5,\mathcal{C}_6)\,.$$

On the basis of the expressions (1.32), (1.33) and (1.34), the system of functional equations (1.29) and (1.30) becomes

$$
\begin{aligned}
&F_1(\mathbf{Z}_1) + F_2(\mathbf{Z}_1) + G_1(\mathbf{Z}_3,\mathbf{Z}_4) + G(\mathbf{Z}_3,\mathbf{Z}_4)\\
&\qquad + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}\,,
\end{aligned}
\tag{1.35}
$$

$$
\begin{aligned}
&F_1(\mathbf{Z}_1) + F_2(\mathbf{Z}_1) + G_1(\mathbf{Z}_3,\mathbf{Z}_4) + G(\mathbf{Z}_3,\mathbf{Z}_4)\\
&\qquad + g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}\,.
\end{aligned}
\tag{1.36}
$$

We put $\mathbf{Z}_r = \mathcal{C}_1$ $(r = 1, 3)$ into (1.36), and further we replace $\mathbf{Z}_4$ by $\mathbf{Z}_3$. Also, we put $\mathbf{Z}_1 = \mathcal{C}_1$ into (1.35). By subtracting the equations obtained, we get

$$f_2(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6) = F_3(\mathbf{Z}_3) - G_1(\mathbf{Z}_3, \mathbf{Z}_4) + G_2(\mathbf{Z}_5, \mathbf{Z}_6) - G(\mathbf{Z}_3, \mathbf{Z}_4)\,, \tag{1.37}$$

where we introduced the notations

$$F_3(\mathbf{Z}_3) = G_1(\mathcal{C}_1, \mathbf{Z}_3) + G(\mathcal{C}_1, \mathbf{Z}_3)\,, \quad G_2(\mathbf{Z}_5, \mathbf{Z}_6) = f_2(\mathcal{C}_1, \mathcal{C}_1, \mathbf{Z}_5, \mathbf{Z}_6)\,.$$

From the equation (1.35), on the basis of the expression (1.37), we get

$$g_2(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{Z}_6) = -F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_1) - F_3(\mathbf{Z}_3) - G_2(\mathbf{Z}_5, \mathbf{Z}_6)\,. \tag{1.38}$$

We substitute the functions f_2 and g_2 determined by (1.37) and (1.38) into the equation (1.36) and find

$$\begin{aligned} &F_1(\mathbf{Z}_1) + F_2(\mathbf{Z}_1) + F_3(\mathbf{Z}_1) - F_1(\mathbf{Z}_3) - F_2(\mathbf{Z}_3) - F_3(\mathbf{Z}_4) \\ &\qquad + G_1(\mathbf{Z}_3, \mathbf{Z}_4) + G(\mathbf{Z}_3, \mathbf{Z}_4) - G_1(\mathbf{Z}_1, \mathbf{Z}_3) - G(\mathbf{Z}_1, \mathbf{Z}_3) = \mathbf{O}\,. \end{aligned} \tag{1.39}$$

If we put $\mathbf{Z}_1 = \mathcal{C}_1$, the equation (1.39) becomes

$$G_1(\mathbf{Z}_3, \mathbf{Z}_4) + G(\mathbf{Z}_3, \mathbf{Z}_4) = F_1(\mathbf{Z}_3) + F_2(\mathbf{Z}_3) + F_3(\mathbf{Z}_3) + F_3(\mathbf{Z}_4) - A\,, \tag{1.40}$$

since

$$G_1(\mathcal{C}_1, \mathbf{Z}_3) + G(\mathcal{C}_1, \mathbf{Z}_3) = F_3(\mathbf{Z}_3)\,,$$

where we have put $A = \sum_{i=1}^{3} F_i(\mathcal{C}_1)$.

On the basis of the expression (1.40), the equalities (1.34) and (1.37) obtain respectively the forms

$$g_1(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = F_2(\mathbf{Z}_2) + F(\mathbf{Z}_3) + F_3(\mathbf{Z}_4) - G_1(\mathbf{Z}_3, \mathbf{Z}_4) - A\,, \tag{1.41}$$

$$f_2(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6) = F_3(\mathbf{Z}_3) - F(\mathbf{Z}_3) - F_3(\mathbf{Z}_4) + G_2(\mathbf{Z}_5, \mathbf{Z}_6) + A\,, \tag{1.42}$$

where we introduced the new function

$$F(\mathbf{Z}_3) = \sum_{i=1}^{3} F_i(\mathbf{Z}_3)\,. \tag{1.43}$$

On the basis of the equalities (1.43), (1.42), (1.41), (1.38), (1.33) and (1.32), we obtain the equalities (1.31). □

Theorem 1.4. *The general solution of the system of functional equations*

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) + f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) \\ + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}\,, \end{aligned} \tag{1.44}$$

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) + f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) \\ + g_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + f_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O}\,, \end{aligned} \tag{1.45}$$

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) + g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + f_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4) \\ + f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) = \mathbf{O} \end{aligned} \tag{1.46}$$

is determined by the equalities

$$\begin{aligned} f_0(\mathbf{Z}_1,\mathbf{Z}_2) &= F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_2)\,, \\ f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= F_2(\mathbf{Z}_1) + G_1(\mathbf{Z}_2,\mathbf{Z}_3)\,, \\ g_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= F_2(\mathbf{Z}_1) + F_1(\mathbf{Z}_2) + F_2(\mathbf{Z}_2) - G_1(\mathbf{Z}_2,\mathbf{Z}_3) + K\,, \\ f_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) &= -F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_1) + G(\mathbf{Z}_3,\mathbf{Z}_4)\,, \\ g_2(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) &= F_1(\mathbf{Z}_1) - F_2(\mathbf{Z}_1) - G(\mathbf{Z}_3,\mathbf{Z}_4) - K\,, \end{aligned} \tag{1.47}$$

where F_1, F_2, G, G_1 are arbitrary functions with values in $\mathcal{V}'$, and K is an arbitrary constant complex vector from $\mathcal{V}'$.

Proof. The equations (1.44) and (1.45) form the system of functional equations given by (1.1) and (1.2). Consequently, the functions determined by the equalities (1.3) satisfy the equations (1.44) and (1.45).

In order for the functions (1.3) to satisfy the equation (1.46) we must have

$$F_1(\mathbf{Z}_1) + F_2(\mathbf{Z}_1) - G_2(\mathbf{Z}_1,\mathbf{Z}_3) = F_1(\mathbf{Z}_2) + F_2(\mathbf{Z}_2) - G_2(\mathbf{Z}_2,\mathbf{Z}_4) = \mathbf{O}\,. \tag{1.48}$$

It is clear that both sides of equality (1.48) are equal to a constant complex vector M, thus

$$G_2(\mathbf{Z}_1,\mathbf{Z}_3) = F_1(\mathbf{Z}_1) + F_2(\mathbf{Z}_1) + M\,, \tag{1.49}$$

where $M = G_2(\mathcal{C}_2,\mathcal{C}_4) - F_1(\mathcal{C}_2) - F_2(\mathcal{C}_2)$.

If we introduce a new function G by

$$G(\mathbf{Z}_1,\mathbf{Z}_2) = G_3(\mathbf{Z}_1,\mathbf{Z}_2) + M$$

and denote $K = A - 2M$, from (1.49) and (1.3) there follows (1.47). □

1.2. Systems in which not all equations contain all the unknown functions.

Theorem 1.5. *The general solution of the system of functional equations*

$$f_0(\mathbf{Z}_1, \mathbf{Z}_2) + f_1(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + g_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) = \mathbf{O}\,, \tag{1.50}$$

$$f_0(\mathbf{Z}_1, \mathbf{Z}_2) + g_1(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) + f_2(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6) = \mathbf{O}, \tag{1.51}$$

is given by the equalities

$$\begin{aligned} f_0(\mathbf{Z}_1, \mathbf{Z}_2) &= -F(\mathbf{Z}_1) - F(\mathbf{Z}_2)\,, \\ f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= F(\mathbf{Z}_1) - G(\mathbf{Z}_2, \mathbf{Z}_3)\,, \\ g_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= F(\mathbf{Z}_1) + G(\mathbf{Z}_2, \mathbf{Z}_3)\,, \\ f_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) &= \mathbf{O}\,, \end{aligned} \tag{1.52}$$

where F and G are arbitrary functions with values in $\mathcal{V}'$.

Proof. The general solution of the functional equation (1.50) is

$$\begin{aligned} f_0(\mathbf{Z}_1, \mathbf{Z}_2) &= H_1(\mathbf{Z}_1) - F_1(\mathbf{Z}_2)\,, \\ f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= F_1(\mathbf{Z}_1) - G_1(\mathbf{Z}_2, \mathbf{Z}_3)\,, \\ g_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= -H_1(\mathbf{Z}_1) + G_1(\mathbf{Z}_2, \mathbf{Z}_3)\,, \end{aligned} \tag{1.53}$$

where F_1, H_1 and G_1 are arbitrary functions with values in $\mathcal{V}'$.

We substitute the functions f_0, f_1 and g_1 determined by the equalities (1.53) into the equation (1.51). We must have

$$F_1(\mathbf{Z}_1) + H_1(\mathbf{Z}_1) - F_1(\mathbf{Z}_2) - H_1(\mathbf{Z}_2) = -f_2(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6)\,. \tag{1.54}$$

It is obvious that both sides of this equality are constant. By putting $\mathbf{Z}_1 = \mathbf{Z}_2$ into (1.54), we get

$$f_2(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6) = \mathbf{O}\,, \tag{1.55}$$

and further

$$F_1(\mathbf{Z}_1) + H_1(\mathbf{Z}_1) = F_1(\mathbf{Z}_2) + H_1(\mathbf{Z}_2) = 2A\,, \tag{1.56}$$

where A is a constant complex vector from $\mathcal{V}'$.

By introducing new functions F and G by the following equalities

$$\begin{aligned} F(\mathbf{Z}_1) &= F_1(\mathbf{Z}_1) - A\,, \\ G(\mathbf{Z}_1, \mathbf{Z}_2) &= G_1(\mathbf{Z}_1, \mathbf{Z}_2) - A\,, \end{aligned}$$

on the basis of the expressions (1.56), (1.55) and (1.53), we obtain (1.52). □

Theorem 1.6. *The general solution of the system of functional equations*

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)=\mathbf{O}\,,$$

$$f_0(\mathbf{Z}_1,\mathbf{Z}_2)+g_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)+f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O}\,,$$

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&+f_1(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+g_1(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4)\\&+f_2(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g_2(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)=\mathbf{O}\,,\end{aligned}$$

$$\vdots$$

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&+\sum_{r=1}^{k-1}g_r(\mathbf{Z}_{r+1},\mathbf{Z}_{r+2},\dots,\mathbf{Z}_{2r+2})\\&+\sum_{r=1}^{k-1}f_r(\mathbf{Z}_1,\mathbf{Z}_3,\dots,\mathbf{Z}_{2r+1},\mathbf{Z}_{2r+2})+f_k(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{2k+2})=\mathbf{O}\,,\end{aligned}$$

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&+\sum_{r=1}^{k}f_r(\mathbf{Z}_{r+1},\mathbf{Z}_{r+2},\dots,\mathbf{Z}_{2r+2})\\&+\sum_{r=1}^{k}g_r(\mathbf{Z}_1,\mathbf{Z}_3,\dots,\mathbf{Z}_{2r+1},\mathbf{Z}_{2r+2})=\mathbf{O}\,,\end{aligned}$$

$$\vdots$$

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&+\sum_{r=1}^{m-1}g_r(\mathbf{Z}_{r+1},\mathbf{Z}_{r+2},\dots,\mathbf{Z}_{2r+2})\\&+\sum_{r=1}^{m-1}f_r(\mathbf{Z}_1,\mathbf{Z}_3,\dots,\mathbf{Z}_{2r+1},\mathbf{Z}_{2r+2})+f_m(\mathbf{Z}_{m+1},\mathbf{Z}_{m+2},\dots,\mathbf{Z}_{2m+2})=\mathbf{O}\,,\end{aligned}$$

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&+\sum_{r=1}^{m}f_r(\mathbf{Z}_{r+1},\mathbf{Z}_{r+2},\dots,\mathbf{Z}_{2r+2})\\&+\sum_{r=1}^{m}g_r(\mathbf{Z}_1,\mathbf{Z}_3,\dots,\mathbf{Z}_{2r+1},\mathbf{Z}_{2r+2})=\mathbf{O}\,,\end{aligned}$$

is given by

$$\begin{aligned}f_0(\mathbf{Z}_1,\mathbf{Z}_2)&=-F(\mathbf{Z}_1)-F(\mathbf{Z}_2)\,,\\f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=F(\mathbf{Z}_1)-G(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\g_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=F(\mathbf{Z}_1)+G(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\f_r=g_r&=\mathbf{O}\quad(2\le r\le m)\,,\end{aligned}$$

where F and G are arbitrary functions with values in $\mathcal{V}'$.

Proof. The proof of this theorem follows from Theorem 1.5. □

We have noticed that this approach to the solution of systems of linear complex vector functional equations is not considered in the references [7, 8, 9].

2. Some Complex Vector Systems of Nonlinear Partial Functional Equations

In this section the unknown functions depend on arguments in an arbitrary finite dimensional complex vector space $\mathcal{V}$ and take values in the field of complex numbers $\mathbf{C}$ so that their products make sense.

2.1. Systems of quadratic functional equations.

Consider the system of functional equations

$$
\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})+f(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4,\dots,\mathbf{Z}_{2n},\mathbf{Z}_2)\\
&+\cdots+f(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\dots,\mathbf{Z}_{2n-2},\mathbf{Z}_{2n-1})+\operatorname{sgn}a\,[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})\\
&+g(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4,\dots,\mathbf{Z}_{2n},\mathbf{Z}_2)+\cdots+g(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\dots,\mathbf{Z}_{2n-2},\mathbf{Z}_{2n-1})]=0\,,\\
&h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})+h(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4,\dots,\mathbf{Z}_{2n},\mathbf{Z}_2)\\
&+\cdots+h(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\dots,\mathbf{Z}_{2n-2},\mathbf{Z}_{2n-1})+k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})\\
&+k(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_4,\dots,\mathbf{Z}_{2n},\mathbf{Z}_2)+\cdots+k(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\dots,\mathbf{Z}_{2n-2},\mathbf{Z}_{2n-1})=0\,,
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})&=[F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_3,\mathbf{Z}_4)+\cdots+F(\mathbf{Z}_{2r-1},\mathbf{Z}_{2r})]\\
&\times[F(\mathbf{Z}_{2r+1},\mathbf{Z}_{2n})+F(\mathbf{Z}_{2r+2},\mathbf{Z}_{2n-1})+\cdots+F(\mathbf{Z}_{r+n},\mathbf{Z}_{r+n+1})]\,,\\
g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})&=[G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_3,\mathbf{Z}_4)+\cdots+G(\mathbf{Z}_{2r-1},\mathbf{Z}_{2r})]\\
&\times[G(\mathbf{Z}_{2r+1},\mathbf{Z}_{2n})+G(\mathbf{Z}_{2r+2},\mathbf{Z}_{2n-1})+\cdots+G(\mathbf{Z}_{r+n},\mathbf{Z}_{r+n+1})]\,,\\
h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})&=[F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_3,\mathbf{Z}_4)+\cdots+F(\mathbf{Z}_{2r-1},\mathbf{Z}_{2r})]\\
&\times[G(\mathbf{Z}_{2r+1},\mathbf{Z}_{2n})+G(\mathbf{Z}_{2r+2},\mathbf{Z}_{2n-1})+\cdots+G(\mathbf{Z}_{r+n},\mathbf{Z}_{r+n+1})]\,,\\
k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n})&=[G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_3,\mathbf{Z}_4)+\cdots+G(\mathbf{Z}_{2r-1},\mathbf{Z}_{2r}]\\
&\times[F(\mathbf{Z}_{2r+1},\mathbf{Z}_{2n})+F(\mathbf{Z}_{2r+2},\mathbf{Z}_{2n-1})+\cdots+F(\mathbf{Z}_{r+n},\mathbf{Z}_{r+n+1})]\,,
\end{aligned}
\tag{2.2}
$$

$F,G:\mathcal{V}^2\to\mathbf{C}$ are arbitrary complex functions, a is a real constant and $n>2$, $1\le r\le n-1$.

Definition 2.1. A solution $(F(\mathbf{U},\mathbf{V}),G(\mathbf{U},\mathbf{V}))$ of system (2.1) is called *isotropic* if $(F(\mathbf{U},\mathbf{V}),G(\mathbf{U},\mathbf{V}))\not\equiv(0,0)$ and

$$F^2(\mathbf{U},\mathbf{V})+G^2(\mathbf{U},\mathbf{V})\equiv 0\,. \tag{2.3}$$

The *general* solution of system (2.1) includes all solutions of this system with the possible exception of the isotropic ones.

Theorem 2.1. *The general solution of the system (2.1) is given by the formulas*

$$F(\mathbf{U},\mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V})\,, \tag{2.4}$$

$$G(\mathbf{U},\mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V})\,, \tag{2.5}$$

where P and Q are arbitrary complex functions defined in $\mathcal{V}$.

Each isotropic **continuous** *solution of (2.1) has the form*

$$F(\mathbf{U},\mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V})\,, \quad G(\mathbf{U},\mathbf{V}) = \pm i(P(\mathbf{U}) - P(\mathbf{V}))$$

($i^2 = -1$*; we take the same sign for all vectors* $\mathbf{U},\mathbf{V} \in \mathcal{V}$*).*

Proof. Let $F(\mathbf{U},\mathbf{V})$, $G(\mathbf{U},\mathbf{V})$ represent an isotropic continuous solution of system (2.1). This means that for any pair of vectors $(\mathbf{U},\mathbf{V}) \in \mathcal{V}^2$ we have

$$G(\mathbf{U},\mathbf{V}) = \pm i F(\mathbf{U},\mathbf{V}) \tag{2.6}$$

(both signs are possible). It is clear that the real dimension of $\mathcal{V}^2$ is $4\dim\mathcal{V}$, the real codimension of the set $\{(\mathbf{U},\mathbf{V}) \in \mathcal{V}^2 :\ F(\mathbf{U},\mathbf{V}) = 0\}$ is 2. Hence its complement is a connected set in which the function $G(\mathbf{U},\mathbf{V})/F(\mathbf{U},\mathbf{V})$ is continuous, thus it can take just one of the values i or $-i$. Henceforth, when we write $\pm i$ (as in (2.6)), we mean only one of these two signs.

By virtue of (2.6) system (2.1) implies

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1},\mathbf{Z}_{2n}) + f(\mathbf{Z}_1,\mathbf{Z}_3,\ldots,\mathbf{Z}_{2n},\mathbf{Z}_2)& \\ + \cdots + f(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1}) = 0\,.& \end{aligned} \tag{2.7}$$

The general solution of this equation for $n > 2$ according to [12] is (2.4):

$$F(\mathbf{U},\mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V})\,,$$

where $P(\mathbf{U}) = F(\mathbf{U},\mathbf{A})$ for some fixed vector $\mathbf{A}$. Then

$$G(\mathbf{U},\mathbf{V}) = \pm i(P(\mathbf{U}) - P(\mathbf{V}))\,.$$

Next we are looking for solutions of system (2.1) not satisfying (2.3). We will distinguish the following cases:

1°. Let $a < 0$. In this case the system of complex vector functional equations (2.1) will be

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n}) &+ f(\mathbf{Z}_1,\mathbf{Z}_3,\ldots,\mathbf{Z}_{2n},\mathbf{Z}_2) \\ &+ \cdots + f(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1}) \\ &- g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n}) - g(\mathbf{Z}_1,\mathbf{Z}_3,\ldots,\mathbf{Z}_{2n},\mathbf{Z}_2) \\ &- \cdots - g(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1}) = 0\,, \\ h(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n}) &+ h(\mathbf{Z}_1,\mathbf{Z}_3,\ldots,\mathbf{Z}_{2n},\mathbf{Z}_2) \\ &+ \cdots + h(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1}) \\ &+ k(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n}) + k(\mathbf{Z}_1,\mathbf{Z}_3,\ldots,\mathbf{Z}_{2n},\mathbf{Z}_2) \\ &+ \cdots + k(\mathbf{Z}_1,\mathbf{Z}_{2n},\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n-1}) = 0\,. \end{aligned} \tag{2.8}$$

If we substitute all the variables $\mathbf{Z}_r$ $(1 \le r \le 2n)$ in the system (2.8) by $\mathbf{U}$, then we obtain

$$F^2(\mathbf{U},\mathbf{U}) - G^2(\mathbf{U},\mathbf{U}) = 0, \qquad 2F(\mathbf{U},\mathbf{U})G(\mathbf{U},\mathbf{U}) = 0\,,$$

so that

$$F(\mathbf{U},\mathbf{U}) = 0, \quad G(\mathbf{U},\mathbf{U}) = 0\,. \tag{2.9}$$

For any nontrivial solution of the system (2.8) there exists at least one pair of vectors $(\mathbf{A},\mathbf{B}) \in \mathcal{V}^2$ such that

$$F^2(\mathbf{A},\mathbf{B}) + G^2(\mathbf{A},\mathbf{B}) \ne 0\,. \tag{2.10}$$

a) Let $r = 1$. If we execute the substitutions $\mathbf{Z}_5 = \mathbf{Z}_6 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{A}$, $\mathbf{Z}_3 = \mathbf{B}$, $\mathbf{Z}_4 = \mathbf{U}$, on the basis of the equalities (2.9) the system (2.8) becomes

$$\begin{aligned} &F(\mathbf{A},\mathbf{B})[F(\mathbf{U},\mathbf{A}) + F(\mathbf{A},\mathbf{U})] - G(\mathbf{A},\mathbf{B})[G(\mathbf{U},\mathbf{A}) + G(\mathbf{A},\mathbf{U})] = 0\,,\\ &G(\mathbf{A},\mathbf{B})[F(\mathbf{U},\mathbf{A}) + F(\mathbf{A},\mathbf{U})] + F(\mathbf{A},\mathbf{B})[G(\mathbf{U},\mathbf{A}) + G(\mathbf{A},\mathbf{U})] = 0\,. \end{aligned} \tag{2.11}$$

In view of (2.10), from (2.11) it follows that

$$F(\mathbf{A},\mathbf{U}) = -F(\mathbf{U},\mathbf{A})\,, \qquad G(\mathbf{A},\mathbf{U}) = -G(\mathbf{U},\mathbf{A})\,. \tag{2.12}$$

By putting $\mathbf{Z}_5 = \mathbf{Z}_6 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Z}_3 = \mathbf{B}$, $\mathbf{Z}_4 = \mathbf{U}$ into (2.8), on the basis of the expressions (2.9) and (2.12) we get

$$\begin{aligned} &F(\mathbf{A},\mathbf{B})[F(\mathbf{U},\mathbf{V}) - F(\mathbf{U},\mathbf{A}) + F(\mathbf{V},\mathbf{A})]\\ &-G(\mathbf{A},\mathbf{B})[G(\mathbf{U},\mathbf{V}) - G(\mathbf{U},\mathbf{A}) + G(\mathbf{V},\mathbf{A})] = 0\,,\\ &G(\mathbf{A},\mathbf{B})[F(\mathbf{U},\mathbf{V}) - F(\mathbf{U},\mathbf{A}) + F(\mathbf{V},\mathbf{A})]\\ &+F(\mathbf{A},\mathbf{B})[G(\mathbf{U},\mathbf{V}) - G(\mathbf{U},\mathbf{A}) + G(\mathbf{V},\mathbf{A})] = 0\,, \end{aligned} \tag{2.13}$$

whence

$$F(\mathbf{U},\mathbf{V}) - F(\mathbf{U},\mathbf{A}) + F(\mathbf{V},\mathbf{A}) = 0\,, \quad G(\mathbf{U},\mathbf{V}) - G(\mathbf{U},\mathbf{A}) + G(\mathbf{V},\mathbf{A}) = 0\,,$$

which implies (2.4), (2.5):

$$F(\mathbf{U},\mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V})\,, \qquad G(\mathbf{U},\mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V})\,,$$

where we have introduced the notations $P(\mathbf{U}) = F(\mathbf{U},\mathbf{A})$, $Q(\mathbf{U}) = G(\mathbf{U},\mathbf{A})$.

b) Let $1 < r < n-1$. For $\mathbf{Z}_4 = \mathbf{Z}_5 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{B}$, $\mathbf{Z}_3 = \mathbf{U}$, the system (2.8) becomes (2.11) and again we have (2.12).

By putting $\mathbf{Z}_5 = \mathbf{Z}_6 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{U}$, $\mathbf{Z}_3 = \mathbf{V}$, $\mathbf{Z}_4 = \mathbf{B}$, on the basis of the equalities (2.9) and (2.12) the system (2.8) takes on the form (2.13). In view of (2.10) this again leads to (2.4), (2.5), where $P(\mathbf{U}) = F(\mathbf{U}, \mathbf{A})$ and $Q(\mathbf{U}) = G(\mathbf{U}, \mathbf{A})$.

c) Let $r = n - 1$. If we substitute all variables $\mathbf{Z}_1$, $\mathbf{Z}_3$, $\ldots$, $\mathbf{Z}_{2n-1}$ (except for $\mathbf{Z}_2$ and $\mathbf{Z}_{2n}$) by $\mathbf{A}$, and if we put $\mathbf{Z}_2 = \mathbf{U}$, $\mathbf{Z}_{2n} = \mathbf{B}$, then the system (2.8) becomes (2.11), which implies (2.12).

If we substitute in the system (2.8)

$$\mathbf{Z}_3 = \mathbf{Z}_4 = \cdots = \mathbf{Z}_{2n-2} = \mathbf{Z}_1 = \mathbf{A}, \quad \mathbf{Z}_2 = \mathbf{V}, \quad \mathbf{Z}_{2n-1} = \mathbf{B}, \quad \mathbf{Z}_{2n} = \mathbf{U},$$

on the basis of equality (2.12) we obtain (2.13), and we find (2.4), (2.5).

Consequently, if $a < 0$, the theorem is proved.

2°. Let $a = 0$. In this case the system (2.1) becomes

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) &+ f(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + f(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) = 0, \\ h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n}) &+ h(\mathbf{Z}_1, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + h(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) \\ &+ k(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n}) + k(\mathbf{Z}_1, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + k(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) = 0. \end{aligned} \tag{2.14}$$

The first equation of the system (2.14) is (2.7), hence its general solution for $n > 2$ according to [12] is (2.4):

$$F(\mathbf{U}, \mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V}),$$

where $P(\mathbf{U}) = F(\mathbf{U}, \mathbf{A})$.

In the second equation of the system (2.14), if we put $\mathbf{Z}_1 = \mathbf{B}$ and substitute all the other variables $\mathbf{Z}_r$ $(2 \le r \le 2n)$ by $\mathbf{A}$ so that $F(\mathbf{A}, \mathbf{B}) \neq 0$, we obtain

$$F(\mathbf{A}, \mathbf{B}) G(\mathbf{A}, \mathbf{A}) = 0,$$

whence

$$G(\mathbf{A}, \mathbf{A}) = 0. \tag{2.15}$$

Now, we will distinguish three cases.

a) If $r = 1$, by putting $\mathbf{Z}_5 = \mathbf{Z}_6 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Z}_3 = \mathbf{B}$, $\mathbf{Z}_4 = \mathbf{U}$, the second equation of the system (2.14) in view of the equalities (2.4) and (2.15) becomes

$$\begin{aligned} F(\mathbf{A}, \mathbf{B}) G(\mathbf{U}, \mathbf{V}) &+ F(\mathbf{A}, \mathbf{B}) G(\mathbf{A}, \mathbf{U}) + F(\mathbf{B}, \mathbf{A}) G(\mathbf{A}, \mathbf{V}) \\ &+ G(\mathbf{A}, \mathbf{B}) F(\mathbf{U}, \mathbf{V}) + G(\mathbf{A}, \mathbf{B}) F(\mathbf{A}, \mathbf{U}) + G(\mathbf{B}, \mathbf{A}) F(\mathbf{A}, \mathbf{V}) \\ &+ F(\mathbf{A}, \mathbf{V}) G(\mathbf{A}, \mathbf{U}) + F(\mathbf{A}, \mathbf{V}) G(\mathbf{U}, \mathbf{A}) = 0. \end{aligned} \tag{2.16}$$

If we put $\mathbf{V} = \mathbf{A}$ into (2.20), on the basis of the expression (2.15) we obtain $G(\mathbf{A}, \mathbf{U}) = -G(\mathbf{U}, \mathbf{A})$, and then (2.16) becomes

$$G(\mathbf{U}, \mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V}),$$

where $Q(\mathbf{U}) = G(\mathbf{U}, \mathbf{A})$.

b) In the case $1 < r < n-1$, if we substitute all variables $\mathbf{Z}_1, \mathbf{Z}_4, \mathbf{Z}_5, \dots, \mathbf{Z}_{2n-1}$, $\mathbf{Z}_{2n}$ (except for $\mathbf{Z}_2$ and $\mathbf{Z}_3$) by $\mathbf{A}$, and if we put $\mathbf{Z}_2 = \mathbf{B}$, $\mathbf{Z}_3 = \mathbf{U}$, the second equation of the system (2.14) becomes

$$F(\mathbf{A}, \mathbf{B})[G(\mathbf{A}, \mathbf{U}) + G(\mathbf{U}, \mathbf{A})] = 0,$$

whence

$$G(\mathbf{A}, \mathbf{U}) = -G(\mathbf{U}, \mathbf{A}). \tag{2.17}$$

For $\mathbf{Z}_5 = \mathbf{Z}_6 = \cdots = \mathbf{Z}_{2n} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{U}$, $\mathbf{Z}_3 = \mathbf{V}$, $\mathbf{Z}_4 = \mathbf{B}$, on the basis of the equality (2.17) and (2.4), the second equation of the system (2.14) takes on the following form

$$G(\mathbf{U}, \mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V}),$$

where $Q(\mathbf{U}) = G(\mathbf{U}, \mathbf{A})$.

c) If $r = n - 1$, for $\mathbf{Z}_3 = \mathbf{Z}_4 = \cdots = \mathbf{Z}_{2n-1} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{U}$, $\mathbf{Z}_{2n} = \mathbf{B}$, by virtue of the equality (2.4), from the second equation of the system (2.14) we obtain (2.17).

By putting $\mathbf{Z}_3 = \mathbf{Z}_4 = \cdots = \mathbf{Z}_{2n-2} = \mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Z}_{2n-1} = \mathbf{B}$, $\mathbf{Z}_{2n} = \mathbf{U}$, on the basis of the equality (2.4) and (2.17) from the second equation of the system (2.14) we find

$$G(\mathbf{U}, \mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V}),$$

where $Q(\mathbf{U}) = G(\mathbf{U}, \mathbf{A})$, i.e. Theorem 2.1 is proved if $a = 0$.

3°. If $a > 0$, then the system (2.1) becomes

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n}) &+ f(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \dots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + f(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \dots, \mathbf{Z}_{2n-1}) \\ &+ g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n}) + g(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \dots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + g(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \dots, \mathbf{Z}_{2n-1}) = 0, \\ h(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n}) &+ h(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \dots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + h(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \dots, \mathbf{Z}_{2n-1}) \\ &+ k(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n}) + k(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \dots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\ &+ \cdots + k(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \dots, \mathbf{Z}_{2n-1}) = 0. \end{aligned} \tag{2.18}$$

If we introduce new functions S and T by the formulas

$$S(\mathbf{U}, \mathbf{V}) = F(\mathbf{U}, \mathbf{V}) + G(\mathbf{U}, \mathbf{V}), \qquad T(\mathbf{U}, \mathbf{V}) = F(\mathbf{U}, \mathbf{V}) - G(\mathbf{U}, \mathbf{V}), \tag{2.19}$$

the system (2.18) becomes

$$
\begin{aligned}
s(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}) +{}& s(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&+ \cdots + s(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) + t(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}) \\
&+ t(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&+ \cdots + t(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) = 0\,, \\
s(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}) +{}& s(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&+ \cdots + s(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) - t(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}) \\
&- t(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&- \cdots - t(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-1}) = 0\,,
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{aligned}
s(\mathbf{Z}_1, \mathbf{Z}_2,\, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) &= [S(\mathbf{Z}_1, \mathbf{Z}_2) + S(\mathbf{Z}_3, \mathbf{Z}_4) + \cdots + S(\mathbf{Z}_{2r-1}, \mathbf{Z}_{2r})] \\
&\times [S(\mathbf{Z}_{2r+1}, \mathbf{Z}_{2n}) + S(\mathbf{Z}_{2r+2}, \mathbf{Z}_{2n-1}) + \cdots + S(\mathbf{Z}_{r+n}, \mathbf{Z}_{r+n+1})]\,, \\
t(\mathbf{Z}_1, \mathbf{Z}_2,\, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) &= [T(\mathbf{Z}_1, \mathbf{Z}_2) + T(\mathbf{Z}_3, \mathbf{Z}_4) + \cdots + T(\mathbf{Z}_{2r-1}, \mathbf{Z}_{2r})] \\
&\times [T(\mathbf{Z}_{2r+1}, \mathbf{Z}_{2n}) + T(\mathbf{Z}_{2r+2}, \mathbf{Z}_{2n-1}) + \cdots + T(\mathbf{Z}_{r+n}, \mathbf{Z}_{r+n+1})]\,.
\end{aligned}
$$

Comparing the equations of the system (2.20), we find

$$
\begin{aligned}
s(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) &+ s(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&+ \cdots + s(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-2}, \mathbf{Z}_{2n-1}) = 0\,, \\
t(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) &+ t(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_{2n}, \mathbf{Z}_2) \\
&+ \cdots + t(\mathbf{Z}_1, \mathbf{Z}_{2n}, \mathbf{Z}_2, \ldots, \mathbf{Z}_{2n-2}, \mathbf{Z}_{2n-1}) = 0\,.
\end{aligned}
\tag{2.21}
$$

The general solution of the system (2.21) is

$$
S(\mathbf{U}, \mathbf{V}) = M(\mathbf{U}) - M(\mathbf{V})\,, \qquad T(\mathbf{U}, \mathbf{V}) = H(\mathbf{U}) - H(\mathbf{V})
\tag{2.22}
$$

where M and H are arbitrary complex functions.

On the basis of the equalities (2.19) and (2.22), we obtain

$$
F(\mathbf{U}, \mathbf{V}) = P(\mathbf{U}) - P(\mathbf{V}), \qquad G(\mathbf{U}, \mathbf{V}) = Q(\mathbf{U}) - Q(\mathbf{V})\,,
$$

where

$$
P(\mathbf{U}) = \frac{1}{2}[M(\mathbf{U}) + H(\mathbf{U})]\,, \qquad Q(\mathbf{U}) = \frac{1}{2}[M(\mathbf{U}) - H(\mathbf{U})]\,.
$$

□

Next we consider the system of functional equations

$$
\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&+f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\
&+a[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)]=0\,,\\
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&+f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\
&+b[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&\quad g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)]=0\,,
\end{aligned}
\tag{2.23}
$$

where $f, g : \mathcal{V}^3 \to \mathbf{C}$ and a and b are real constants.

Definition 2.2. Let $b^2/4+a<0$. A solution $(f(\mathbf{U},\mathbf{V},\mathbf{W}), g(\mathbf{U},\mathbf{V},\mathbf{W}))$ of system (2.23) is called *isotropic* if

$$(f(\mathbf{U},\mathbf{V},\mathbf{W}), g(\mathbf{U},\mathbf{V},\mathbf{W})) \not\equiv (0,0)\,,$$

$$f^2(\mathbf{U},\mathbf{V},\mathbf{W})+bf(\mathbf{U},\mathbf{V},\mathbf{W})g(\mathbf{U},\mathbf{V},\mathbf{W}))-ag^2(\mathbf{U},\mathbf{V},\mathbf{W}))\equiv 0\,. \tag{2.24}$$

The *general* solution of system (2.1) includes all solutions of this system with the possible exception of the isotropic ones.

Theorem 2.2. *The general solution of the system* (2.23) *is given by the following formulas*

1° *if* $b^2/4+a=-p^2 \quad (p>0)$*:*

$$
\begin{aligned}
2pf(\mathbf{U},\mathbf{V},\mathbf{W}) = {}&\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),(2p+b)H_3(\mathbf{W})]\\
&-\Delta[K_1(\mathbf{U}),K_2(\mathbf{V}),2pH_3(\mathbf{W})+bK_3(\mathbf{W})]\\
&+\Delta[H_1(\mathbf{U}),K_2(\mathbf{V}),2pK_3(\mathbf{W})-bH_3(\mathbf{W})]\\
&+\Delta[K_1(\mathbf{U}),H_2(\mathbf{V}),2pK_3(\mathbf{W})-bH_3(\mathbf{W})]\,,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
pg(\mathbf{U},\mathbf{V},\mathbf{W}) = {}&\Delta[K_1(\mathbf{U}),K_2(\mathbf{V}),K_3(\mathbf{W})]-\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]\\
&+\Delta[K_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]+\Delta[H_1(\mathbf{U}),K_2(\mathbf{V}),H_3(\mathbf{W})]
\end{aligned}
\tag{2.26}
$$

for the continuous isotropic solutions we have

$$
\begin{aligned}
2pf(\mathbf{U},\mathbf{V},\mathbf{W}) &= (2p\mp bi)\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]\,,\\
pg(\mathbf{U},\mathbf{V},\mathbf{W}) &= \pm\, i\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]);
\end{aligned}
\tag{2.27}
$$

2° *if* $b^2/4 + a = 0$:

$$g(\mathbf{U},\mathbf{V},\mathbf{W}) = \Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]$$

$$+\,\Delta[H_1(\mathbf{U}),K_2(\mathbf{V}),H_3(\mathbf{W})] + \Delta[K_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]\,, \tag{2.28}$$

$$f(\mathbf{U},\mathbf{V},\mathbf{W}) = \Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W}) - \frac{b}{2}K_3(\mathbf{W})]$$

$$-\,\frac{b}{2}\Delta[H_1(\mathbf{U}),K_2(\mathbf{V}),H_3(\mathbf{W})] - \frac{b}{2}\Delta[K_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]\,; \tag{2.29}$$

3° *if* $b^2/4 + a = p^2 \quad (p > 0)$:

$$4pf(\mathbf{U},\mathbf{V},\mathbf{W}) = (2p-b)\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]$$

$$+\,(2p+b)\Delta[K_1(\mathbf{U}),K_2(\mathbf{V}),K_3(\mathbf{W})]\,, \tag{2.30}$$

$$2pg(\mathbf{U},\mathbf{V},\mathbf{W}) = \Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})]$$

$$-\,\Delta[K_1(\mathbf{U}),K_2(\mathbf{V}),K_3(\mathbf{W})]\,, \tag{2.31}$$

where $H_r,\ K_j\ (r,j = 1,2,3)$ *are arbitrary complex functions defined in* $\mathcal{V}$, *and*

$$\Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})] = \begin{vmatrix} H_1(\mathbf{U}) & H_2(\mathbf{U}) & H_3(\mathbf{U}) \\ H_1(\mathbf{V}) & H_2(\mathbf{V}) & H_3(\mathbf{V}) \\ H_1(\mathbf{W}) & H_2(\mathbf{W}) & H_3(\mathbf{W}) \end{vmatrix}\,.$$

Proof. If we introduce the function h by

$$h(\mathbf{U},\mathbf{V},\mathbf{W}) = f(\mathbf{U},\mathbf{V},\mathbf{W}) + \frac{b}{2}g(\mathbf{U},\mathbf{V},\mathbf{W}), \tag{2.32}$$

the system of functional equations (2.23) becomes

$$\begin{aligned} &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) \\ &+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) \\ &+(b^2/4 + a)[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) \\ &+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)] = 0\,, \\ &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) \\ &+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) \\ &+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) \\ &+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,. \end{aligned} \tag{2.33}$$

According to [1,17], we will distinguish three cases:

1°. Let $b^2/4 + a = -p^2$ $(p > 0)$. In this case the system (2.33) will be

$$\begin{aligned} &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\ &-[k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)k(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+k(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)k(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)k(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+k(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)k(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)]=0\,,\\ &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)k(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)k(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)k(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)k(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\ &+k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+k(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+k(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+k(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)=0\,, \end{aligned} \tag{2.34}$$

where

$$k(\mathbf{U},\mathbf{V},\mathbf{W}) = pg(\mathbf{U},\mathbf{V},\mathbf{W}). \tag{2.35}$$

Suppose that $(f(\mathbf{U},\mathbf{V},\mathbf{W}), g(\mathbf{U},\mathbf{V},\mathbf{W}))$ is a continuous isotropic solution of system (2.23). The relation (2.24) by virtue of the changes (2.32) and (2.35) becomes

$$h^2(\mathbf{U},\mathbf{V},\mathbf{W}) + k^2(\mathbf{U},\mathbf{V},\mathbf{W}) \equiv 0\,,$$

i.e.

$$k(\mathbf{U},\mathbf{V},\mathbf{W}) = \pm i h(\mathbf{U},\mathbf{V},\mathbf{W})\,. \tag{2.36}$$

As in Theorem 2.1 we must take the same sign for all triples of vectors $(\mathbf{U},\mathbf{V},\mathbf{W}) \in \mathcal{V}^3$.

In view of (2.36) system (2.34) becomes

$$\begin{aligned} &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &\quad+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)=0\,. \end{aligned} \tag{2.37}$$

The general solution of the equation (2.37) according to [4] is given by

$$h(\mathbf{U},\mathbf{V},\mathbf{W}) = \Delta[H_1(\mathbf{U}), H_2(\mathbf{V}), H_3(\mathbf{W})] \quad (H_i : \mathcal{V} \to \mathbf{C}\,,\ i=1,2,3)\,, \tag{2.38}$$

where

$$H_1(\mathbf{U}) = \frac{h(\mathcal{A},\mathcal{B},\mathbf{U})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,,\quad H_2(\mathbf{U}) = \frac{h(\mathcal{A},\mathbf{U},\mathcal{C})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,,\quad H_3(\mathbf{U}) = h(\mathcal{B},\mathbf{U},\mathcal{C})\,,$$

are arbitrary complex functions and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are vectors from $\mathcal{V}$ such that $h(\mathcal{A},\mathcal{B},\mathcal{C}) \neq 0$.

From (2.32), (2.35), (2.36) we find

$$2pf(\mathbf{U},\mathbf{V},\mathbf{W}) = (2p \mp bi)h(\mathbf{U},\mathbf{V},\mathbf{W})\,, \quad pg(\mathbf{U},\mathbf{V},\mathbf{W}) = \pm ih(\mathbf{U},\mathbf{V},\mathbf{W})\,,$$

and (2.38) leads to (2.27).

Next we are looking for solutions of (2.23) not satisfying (2.36).
By putting $\mathbf{Z}_1 = \mathbf{Z}_2 = \cdots = \mathbf{Z}_6 = \mathbf{U}$, the system (2.34) becomes

$$h^2(\mathbf{U},\mathbf{U},\mathbf{U}) - k^2(\mathbf{U},\mathbf{U},\mathbf{U}) = 0\,, \quad h(\mathbf{U},\mathbf{U},\mathbf{U})k(\mathbf{U},\mathbf{U},\mathbf{U}) = 0\,,$$

whence

$$h(\mathbf{U},\mathbf{U},\mathbf{U}) = 0\,, \qquad k(\mathbf{U},\mathbf{U},\mathbf{U}) = 0\,. \tag{2.39}$$

For $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_4 = \mathbf{U},\ \mathbf{Z}_5 = \mathbf{Z}_6 = \mathbf{V}$, the system (2.34) yields

$$h^2(\mathbf{U},\mathbf{U},\mathbf{V}) - k^2(\mathbf{U},\mathbf{U},\mathbf{V}) = 0\,, \quad h(\mathbf{U},\mathbf{U},\mathbf{V})k(\mathbf{U},\mathbf{U},\mathbf{V}) = 0\,,$$

so that we obtain

$$h(\mathbf{U},\mathbf{U},\mathbf{V}) = 0\,, \qquad k(\mathbf{U},\mathbf{U},\mathbf{V}) = 0\,. \tag{2.40}$$

If we put $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathbf{Z}_5 = \mathbf{Z}_6 = \mathbf{U},\ \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{V}$, from (2.34) we obtain

$$\begin{aligned} 2h^2(\mathbf{V},\mathbf{U},\mathbf{U}) + h(\mathbf{U},\mathbf{V},\mathbf{U})h(\mathbf{V},\mathbf{U},\mathbf{U}) - 2k^2(\mathbf{V},\mathbf{U},\mathbf{U})& \\ -k(\mathbf{U},\mathbf{V},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U}) = 0\,,& \\ 4h(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U}) + h(\mathbf{U},\mathbf{V},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U})& \\ +h(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,.& \end{aligned} \tag{2.41}$$

By putting $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{V},\ \mathbf{Z}_2 = \mathbf{Z}_4 = \mathbf{Z}_5 = \mathbf{Z}_6 = \mathbf{U}$ into (2.34), we obtain

$$\begin{aligned} h^2(\mathbf{V},\mathbf{U},\mathbf{U}) + 2h(\mathbf{V},\mathbf{U},\mathbf{U})h(\mathbf{U},\mathbf{V},\mathbf{U})& \\ - k^2(\mathbf{V},\mathbf{U},\mathbf{U}) - 2k(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,,& \\ h(\mathbf{U},\mathbf{V},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U}) + h(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U})& \\ + h(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,.& \end{aligned} \tag{2.42}$$

Comparing (2.41) and (2.42), we find

$$h^2(\mathbf{V},\mathbf{U},\mathbf{U}) - k^2(\mathbf{V},\mathbf{U},\mathbf{U}) = 0\,, \quad h(\mathbf{V},\mathbf{U},\mathbf{U})k(\mathbf{V},\mathbf{U},\mathbf{U}) = 0\,,$$

i.e. we obtain

$$h(\mathbf{V},\mathbf{U},\mathbf{U}) = 0\,, \qquad k(\mathbf{V},\mathbf{U},\mathbf{U}) = 0\,. \tag{2.43}$$

By the substitutions $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathbf{V}$, $\mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_5 = \mathbf{Z}_6 = \mathbf{U}$, the system (2.34) becomes

$$h^2(\mathbf{U},\mathbf{V},\mathbf{U}) - k^2(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,, \qquad h(\mathbf{U},\mathbf{V},\mathbf{U})k(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,,$$

i.e.

$$h(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,, \qquad k(\mathbf{U},\mathbf{V},\mathbf{U}) = 0\,. \tag{2.44}$$

For any solution of the system (2.34) there exist at least three vectors $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in \mathcal{V}$ such that $h^2(\mathcal{A},\mathcal{B},\mathcal{C}) + k^2(\mathcal{A},\mathcal{B},\mathcal{C}) \neq 0$.

If we put $\mathbf{Z}_1 = \mathbf{U},\ \mathbf{Z}_2 = \mathbf{V},\ \mathbf{Z}_3 = \mathcal{A},\ \mathbf{Z}_4 = \mathcal{B},\ \mathbf{Z}_5 = \mathbf{Z}_6 = \mathcal{C}$ into (2.34), on the basis of the equalities (2.39), (2.40), (2.43) and (2.44) we obtain

$$h(\mathbf{U},\mathbf{V},\mathcal{C}) = -h(\mathbf{V},\mathbf{U},\mathcal{C})\,, \qquad k(\mathbf{U},\mathbf{V},\mathcal{C}) = -k(\mathbf{V},\mathbf{U},\mathcal{C})\,. \tag{2.45}$$

Using (2.45), the system (2.34) for $\mathbf{Z}_1 = \mathcal{A},\ \mathbf{Z}_2 = \mathcal{B},\ \mathbf{Z}_3 = \mathbf{Z}_6 = \mathcal{C},\ \mathbf{Z}_4 = \mathbf{U},\ \mathbf{Z}_5 = \mathbf{V}$ yields

$$h(\mathbf{U},\mathbf{V},\mathcal{C}) = h(\mathcal{C},\mathbf{U},\mathbf{V})\,, \qquad k(\mathbf{U},\mathbf{V},\mathcal{C}) = k(\mathcal{C},\mathbf{U},\mathbf{V})\,. \tag{2.46}$$

If we put $\mathbf{Z}_1 = \mathcal{A},\ \mathbf{Z}_2 = \mathcal{B},\ \mathbf{Z}_3 = \mathbf{Z}_5 = \mathcal{C},\ \mathbf{Z}_4 = \mathbf{U},\ \mathbf{Z}_6 = \mathbf{V}$, the system (2.34) becomes

$$h(\mathcal{C},\mathbf{U},\mathbf{V}) = -h(\mathbf{U},\mathcal{C},\mathbf{V}), \qquad k(\mathcal{C},\mathbf{U},\mathbf{V}) = -k(\mathbf{U},\mathcal{C},\mathbf{V}). \tag{2.47}$$

On the basis of the equalities (2.44), (2.45) and (2.46), if we put $\mathbf{Z}_1 = \mathcal{A},\ \mathbf{Z}_2 = \mathcal{B},\ \mathbf{Z}_3 = \mathcal{C},\ \mathbf{Z}_4 = \mathbf{U},\ \mathbf{Z}_5 = \mathbf{V},\ \mathbf{Z}_6 = \mathbf{W}$, the system (2.34) yields

$$\begin{aligned} h(\mathbf{U},\mathbf{V},\mathbf{W}) = {} & q[H_1(\mathbf{U})F(\mathbf{V},\mathbf{W}) - H_1(\mathbf{V})F(\mathbf{U},\mathbf{W}) + H_1(\mathbf{W})F(\mathbf{U},\mathbf{V}) \\ & - K_1(\mathbf{U})G(\mathbf{V},\mathbf{W}) + K_1(\mathbf{V})G(\mathbf{U},\mathbf{W}) - K_1(\mathbf{W})G(\mathbf{U},\mathbf{V})] \\ & + r[H_1(\mathbf{U})G(\mathbf{V},\mathbf{W}) - H_1(\mathbf{V})G(\mathbf{U},\mathbf{W}) + H_1(\mathbf{W})G(\mathbf{U},\mathbf{V}) \\ & + K_1(\mathbf{U})F(\mathbf{V},\mathbf{W}) - K_1(\mathbf{V})F(\mathbf{U},\mathbf{W}) + K_1(\mathbf{W})F(\mathbf{U},\mathbf{V})]\,, \end{aligned} \tag{2.48}$$

$$\begin{aligned} k(\mathbf{U},\mathbf{V},\mathbf{W}) = {} & q[H_1(\mathbf{U})G(\mathbf{V},\mathbf{W}) - H_1(\mathbf{V})G(\mathbf{U},\mathbf{W}) + H_1(\mathbf{W})G(\mathbf{U},\mathbf{V}) \\ & + K_1(\mathbf{U})F(\mathbf{V},\mathbf{W}) - K_1(\mathbf{V})F(\mathbf{U},\mathbf{W}) + K_1(\mathbf{W})F(\mathbf{U},\mathbf{V})] \\ & + r[H_1(\mathbf{U})F(\mathbf{V},\mathbf{W}) - H_1(\mathbf{V})F(\mathbf{U},\mathbf{W}) + H_1(\mathbf{W})F(\mathbf{U},\mathbf{V}) \\ & - K_1(\mathbf{U})G(\mathbf{V},\mathbf{W}) + K_1(\mathbf{V})G(\mathbf{U},\mathbf{W}) - K_1(\mathbf{W})G(\mathbf{U},\mathbf{V})]\,, \end{aligned} \tag{2.49}$$

where we have introduced the notations

$$\begin{aligned} & H_1(\mathbf{U}) = h(\mathcal{A},\mathcal{B},\mathbf{U})\,, && K_1(\mathbf{U}) = k(\mathcal{A},\mathcal{B},\mathbf{U})\,, \\ & F(\mathbf{U},\mathbf{V}) = h(\mathbf{U},\mathbf{V},\mathcal{C})\,, && G(\mathbf{U},\mathbf{V}) = k(\mathbf{U},\mathbf{V},\mathcal{C})\,, \\ & q = \frac{h(\mathcal{A},\mathcal{B},\mathcal{C})}{h^2(\mathcal{A},\mathcal{B},\mathcal{C}) + k^2(\mathcal{A},\mathcal{B},\mathcal{C})}\,, && r = \frac{k(\mathcal{A},\mathcal{B},\mathcal{C})}{h^2(\mathcal{A},\mathcal{B},\mathcal{C}) + k^2(\mathcal{A},\mathcal{B},\mathcal{C})}\,. \end{aligned} \tag{2.50}$$

For $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathcal{C}$, $\mathbf{Z}_2 = \mathbf{S}$, $\mathbf{Z}_3 = \mathbf{T}$, $\mathbf{Z}_5 = \mathbf{U}$, $\mathbf{Z}_6 = \mathbf{V}$ in view of notations (2.50), the system (2.34) becomes

$$\begin{aligned} &F(\mathbf{S},\mathbf{T})F(\mathbf{U},\mathbf{V}) + F(\mathbf{S},\mathbf{U})F(\mathbf{V},\mathbf{T}) + F(\mathbf{S},\mathbf{V})F(\mathbf{T},\mathbf{U}) \\ &-G(\mathbf{S},\mathbf{T})G(\mathbf{U},\mathbf{V}) - G(\mathbf{S},\mathbf{U})G(\mathbf{V},\mathbf{T}) - G(\mathbf{S},\mathbf{V})G(\mathbf{T},\mathbf{U}) = 0\,, \\ &F(\mathbf{S},\mathbf{T})G(\mathbf{U},\mathbf{V}) + F(\mathbf{S},\mathbf{U})G(\mathbf{V},\mathbf{T}) + F(\mathbf{S},\mathbf{V})G(\mathbf{T},\mathbf{U}) \\ &+G(\mathbf{S},\mathbf{T})F(\mathbf{U},\mathbf{V}) + G(\mathbf{S},\mathbf{U})F(\mathbf{V},\mathbf{T}) + G(\mathbf{S},\mathbf{V})F(\mathbf{T},\mathbf{U}) = 0\,. \end{aligned} \tag{2.51}$$

The general solution of the system (2.51) according to [17] is given by the formulas

$$\begin{aligned} F(\mathbf{U},\mathbf{V}) =& \Delta[H_1(\mathbf{U}), qH_3'(\mathbf{V}) + rK_3'(\mathbf{V})] \\ &+ \Delta[K_2(\mathbf{U}), rH_3'(\mathbf{V}) - qK_3'(\mathbf{V})]\,, \\ G(\mathbf{U},\mathbf{V}) =& \Delta[K_2(\mathbf{U}), qH_3'(\mathbf{V}) + rK_3'(\mathbf{V})] \\ &+ \Delta[H_2(\mathbf{U}), qK_3'(\mathbf{V}) - rH_3'(\mathbf{V})]\,. \end{aligned} \tag{2.52}$$

On the basis of the equalities (2.32), (2.35), (2.48), (2.49) and (2.52) we obtain (2.25) and (2.26), with the following notations

$$H_3(\mathbf{U}) = (q^2 - r^2)H_3'(\mathbf{U}) + 2qrK_3'(\mathbf{U})\,, \quad K_3(\mathbf{U}) = (r^2 - q^2)K_3'(\mathbf{U}) + 2qrH_3'(\mathbf{U})\,.$$

The functions f and g given by the formulas (2.25) and (2.26) are really solutions of the system (2.23). Consequently, in the case $b^2/4 + a = -p^2$ $(p > 0)$ all functions f and g which are solutions of the system (2.23) and do not satisfy (2.24) are determined by the formulas (2.25) and (2.26). The functions H_r, K_j $(r, j = 1, 2, 3)$ which appear in these solutions are arbitrary complex functions defined in $\mathcal{V}$.

2°. Let $b^2/4 + a = 0$. In this case the first equation of the system (2.33) takes on the form (2.37):

$$\begin{aligned} &h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6) \\ &+h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + h(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,, \end{aligned}$$

and its general solution according to [4] is given by (2.38):

$$h(\mathbf{U},\mathbf{V},\mathbf{W}) = \Delta[H_1(\mathbf{U}), H_2(\mathbf{V}), H_3(\mathbf{W})] \quad (H_i : \mathcal{V} \to \mathbf{C},\ i = 1, 2, 3)\,,$$

where

$$H_1(\mathbf{U}) = \frac{h(\mathcal{A},\mathcal{B},\mathbf{U})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,, \quad H_2(\mathbf{U}) = \frac{h(\mathcal{A},\mathbf{U},\mathcal{C})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,, \quad H_3(\mathbf{U}) = h(\mathcal{B},\mathbf{U},\mathcal{C})\,,$$

are arbitrary complex functions and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are vectors from $\mathcal{V}$ such that $h(\mathcal{A},\mathcal{B},\mathcal{C}) \neq 0$.

If we put $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_4 = \mathbf{Z}_5 = \mathcal{C}$, $\mathbf{Z}_3 = \mathbf{U}$, $\mathbf{Z}_6 = \mathbf{V}$, the second equation of the system (2.33) yields

$$h(\mathbf{U},\mathcal{C},\mathbf{V})g(\mathcal{C},\mathcal{C},\mathcal{C}) = 0\,. \tag{2.53}$$

Using the equality $h(\mathbf{U},\mathcal{C},\mathbf{V}) = -h(\mathbf{U},\mathbf{V},\mathcal{C})$, which follows from (2.38), and by putting $\mathbf{U} = \mathcal{A}$, $\mathbf{V} = \mathcal{B}$, from (2.53) we obtain $g(\mathcal{C},\mathcal{C},\mathcal{C}) = 0$.

For $\mathbf{Z}_1 = \mathcal{A}$, $\mathbf{Z}_2 = \mathcal{B}$, $\mathbf{Z}_3 = \mathbf{Z}_4 = \mathbf{Z}_5 = \mathcal{C}$, $\mathbf{Z}_6 = \mathbf{U}$ the second equation of the system (2.33) becomes $g(\mathcal{C},\mathcal{C},\mathbf{U}) = 0$. If we put $\mathbf{Z}_1 = \mathcal{A}$, $\mathbf{Z}_2 = \mathcal{B}$, $\mathbf{Z}_3 = \mathbf{Z}_5 = \mathbf{Z}_6 = \mathcal{C}$, $\mathbf{Z}_4 = \mathbf{U}$, the same equation yields $g(\mathbf{U},\mathcal{C},\mathcal{C}) = 0$.

Finally, the substitution $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathbf{Z}_5 = \mathcal{C}$, $\mathbf{Z}_2 = \mathbf{U}$, $\mathbf{Z}_3 = \mathcal{A}$, $\mathbf{Z}_6 = \mathcal{B}$ reduces the second equation of the system (2.33) to $g(\mathcal{C},\mathbf{U},\mathcal{C}) = 0$.

By putting $\mathbf{Z}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Z}_3 = \mathcal{A}$, $\mathbf{Z}_4 = \mathcal{B}$, $\mathbf{Z}_5 = \mathbf{Z}_6 = \mathcal{C}$, on the basis of the above result we obtain

$$g(\mathbf{U},\mathbf{V},\mathcal{C}) = -g(\mathbf{V},\mathbf{U},\mathcal{C})\,. \tag{2.54}$$

For $\mathbf{Z}_1 = \mathcal{A}$, $\mathbf{Z}_2 = \mathcal{B}$, $\mathbf{Z}_3 = \mathbf{Z}_6 = \mathcal{C}$, $\mathbf{Z}_4 = \mathbf{U}$, $\mathbf{Z}_5 = \mathbf{V}$, the same equation yields

$$g(\mathbf{U},\mathbf{V},\mathcal{C}) = g(\mathcal{C},\mathbf{U},\mathbf{V}). \tag{2.55}$$

Also, based on the previous results, for $\mathbf{Z}_1 = \mathcal{A}$, $\mathbf{Z}_2 = \mathcal{B}$, $\mathbf{Z}_3 = \mathbf{Z}_5 = \mathcal{C}$, $\mathbf{Z}_4 = \mathbf{U}$, $\mathbf{Z}_6 = \mathbf{V}$, the second equation of the system (2.33) is reduced to

$$g(\mathcal{C},\mathbf{U},\mathbf{V}) = -g(\mathbf{U},\mathcal{C},\mathbf{V})\,. \tag{2.56}$$

By putting $\mathbf{Z}_1 = \mathcal{A}$, $\mathbf{Z}_2 = \mathcal{B}$, $\mathbf{Z}_3 = \mathcal{C}$, $\mathbf{Z}_4 = \mathbf{U}$, $\mathbf{Z}_5 = \mathbf{V}$, $\mathbf{Z}_6 = \mathbf{W}$, on the basis of (2.54), (2.55) and (2.56) the second equation of the system (2.33) becomes

$$\begin{aligned} h(\mathcal{A},\mathcal{B},\mathcal{C})[g(\mathbf{U},\mathbf{V},\mathbf{W}) &- H_1(\mathbf{U})g(\mathbf{V},\mathbf{W},\mathcal{C}) + H_1(\mathbf{V})g(\mathbf{U},\mathbf{W},\mathcal{C}) \\ &- H_1(\mathbf{W})g(\mathbf{U},\mathbf{V},\mathcal{C})] + g(\mathcal{A},\mathcal{B},\mathcal{C})h(\mathbf{U},\mathbf{V},\mathbf{W}) \\ &- g(\mathcal{A},\mathcal{B},\mathbf{U})h(\mathbf{V},\mathbf{W},\mathcal{C}) + g(\mathcal{A},\mathcal{B},\mathbf{V})h(\mathbf{U},\mathbf{W},\mathcal{C}) \\ &- g(\mathcal{A},\mathcal{B},\mathbf{W})h(\mathbf{U},\mathbf{V},\mathcal{C}) = 0\,. \end{aligned} \tag{2.57}$$

For $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathcal{C}$, $\mathbf{Z}_2 = \mathbf{S}$, $\mathbf{Z}_3 = \mathbf{T}$, $\mathbf{Z}_5 = \mathbf{U}$, $\mathbf{Z}_6 = \mathbf{V}$ and with the notations (2.50) the system (2.33) is reduced to

$$\begin{aligned} F(\mathbf{S},\mathbf{T})F(\mathbf{U},\mathbf{V}) + F(\mathbf{S},\mathbf{U})F(\mathbf{V},\mathbf{T}) + F(\mathbf{S},\mathbf{V})F(\mathbf{T},\mathbf{U}) &= 0\,, \\ F(\mathbf{S},\mathbf{T})G(\mathbf{U},\mathbf{V}) + F(\mathbf{S},\mathbf{U})G(\mathbf{V},\mathbf{T}) + F(\mathbf{S},\mathbf{V})G(\mathbf{T},\mathbf{U})& \\ +G(\mathbf{S},\mathbf{T})F(\mathbf{U},\mathbf{V}) + G(\mathbf{S},\mathbf{U})F(\mathbf{V},\mathbf{T}) + G(\mathbf{S},\mathbf{V})F(\mathbf{T},\mathbf{U}) &= 0\,, \end{aligned}$$

The solution of this system according to [17] is given by the formulas

$$\begin{aligned} F(\mathbf{U},\mathbf{V}) &= \Delta[H_2(\mathbf{U}),H_3(\mathbf{V})], \\ G(\mathbf{U},\mathbf{V}) &= \Delta[K_2(\mathbf{U}),H_3(\mathbf{V})] + \Delta[H_2(\mathbf{U}),K_3'(\mathbf{V}) - mH_3'(\mathbf{V})]\,, \end{aligned} \tag{2.58}$$

where

$$K_2(\mathbf{U}) = \frac{g(\mathcal{A},\mathbf{U},\mathcal{C})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,, \quad K_3'(\mathbf{U}) = g(\mathcal{B},\mathbf{U},\mathcal{C})\,, \quad m = \frac{g(\mathcal{A},\mathcal{B},\mathcal{C})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,.$$

On the basis of the equalities (2.57) and (2.58) we obtain (2.28), where

$$K_3(\mathbf{U}) = K_3'(\mathbf{U}) - 2mH_3(\mathbf{U})\,, \quad K_1(\mathbf{U}) = \frac{g(\mathcal{A},\mathcal{B},\mathbf{U})}{h(\mathcal{A},\mathcal{B},\mathcal{C})}\,.$$

On the basis of the expressions (2.32), (2.52) and (2.28) we get (2.29).

The functions (2.28) and (2.29) are really solutions of the system (2.23). Consequently, the general solution of the system (2.23) in the case $b^2+4a=0$ is given by the formulas (2.28) and (2.29), where $H_r,\ K_j\ (r,j=1,2,3)$ are arbitrary complex functions defined in $\mathcal{V}$ and m is an arbitrary constant.

3°. Let $b^2/4 + a = p^2$ $(p>0)$. Introducing the new functions M and N by the formulas

$$\begin{aligned} M(\mathbf{U},\mathbf{V},\mathbf{W}) &= h(\mathbf{U},\mathbf{V},\mathbf{W}) + pg(\mathbf{U},\mathbf{V},\mathbf{W})\,,\\ N(\mathbf{U},\mathbf{V},\mathbf{W}) &= h(\mathbf{U},\mathbf{V},\mathbf{W}) - pg(\mathbf{U},\mathbf{V},\mathbf{W})\,, \end{aligned} \tag{2.59}$$

the system (2.33) becomes

$$\begin{aligned} &M(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)M(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_5)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\ &+N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)N(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)N(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,,\\ &M(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)M(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+M(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\ &-N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)N(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) - N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)N(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &-N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) - N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,, \end{aligned} \tag{2.60}$$

Comparing the equations of the system (2.60), we find

$$\begin{aligned} &M(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)M(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+M(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + M(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)M(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,,\\ &N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)N(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6) + N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)N(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\ &+N(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6) + N(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)N(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = 0\,. \end{aligned} \tag{2.61}$$

The general solution of this system according to [4] will be

$$\begin{aligned} M(\mathbf{U},\mathbf{V},\mathbf{W}) &= \Delta[H_1(\mathbf{U}),H_2(\mathbf{V}),H_3(\mathbf{W})],\\ N(\mathbf{U},\mathbf{V},\mathbf{W}) &= \Delta[K_1(\mathbf{U}),K_2(\mathbf{V}),K_3(\mathbf{W})]\,, \end{aligned} \tag{2.62}$$

where $H_r,\ K_j\ (r,j=1,2,3)$ are arbitrary complex functions defined in $\mathcal{V}$.

On the basis of the equalities (2.32), (2.59) and (2.62), we find (2.30) and (2.31).

The functions (2.30) and (2.31) are solutions of the system (2.23).

Consequently, the general solution of the system (2.23) in the case $b^2/4 + a = p^2\ (p>0)$ is given by the formulas (2.30) and (2.31) where $H_r,\ K_j\ (r,j=1,2,3)$ are arbitrary complex functions defined in $\mathcal{V}$. □

Now we will prove the following general result.

Theorem 2.3. *The general solution of the system of functional equations*

$$\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_n)f(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n})\\
&-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1})f(\mathbf{Z}_n,\mathbf{Z}_{n+2},\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2})f(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-\cdots-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n})f(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_n)\\
&+\operatorname{sgn} a\,[g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_n)g(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n})\\
&-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1})g(\mathbf{Z}_n,\mathbf{Z}_{n+2},\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2})g(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-\cdots-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n})g(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_n)]=0\,,\\
&f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_n)g(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n})\\
&-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1})g(\mathbf{Z}_n,\mathbf{Z}_{n+2},\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2})g(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-\cdots-f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n})g(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_n)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_n)f(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n})\\
&-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1})f(\mathbf{Z}_n,\mathbf{Z}_{n+2},\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2})f(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\dots,\mathbf{Z}_{2n})\\
&-\cdots-g(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n})f(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\dots,\mathbf{Z}_{2n-1},\mathbf{Z}_n)=0\,,
\end{aligned}\tag{2.63}$$

where $f,g:\mathcal{V}^n\to\mathbf{C}$ and a is a real constant, is given by the following formulas:

1°. *if* $a<0$,

$$\begin{aligned}f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)&=\Delta[H_1(\mathbf{U}_1),H_2(\mathbf{U}_2),\dots,H_n(\mathbf{U}_n)]\\&\quad+\Delta[K_1(\mathbf{U}_1),K_2(\mathbf{U}_2),\dots,K_n(\mathbf{U}_n)]\,,\end{aligned}\tag{2.64}$$

$$\begin{aligned}g(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)&=i\{\Delta[K_1(\mathbf{U}_1),K_2(\mathbf{U}_2),\dots,K_n(\mathbf{U}_n)]\\&\quad-\Delta[H_1(\mathbf{U}_1),H_2(\mathbf{U}_2),\dots,H_n(\mathbf{U}_n)]\};\end{aligned}\tag{2.65}$$

2°. *if* $a>0$,

$$\begin{aligned}f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)&=\Delta[H_1(\mathbf{U}_1),H_2(\mathbf{U}_2),\dots,H_n(\mathbf{U}_n)]\\&\quad+\Delta[K_1(\mathbf{U}_1),K_2(\mathbf{U}_2),\dots,K_n(\mathbf{U}_n)]\,,\end{aligned}\tag{2.66}$$

$$g(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) = \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \dots, H_n(\mathbf{U}_n)] - \Delta[K_1(\mathbf{U}_1), K_2(\mathbf{U}_2), \dots, K_n(\mathbf{U}_n)]; \tag{2.67}$$

3° *if* $a = 0$,

$$f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) = \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \dots, H_n(\mathbf{U}_n)] = \begin{vmatrix} H_1(\mathbf{U}_1) & H_1(\mathbf{U}_2) & \dots & H_1(\mathbf{U}_n) \\ H_2(\mathbf{U}_1) & H_2(\mathbf{U}_2) & \dots & H_2(\mathbf{U}_n) \\ \vdots & & & \\ H_n(\mathbf{U}_1) & H_n(\mathbf{U}_2) & \dots & H_n(\mathbf{U}_n) \end{vmatrix}, \tag{2.68}$$

$$\begin{aligned} g(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) = {} & \Delta[K_1(\mathbf{U}_1), H_2(\mathbf{U}_2), H_3(\mathbf{U}_3), \dots, H_n(\mathbf{U}_n)] \\ & + \Delta[H_1(\mathbf{U}_1), K_2(\mathbf{U}_2), H_3(\mathbf{U}_3), \dots, H_n(\mathbf{U}_n)] \\ & + \cdots + \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \dots, H_{n-1}(\mathbf{U}_{n-1}), K_n(\mathbf{U}_n)], \end{aligned} \tag{2.69}$$

where $H_r,\ K_j$ $(1 \le r, j \le n)$ *are arbitrary complex functions defined in* $\mathcal{V}$.

Proof. 1°. Let $a < 0$. Introducing new functions M and N by the formulas

$$\begin{aligned} M(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) &= f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) + ig(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n), \\ N(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) &= f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) - ig(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n), \end{aligned} \tag{2.70}$$

$(i^2 = -1)$ the system (2.63) becomes

$$\begin{aligned} & M(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_n) M(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \quad = M(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+1}) M(\mathbf{Z}_n, \mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \qquad + M(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2}) M(\mathbf{Z}_{n+1}, \mathbf{Z}_n, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \qquad + \cdots + M(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{2n}) M(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_n), \\ & N(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_n) N(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \quad = N(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+1}) N(\mathbf{Z}_n, \mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \qquad + N(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2}) N(\mathbf{Z}_{n+1}, \mathbf{Z}_n, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \qquad + \cdots + N(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{2n}) N(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_n). \end{aligned} \tag{2.71}$$

On the basis of the equalities (2.71) and (2.70), we find (2.64) and (2.65), where $H_r,\ K_j : \mathcal{V} \to \mathbf{C}$ are arbitrary complex functions.

2°. Let $a > 0$. Introducing the functions M and N by

$$\begin{aligned} M(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) &= f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) + g(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n), \\ N(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) &= f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) - g(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n), \end{aligned} \tag{2.72}$$

the system (2.63) is reduced to (2.71). On the basis of the equalities (2.71) and (2.72) there follow (2.66) and (2.67).

3°. Let $a = 0$. In this case, the first equation of the system (2.63) is reduced to

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_n) f(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n}) \\ & \quad = f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+1}) f(\mathbf{Z}_n, \mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \quad\quad + f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2}) f(\mathbf{Z}_{n+1}, \mathbf{Z}_n, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n}) \\ & \quad\quad + \cdots + f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{2n}) f(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_n) . \end{aligned} \tag{2.73}$$

According to [18], the general solution of the equation (2.73) is given by (2.68). Further in the proof of this theorem, we will use the following

Lemma. *If* $\mathbf{A}_r \in \mathcal{V}$ $(1 \le r \le n)$ *are such that*

$$f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \neq 0 ,$$

then the following equality holds

$$f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) = 0 . \tag{2.74}$$

Proof of the Lemma. By putting

$$\begin{gathered} \mathbf{Z}_r = \mathbf{A}_r \qquad (1 \le r \le n) , \\ \mathbf{Z}_{n+1} = \mathbf{Z}_{2n} = \mathbf{A}_n , \quad \mathbf{Z}_j = \mathbf{U}_{j-n-1} \quad (j = n+2, n+3, \dots, 2n-1) , \end{gathered}$$

the equality (2.73) becomes

$$\begin{aligned} & f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n) f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\ & + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_1) f(\mathbf{A}_n, \mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\ & + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_2) f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{A}_n, \mathbf{U}_3, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\ & + \cdots + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_{n-2}) f(\mathbf{A}_n, \mathbf{U}_1, \dots, \mathbf{U}_{n-3}, \mathbf{A}_n, \mathbf{A}_n) = 0 . \end{aligned} \tag{2.75}$$

Let $E_{n-2} = \{1, 2, 3, \dots, n-2\}$, and let S_ν $(0 < \nu \le n-2)$ be a subset of E_{n-2} which contains ν elements. For $\nu = n-2$ we have $S_{n-2} = E_{n-2}$. Replacing into (2.73) all variables by $\mathbf{A}_n$, we obtain

$$f(\mathbf{A}_n, \mathbf{A}_n, \dots, \mathbf{A}_n) = 0 . \tag{2.76}$$

We assume that

$$f(\mathbf{A}_n, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{n-2}, \mathbf{A}_n) = 0 \tag{2.77}$$

holds, where

$$\mathbf{V}_r = \begin{cases} \mathbf{A}_n\,, & r \in S_\nu\,, \\ \mathbf{Y}_r\,, & r \in E_{n-2}\backslash S_\nu\,. \end{cases} \tag{2.78}$$

Using this assumption, we will prove that

$$f(\mathbf{A}_n, \mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0\,, \tag{2.79}$$

where

$$\mathbf{W}_r = \begin{cases} \mathbf{A}_n\,, & r \in S_{\nu-1}, \\ \mathbf{Y}_r\,, & r \in E_{n-2}\backslash S_{\nu-1}\,. \end{cases} \tag{2.80}$$

Putting $\mathbf{U}_r = \mathbf{W}_r$ $(1 \le r \le n-2)$ into (2.75), on the basis of the hypothesis (2.77) we obtain

$$\nu f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n) f(\mathbf{A}_n, \mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0\,.$$

Hence, we obtain

$$f(\mathbf{A}_n, \mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0$$

if the hypothesis (2.77) is true.

Consequently, by induction we proved that

$$f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) = 0$$

if there exist just ν $(0 \le \nu \le n-2)$ elements among $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-2}$ equal to $\mathbf{A}_n$.

For $\nu = 0$, the lemma follows. □

We will prove that (2.69) holds by induction.

If $n = 2$, then

$$f(\mathbf{U}_1, \mathbf{U}_2) = \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_1)]\,, \tag{2.81}$$

where

$$H_1(\mathbf{U}_1) = \frac{f(\mathbf{A}, \mathbf{U}_1)}{f(\mathbf{A}, \mathbf{B})}, \qquad H_2(\mathbf{U}_2) = f(\mathbf{B}, \mathbf{U}_2)$$

and

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_1, \mathbf{Z}_3)g(\mathbf{Z}_2, \mathbf{Z}_4) - f(\mathbf{Z}_1, \mathbf{Z}_4)g(\mathbf{Z}_3, \mathbf{Z}_2) \\ + g(\mathbf{Z}_1, \mathbf{Z}_2)f(\mathbf{Z}_3, \mathbf{Z}_4) - g(\mathbf{Z}_1, \mathbf{Z}_3)f(\mathbf{Z}_2, \mathbf{Z}_4) \\ - g(\mathbf{Z}_1, \mathbf{Z}_4)f(\mathbf{Z}_3, \mathbf{Z}_2) = 0\,. \end{aligned} \tag{2.82}$$

For any nontrivial solution of the equation (2.82) there exists at least one pair of complex vectors $(\mathbf{A},\ \mathbf{B})$ such that $f(\mathbf{A},\mathbf{B}) \neq 0$.

If we put $\mathbf{Z}_1 = \mathbf{A},\ \mathbf{Z}_3 = \mathbf{B},\ \mathbf{Z}_2 = \mathbf{U}_1$ and $\mathbf{Z}_4 = \mathbf{U}_2$, then the equation (2.82) takes the following form

$$\begin{aligned} g(\mathbf{U}_1,\mathbf{U}_2) = {} & \frac{f(\mathbf{A},\mathbf{U}_1)}{f(\mathbf{A},\mathbf{B})} g(\mathbf{B},\mathbf{U}_2) - \frac{f(\mathbf{A},\mathbf{U}_2)}{f(\mathbf{A},\mathbf{B})} g(\mathbf{B},\mathbf{U}_1) \\ & + \frac{g(\mathbf{A},\mathbf{U}_1)}{f(\mathbf{A},\mathbf{B})} f(\mathbf{B},\mathbf{U}_2) - \frac{g(\mathbf{A},\mathbf{U}_2)}{f(\mathbf{A},\mathbf{B})} f(\mathbf{B},\mathbf{U}_1) - \frac{g(\mathbf{A},\mathbf{B})}{f(\mathbf{A},\mathbf{B})} f(\mathbf{U}_1,\mathbf{U}_2). \end{aligned} \tag{2.83}$$

Using the notations

$$\frac{g(\mathbf{A},\mathbf{U}_1)}{f(\mathbf{A},\mathbf{B})} = K_1(\mathbf{U}_1), \quad g(\mathbf{B},\mathbf{U}_1) = K_2'(\mathbf{U}_1), \quad \frac{g(\mathbf{A},\mathbf{B})}{f(\mathbf{A},\mathbf{B})} = -m,$$

on the basis of the relations (2.83) and (2.81) we find

$$g(\mathbf{U}_1,\mathbf{U}_2) = \Delta[H_1(\mathbf{U}_1), K_2'(\mathbf{U}_2)] + \Delta[K_1(\mathbf{U}_1), H_2(\mathbf{U}_2)] + m\Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2)],$$

i.e.

$$g(\mathbf{U}_1,\mathbf{U}_2) = \Delta[K_1(\mathbf{U}_1), H_2(\mathbf{U}_2)] + \Delta[H_1(\mathbf{U}_1), K_2(\mathbf{U}_2)],$$

where

$$K_2(\mathbf{U}) = K_2'(\mathbf{U}) + mH_2(\mathbf{U}).$$

Assume that this theorem holds for $n-1$, so that the general solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{n-1}) g(\mathbf{Z}_n,\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-2}) \\ & - f(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_n) g(\mathbf{Z}_{n-1},\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-2}) \\ & - f(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{n+1}) g(\mathbf{Z}_n,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n-2}) \\ & - \cdots - f(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{2n-2}) g(\mathbf{Z}_n,\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-3},\mathbf{Z}_{n-1}) \\ & + g(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{n-1}) f(\mathbf{Z}_n,\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-2}) \\ & - g(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_n) f(\mathbf{Z}_{n-1},\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-2}) \\ & - g(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{n+1}) f(\mathbf{Z}_n,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n-2}) \\ & - \cdots - g(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2},\mathbf{Z}_{2n-2}) f(\mathbf{Z}_n,\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-3},\mathbf{Z}_{n-1}) = 0 \end{aligned} \tag{2.84}$$

is introduced by

$$\begin{aligned} g(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_{n-1}) = {} & \Delta[K_1(\mathbf{U}_1), H_2(\mathbf{U}_2),\ldots,H_{n-1}(\mathbf{U}_{n-1})] \\ & + \Delta[H_1(\mathbf{U}_1), K_2(\mathbf{U}_2), H_3(\mathbf{U}_3),\ldots,H_{n-1}(\mathbf{U}_{n-1})] \\ & + \cdots + \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \\ & \quad \ldots, H_{n-2}(\mathbf{U}_{n-2}), K_{n-1}(\mathbf{U}_{n-1})]. \end{aligned} \tag{2.85}$$

Let $f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \neq 0$. If we put $\mathbf{Z}_1 = \mathbf{A}_n,\ \mathbf{Z}_{n+1} = \mathbf{A}_n$,

$$f(\mathbf{A}_n, \mathbf{U}_2, \mathbf{U}_3, \dots, \mathbf{U}_n) = F(\mathbf{U}_2, \mathbf{U}_3, \dots, \mathbf{U}_n)\,,$$
$$g(\mathbf{A}_n, \mathbf{U}_2, \mathbf{U}_3, \dots, \mathbf{U}_n) = G(\mathbf{U}_2, \mathbf{U}_3, \dots, \mathbf{U}_n)\,,$$

the second equation of the system (2.63), according to the lemma, becomes

$$\begin{aligned}
&F(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_n)(\mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n})\\
&- F(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2})G(\mathbf{Z}_n, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n})\\
&- F(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+3})G(\mathbf{Z}_{n+2}, \mathbf{Z}_n, \mathbf{Z}_{n+4}, \dots, \mathbf{Z}_{2n})\\
&- \cdots - F(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{2n})G(\mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n})\\
&+ G(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_n)F(\mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n})\\
&- G(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2})F(\mathbf{Z}_n, \mathbf{Z}_{n+3}, \dots, \mathbf{Z}_{2n})\\
&- G(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+3})F(\mathbf{Z}_{n+2}, \mathbf{Z}_n, \mathbf{Z}_{n+4}, \dots, \mathbf{Z}_{2n})\\
&- \cdots - G(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_{2n})F(\mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_n) = 0\,.
\end{aligned} \tag{2.86}$$

On the basis of the equalities (2.84) and (2.85), we find the general solution of the equation (2.86) which is

$$\begin{aligned}
G(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-1}) &= g(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-1})\\
&= \Delta[(K_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \dots, H_{n-1}(\mathbf{U}_{n-1})]\\
&\quad + \Delta[H_1(\mathbf{U}_1), K_2(\mathbf{U}_2), H_3(\mathbf{U}_3), \dots, H_{n-1}(\mathbf{U}_{n-1})]\\
&\quad + \cdots + \Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \dots, H_{n-2}(\mathbf{U}_{n-2}), K_{n-1}(\mathbf{U}_{n-1})]\,,
\end{aligned} \tag{2.87}$$

where $H_r(\mathbf{U}), K_j(\mathbf{U})$ $(1 \leq r, j \leq n-1)$ are arbitrary complex functions defined in $\mathcal{V}$.

If we put

$$\mathbf{Z}_r = \mathbf{A}_r \quad (1 \leq r \leq n-1)\,,$$
$$\mathbf{Z}_n = \mathbf{U}_1\,, \quad \mathbf{Z}_{n+1} = \mathbf{A}_n\,, \quad \mathbf{Z}_{n+k} = \mathbf{U}_k \quad (2 \leq k \leq n)$$

into the second equation of the system (2.63), we obtain

$$\begin{aligned}
&f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)g(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n)\\
&\quad = f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_1)g(\mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_n)\\
&\quad\quad - f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_2)g(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_3, \dots, \mathbf{U}_n)\\
&\quad\quad - \cdots - f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_n)g(\mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_{n-1}, \mathbf{U}_1)\\
&\quad\quad + g(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_1)f(\mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_n)\\
&\quad\quad - g(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_2)f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_3, \dots, \mathbf{U}_n)\\
&\quad\quad - \cdots - g(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_n)f(\mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_{n-1}, \mathbf{U}_1)\\
&\quad\quad - g(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n)\,.
\end{aligned} \tag{2.88}$$

On the basis of the properties (2.68) and (2.87), we obtain

$$
\begin{aligned}
& f(\mathbf{A}_n, \ldots, \mathbf{U}_{r-1}, \mathbf{U}_r, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_j, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1}) \\
& \quad = - f(\mathbf{A}_n, \ldots, \mathbf{U}_{r-1}, \mathbf{U}_j, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_r, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1}), \\
& g(\mathbf{A}_n, \ldots, \mathbf{U}_{r-1}, \mathbf{U}_r, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_j, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1}) \\
& \quad = - g(\mathbf{A}_n, \ldots, \mathbf{U}_{r-1}, \mathbf{U}_j, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_r, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1}) \\
& \qquad\qquad (1 \le r < j \le n-1)\,.
\end{aligned}
\tag{2.89}
$$

By introducing the notations

$$
\begin{aligned}
& \frac{f(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}, \mathbf{U})}{f(\mathbf{A}_1, \ldots, \mathbf{A}_n)} = H_1(\mathbf{U})\,, \quad \frac{g(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}, \mathbf{U})}{f(\mathbf{A}_1, \ldots, \mathbf{A}_n)} = K_1(\mathbf{U})\,, \\
& \frac{g(\mathbf{A}_1, \ldots, \mathbf{A}_n)}{f(\mathbf{A}_1, \ldots, \mathbf{A}_n)} = -m\,,
\end{aligned}
$$

on the basis of the equalities (2.89), (2.88), (2.87) and (2.68) we find

$$
\begin{aligned}
g(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n) = & \Delta[K_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \ldots, H_n(\mathbf{U}_n)] \\
& + \Delta[H_1(\mathbf{U}_1), K_2(\mathbf{U}_2), H_3(\mathbf{U}_3), \ldots, H_n(\mathbf{U}_n)] \\
& + \cdots + \Delta[H_1(\mathbf{U}_1), \ldots, H_{n-1}(\mathbf{U}_{n-1}), K_n(\mathbf{U}_n)] \\
& + m\Delta[H_1(\mathbf{U}_1), H_2(\mathbf{U}_2), \ldots, H_n(\mathbf{U}_n)]\,,
\end{aligned}
$$

i.e. $g(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n)$ is given by (2.69), where $K_n(\mathbf{U}_n) + mH_n(\mathbf{U}_n)$ is replaced by $K_n(\mathbf{U}_n)$.

Now we will prove that the functions (2.64) and (2.65), (2.66) and (2.67), (2.68) and (2.69) are solutions of the system (2.63).

We introduce the following notation

$$
D(F_1, F_2, \ldots, F_n, G_1, G_2, \ldots, G_n)
$$

$$
= \begin{vmatrix}
F_1(\mathbf{U}_1) & F_2(\mathbf{U}_1) & \ldots & F_n(\mathbf{U}_1) & 0 & 0 & \ldots & 0 \\
F_1(\mathbf{U}_2) & F_2(\mathbf{U}_2) & \ldots & F_n(\mathbf{U}_2) & 0 & 0 & \ldots & 0 \\
\vdots & & & & & & & \\
F_1(\mathbf{U}_{n-1}) & F_2(\mathbf{U}_{n-1}) & \ldots & F_n(\mathbf{U}_{n-1}) & 0 & 0 & \ldots & 0 \\
F_1(\mathbf{U}_n) & F_2(\mathbf{U}_n) & \ldots & F_n(\mathbf{U}_n) & G_1(\mathbf{U}_n) & G_2(\mathbf{U}_n) & \ldots & G_n(\mathbf{U}_n) \\
F_1(\mathbf{U}_{n+1}) & F_2(\mathbf{U}_{n+1}) & \ldots & F_n(\mathbf{U}_{n+1}) & G_1(\mathbf{U}_{n+1}) & G_2(\mathbf{U}_{n+1}) & \ldots & G_n(\mathbf{U}_{n+1}) \\
\vdots & & & & & & & \\
F_1(\mathbf{U}_{2n}) & F_2(\mathbf{U}_{2n}) & \ldots & F_n(\mathbf{U}_{2n}) & G_1(\mathbf{U}_{2n}) & G_2(\mathbf{U}_{2n}) & \ldots & G_n(\mathbf{U}_{2n})
\end{vmatrix}.
$$

Let

$$D(H_1, H_2, \dots, H_n, H_1, H_2, \dots, H_n) = 0\,,$$

then it will be

$$\begin{aligned}
&D(H_1, H_2, \dots, H_n, H_1, H_2, \dots, H_n) + D(H_1, H_2, \dots, H_n, K_1, K_2, \dots, K_n)\\
&+ D(K_1, K_2, \dots, K_n, H_1, H_2, \dots, H_n) + D(K_1, K_2, \dots, K_n, K_1, K_2, \dots, K_n)\\
&+ D(H_1, H_2, \dots, H_n, H_1, H_2, \dots, H_n) - D(H_1, H_2, \dots, H_n, K_1, K_2, \dots, K_n)\\
&- D(K_1, K_2, \dots, K_n, H_1, H_2, \dots, H_n) + D(K_1, K_2, \dots, K_n, K_1, K_2, \dots, K_n) = 0\,,\\
&D(H_1, H_2, \dots, H_n, H_1, H_2, \dots, H_n) + D(K_1, K_2, \dots, K_n, H_1, H_2, \dots, H_n)\\
&- D(H_1, H_2, \dots, H_n, K_1, K_2, \dots, K_n) - D(K_1, K_2, \dots, K_n, K_1, K_2, \dots, K_n)\\
&+ D(H_1, H_2, \dots, H_n, H_1, H_2, \dots, H_n) - D(K_1, K_2, \dots, K_n, H_1, H_2, \dots, H_n)\\
&+ D(H_1, H_2, \dots, H_n, K_1, K_2, \dots, K_n) - D(K_1, K_2, \dots, K_n, K_1, K_2, \dots, K_n) = 0\,.
\end{aligned}$$

By the Laplace rule, if we evaluate the determinants which appear in the previous identities, we obtain that the functions (2.66) and (2.67) in the case $a > 0$ are a solution of the system (2.63).

In the case $a < 0$, we can analogously show that the functions (2.64) and (2.65) represent a solution of the system (2.63).

If $a = 0$, we will show that the functions (2.68) and (2.69) are a solution of the system (2.63).

The function (2.68) satisfies the first equation of the system (2.63).

Now we will prove that the functions (2.68) and (2.69) satisfy the second equation of the system (2.63).

Let us consider in this case the following identity

$$\begin{aligned}
&D(H_1 + K_1, H_2, \dots, H_n, H_1 + K_1, H_2, \dots, H_n)\\
&+ D(H_1, H_2 + K_2, H_3, \dots, H_n, H_1, H_2 + K_2, H_3, \dots, H_n)\\
&+ \dots + D(H_1, H_2, \dots, H_n + K_n, H_1, H_2, \dots, H_n + K_n) = 0\,.
\end{aligned} \tag{2.90}$$

By the evaluation of the determinants which are found in the identity (2.90), we obtain that (2.68) and (2.69) satisfy the second equation of the system (2.63). Consequently, the functions (2.68) and (2.69) represent a solution of the system (2.63) if $a = 0$. □

2.2. Systems of higher order functional equations.

Theorem 2.4. *The general solution of the system of functional equations*

$$
\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4)+f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2)+f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3)=0\,,\\
&g(\mathbf{Z}_1,\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4)+g(\mathbf{Z}_1,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_2)+g(\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_2,\mathbf{Z}_3)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2)f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)+g(\mathbf{Z}_1,\mathbf{Z}_3)f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_4)f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)+g(\mathbf{Z}_3,\mathbf{Z}_4)f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_2)\\
&+g(\mathbf{Z}_4,\mathbf{Z}_2)f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_3)+g(\mathbf{Z}_2,\mathbf{Z}_3)f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_4)\\
&+(3+2r)f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_2)f(\mathbf{Z}_2,\mathbf{Z}_3)\\
&\times[f^2(\mathbf{Z}_1,\mathbf{Z}_2)f^2(\mathbf{Z}_3,\mathbf{Z}_4)+f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_3,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_2)\\
&+f^2(\mathbf{Z}_1,\mathbf{Z}_3)f^2(\mathbf{Z}_4,\mathbf{Z}_2)]^r=0\,,
\end{aligned}
$$

$$
\begin{aligned}
&h(\mathbf{Z}_1,\mathbf{Z}_2)h(\mathbf{Z}_3,\mathbf{Z}_4)+h(\mathbf{Z}_1,\mathbf{Z}_3)h(\mathbf{Z}_4,\mathbf{Z}_2)+h(\mathbf{Z}_1,\mathbf{Z}_4)h(\mathbf{Z}_2,\mathbf{Z}_3)\\
&+h(\mathbf{Z}_1,\mathbf{Z}_2)[g(\mathbf{Z}_3,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)]^{3+2j}\\
&+h(\mathbf{Z}_1,\mathbf{Z}_3)[g(\mathbf{Z}_4,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)]^{3+2j}\\
&+h(\mathbf{Z}_1,\mathbf{Z}_4)[g(\mathbf{Z}_2,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)]^{3+2j}\\
&+(3+2j)[g(\mathbf{Z}_1,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_2)][g(\mathbf{Z}_1,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_3)]\\
&\times[g(\mathbf{Z}_1,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_4)][g(\mathbf{Z}_3,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)]\\
&\times[g(\mathbf{Z}_4,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)][g(\mathbf{Z}_2,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)]\\
&\times\{[g(\mathbf{Z}_1,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_4)]^2[g(\mathbf{Z}_2,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)]^2\\
&-[g(\mathbf{Z}_1,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_2)][g(\mathbf{Z}_1,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_3)]\\
&\times[g(\mathbf{Z}_3,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)][g(\mathbf{Z}_4,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)]\}^j=0\,,
\end{aligned}
\tag{2.91}
$$

where $f,\ g,\ h:\ \mathcal{V}^2\to\mathbf{C}$ *are arbitrary complex functions and* $r,j\in\{0,1,2\}$, *is given by the formulas*

$$
f(\mathbf{U},\mathbf{V})=\begin{vmatrix}F_1(\mathbf{U}) & F_2(\mathbf{U})\\ F_1(\mathbf{V}) & F_2(\mathbf{V})\end{vmatrix}\,,
\tag{2.92}
$$

$$
g(\mathbf{U},\mathbf{V})=\begin{vmatrix}G_1(\mathbf{U}) & G_2(\mathbf{U})\\ G_1(\mathbf{V}) & G_2(\mathbf{V})\end{vmatrix}-\begin{vmatrix}F_1(\mathbf{U}) & F_2(\mathbf{U})\\ F_1(\mathbf{V}) & F_2(\mathbf{V})\end{vmatrix}^{3+2r}\,,\qquad r\in\{0,1,2\}
\tag{2.93}
$$

$$
h(\mathbf{U},\mathbf{V})=\begin{vmatrix}H_1(\mathbf{U}) & H_2(\mathbf{U})\\ H_1(\mathbf{V}) & H_2(\mathbf{V})\end{vmatrix}-\begin{vmatrix}G_1(\mathbf{U}) & G_2(\mathbf{U})\\ G_1(\mathbf{V}) & G_2(\mathbf{V})\end{vmatrix}^{3+2j}\,,\qquad j\in\{0,1,2\}
\tag{2.94}
$$

where $F_k,\ G_k,\ H_k\ (k=1,2)$ *are arbitrary complex functions defined in* $\mathcal{V}$.

Proof. The first functional equation of the system (2.91) according to [13, 14] has a general solution given by (2.92).

The complex function $f(\mathbf{U},\mathbf{V})$ satisfied the condition

$$f(\mathbf{U},\mathbf{U})\equiv 0\,,\quad f(\mathbf{U},\mathbf{V})+f(\mathbf{V},\mathbf{U})=0\,. \tag{2.95}$$

The functions $g(\mathbf{U},\mathbf{V})$ and $h(\mathbf{U},\mathbf{V})$ from the second, respectively third equation of the system (2.91) also satisfy the same conditions for $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{Z}_3=\mathbf{Z}_4=\mathbf{U}\neq\mathbf{O}$ or $\mathbf{Z}_2=\mathbf{Z}_3$.

Since $f^{3+2r}(\mathbf{Z}_p,\mathbf{Z}_k)$ have distinct powers in the second equation of the system (2.91) for $r\in\{0,1,2\}$, we obtain that the first equation of the system (2.91) implies

$$\begin{aligned}&f^3(\mathbf{Z}_1,\mathbf{Z}_2)f^3(\mathbf{Z}_3,\mathbf{Z}_4)+f^3(\mathbf{Z}_1,\mathbf{Z}_3)f^3(\mathbf{Z}_4,\mathbf{Z}_2)+f^3(\mathbf{Z}_1,\mathbf{Z}_4)f^3(\mathbf{Z}_2,\mathbf{Z}_3)\\&=3f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_2)f(\mathbf{Z}_2,\mathbf{Z}_3)\,,\\[1em]&f^5(\mathbf{Z}_1,\mathbf{Z}_2)f^5(\mathbf{Z}_3,\mathbf{Z}_4)+f^5(\mathbf{Z}_1,\mathbf{Z}_3)f^5(\mathbf{Z}_4,\mathbf{Z}_2)+f^5(\mathbf{Z}_1,\mathbf{Z}_4)f^5(\mathbf{Z}_2,\mathbf{Z}_3)\\&=\frac{5}{2}f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_2)f(\mathbf{Z}_2,\mathbf{Z}_3)\\&\times[f^2(\mathbf{Z}_1,\mathbf{Z}_2)f^2(\mathbf{Z}_3,\mathbf{Z}_4)+f^2(\mathbf{Z}_1,\mathbf{Z}_3)f^2(\mathbf{Z}_4,\mathbf{Z}_2)+f^2(\mathbf{Z}_1,\mathbf{Z}_4)f^2(\mathbf{Z}_2,\mathbf{Z}_3)],\\[1em]&f^7(\mathbf{Z}_1,\mathbf{Z}_2)f^7(\mathbf{Z}_3,\mathbf{Z}_4)+f^7(\mathbf{Z}_1,\mathbf{Z}_3)f^7(\mathbf{Z}_4,\mathbf{Z}_2)+f^7(\mathbf{Z}_1,\mathbf{Z}_4)f^7(\mathbf{Z}_2,\mathbf{Z}_3)\\&=\frac{7}{4}f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_2)f(\mathbf{Z}_2,\mathbf{Z}_3)\\&\times[f^2(\mathbf{Z}_1,\mathbf{Z}_2)f^2(\mathbf{Z}_3,\mathbf{Z}_4)+f^2(\mathbf{Z}_1,\mathbf{Z}_3)f^2(\mathbf{Z}_4,\mathbf{Z}_2)+f^2(\mathbf{Z}_1,\mathbf{Z}_4)f^2(\mathbf{Z}_2,\mathbf{Z}_3)]^2.\end{aligned} \tag{2.96}$$

If we apply the formulas (2.96) to the second functional equation of the system (2.91), then we obtain the following equation

$$\begin{aligned}&[g(\mathbf{Z}_1,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_2)][g(\mathbf{Z}_3,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)]\\&\quad+[g(\mathbf{Z}_1,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_3)][g(\mathbf{Z}_4,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)]\\&\quad+[g(\mathbf{Z}_1,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_4)][g(\mathbf{Z}_2,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)]=0\,.\end{aligned} \tag{2.97}$$

By repeating the previous procedure to the last equation of the system (2.91), we obtain

$$\begin{aligned}&\{h(\mathbf{Z}_1,\mathbf{Z}_2)+[g(\mathbf{Z}_1,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_2)]^{3+2j}\}\\&\times\{h(\mathbf{Z}_3,\mathbf{Z}_4)+[g(\mathbf{Z}_3,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_3,\mathbf{Z}_4)]^{3+2j}\}\\&+\{h(\mathbf{Z}_1,\mathbf{Z}_3)+[g(\mathbf{Z}_1,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_3)]^{3+2j}\}\\&\times\{h(\mathbf{Z}_4,\mathbf{Z}_2)+[g(\mathbf{Z}_4,\mathbf{Z}_2)+f^{3+2r}(\mathbf{Z}_4,\mathbf{Z}_2)]^{3+2j}\}\\&+\{h(\mathbf{Z}_1,\mathbf{Z}_4)+[g(\mathbf{Z}_1,\mathbf{Z}_4)+f^{3+2r}(\mathbf{Z}_1,\mathbf{Z}_4)]^{3+2j}\}\\&\times\{h(\mathbf{Z}_2,\mathbf{Z}_3)+[g(\mathbf{Z}_2,\mathbf{Z}_3)+f^{3+2r}(\mathbf{Z}_2,\mathbf{Z}_3)]^{3+2j}\}=0\,.\end{aligned} \tag{2.98}$$

The system of vector functional equations (2.91) with the unknown functions $f,\ g,\ h:\ \mathcal{V}^2 \to \mathbf{C}$ and $r, j \in \{0,1,2\}$ is equivalent to the system formed by the first equation of the system (2.91) and the equations (2.97) and (2.98), whose general solution is given by (2.92), (2.93) and (2.94). □

Theorem 2.5. *The general solution of the following system of functional equations*

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)$$
$$+f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)=0\,,$$

$$\begin{aligned}
&g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f^3(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)f^3(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f^3(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)+g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)f^3(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)\\
&+g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f^3(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)f^3(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)\\
&+g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f^3(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)+g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)f^3(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)\\
&+3f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)\\
&\times[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)]\\
&+3f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)f(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)\\
&\times[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)]=0\,,
\end{aligned}\tag{2.99}$$

where $f,\ g:\mathcal{V}^3\to\mathbf{C}$ *are arbitrary complex functions, is given by the following formulas*

$$f(\mathbf{U},\mathbf{V},\mathbf{W})=\begin{vmatrix}F_1(\mathbf{U}) & F_2(\mathbf{U}) & F_3(\mathbf{U})\\ F_1(\mathbf{V}) & F_2(\mathbf{V}) & F_3(\mathbf{V})\\ F_1(\mathbf{W}) & F_2(\mathbf{W}) & F_3(\mathbf{W})\end{vmatrix},\tag{2.100}$$

(2.101)
$$g(\mathbf{U},\mathbf{V},\mathbf{W})=\begin{vmatrix}G_1(\mathbf{U}) & G_2(\mathbf{U}) & G_3(\mathbf{U})\\ G_1(\mathbf{V}) & G_2(\mathbf{V}) & G_3(\mathbf{V})\\ G_1(\mathbf{W}) & G_2(\mathbf{W}) & G_3(\mathbf{W})\end{vmatrix}-\begin{vmatrix}F_1(\mathbf{U}) & F_2(\mathbf{U}) & F_3(\mathbf{U})\\ F_1(\mathbf{V}) & F_2(\mathbf{V}) & F_3(\mathbf{V})\\ F_1(\mathbf{W}) & F_2(\mathbf{W}) & F_3(\mathbf{W})\end{vmatrix}^3,$$

where $F_r,\ G_r\ (r=1,2,3)$ *are arbitrary complex functions defined in* $\mathcal{V}$.

Proof. For the first equation of the system (2.99), the general solution is given by (2.100).

We can rearrange the terms in the second equation of the system (2.99) as follows

$$\begin{aligned}
&[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f^3(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)][g(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)+f^3(\mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6)]\\
&+[g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)+f^3(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_4)][g(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)+f^3(\mathbf{Z}_3,\mathbf{Z}_5,\mathbf{Z}_6)]\\
&+[g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)+f^3(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_5)][g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)+f^3(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6)]\\
&+[g(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)+f^3(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_6)][g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)+f^3(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5)]=0\,,
\end{aligned}\tag{2.102}$$

by using the identity

$$A^3 + B^3 + C^3 + D^3 = 3AB(C+D) + 3CD(A+B) \quad \text{if} \quad A+B+C+D=0\,.$$

Therefore, the system (2.99) with two unknown functions $f,\ g:\ \mathcal{V}^3 \to \mathbf{C}$ is equivalent to the system formed by the first equation of the system (2.99) and the equation (2.102), whose general solution is given by (2.100) and (2.101). □

Theorem 2.6. *The general continuous solution of the following system of functional equations*

$$\begin{aligned}
&f_{11}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) + f_{21}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) = f_{31}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)\,,\\
&f_{12}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) + 2f_{11}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{21}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + f_{22}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) = f_{32}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)\,,\\
&f_{13}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) + 3f_{12}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{21}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 3f_{11}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{22}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) + f_{23}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad = f_{33}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)\,,\\
&f_{14}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) + 4f_{13}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{21}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 6f_{12}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{22}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 4f_{11}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{23}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + f_{24}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) = f_{34}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)\,,\\
&f_{15}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) + 5f_{14}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{21}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 10f_{13}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{22}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 10f_{12}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{23}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + 5f_{11}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)f_{24}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)\\
&\qquad + f_{25}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) = f_{35}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)\,,
\end{aligned} \tag{2.103}$$

where $f_{1r},\ f_{2r},\ f_{3r}:\ \mathcal{V}^2 \to \mathbf{C}$ $(1 \le r \le 5)$ *are arbitrary complex functions, is given by the formulas*

$$\begin{aligned}
f_{r1}(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(2\mathbf{U}-\mathbf{V}) + F_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(2\mathbf{U}-\mathbf{V})\\
&\quad + G_r(\mathbf{U}+\mathbf{V}) \qquad (r=1,2)\,,\\
f_{31}(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(\mathbf{V}-2\mathbf{U}) + F_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(\mathbf{V}-2\mathbf{U})\\
&\quad + G_1(\mathbf{U}+\mathbf{V}) + G_2(\mathbf{U}+\mathbf{V})\,,
\end{aligned} \tag{2.104}$$

$$\begin{aligned}
f_{r2}(\mathbf{U},\mathbf{V}) &= L_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(2\mathbf{U}-\mathbf{V}) + L_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(2\mathbf{U}-\mathbf{V})\\
&\quad + H_r(\mathbf{U}+\mathbf{V}) + f_{r1}^2(\mathbf{U},\mathbf{V}) \qquad (r=1,2)\,,\\
f_{32}(\mathbf{U},\mathbf{V}) &= L_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(\mathbf{V}-2\mathbf{U}) + L_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(\mathbf{V}-2\mathbf{U})\\
&\quad + H_1(\mathbf{U}+\mathbf{V}) + H_2(\mathbf{U}+\mathbf{V}) + f_{31}^2(\mathbf{U},\mathbf{V})\,,
\end{aligned} \tag{2.105}$$

$$\begin{aligned}
f_{r3}(\mathbf{U},\mathbf{V}) &= M_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(2\mathbf{U}-\mathbf{V}) + M_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(2\mathbf{U}-\mathbf{V}) \\
&\quad + K_r(\mathbf{U}+\mathbf{V}) + 3f_{r2}^2(\mathbf{U},\mathbf{V})f_{r1}(\mathbf{U},\mathbf{V}) - 2f_{r1}^3(\mathbf{U},\mathbf{V}) \\
&\qquad (r=1,2), \\
f_{33}(\mathbf{U},\mathbf{V}) &= M_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(\mathbf{V}-2\mathbf{U}) + M_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(\mathbf{V}-2\mathbf{U}) \\
&\quad + K_1(\mathbf{U}+\mathbf{V}) + K_2(\mathbf{U}+\mathbf{V}) \\
&\quad + 3f_{32}(\mathbf{U},\mathbf{V})f_{31}(\mathbf{U},\mathbf{V}) - 2f_{31}^3(\mathbf{U},\mathbf{V})\,,
\end{aligned} \tag{2.106}$$

$$\begin{aligned}
f_{r4}(\mathbf{U},\mathbf{V}) &= P_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(2\mathbf{U}-\mathbf{V}) + P_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(2\mathbf{U}-\mathbf{V}) \\
&\quad + Q_r(\mathbf{U}+\mathbf{V}) + 4f_{r3}(\mathbf{U},\mathbf{V})f_{r1}(\mathbf{U},\mathbf{V}) \\
&\quad - 12f_{r2}(\mathbf{U},\mathbf{V})f_{r1}^2(\mathbf{U},\mathbf{V}) + 3f_{r2}^2(\mathbf{U},\mathbf{V}) + 6f_{r1}^4(\mathbf{U},\mathbf{V}) \\
&\qquad (r=1,2), \\
f_{34}(\mathbf{U},\mathbf{V}) &= P_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(\mathbf{V}-2\mathbf{U}) + P_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(\mathbf{V}-2\mathbf{U}) \\
&\quad + Q_1(\mathbf{U}+\mathbf{V}) + Q_2(\mathbf{U}+\mathbf{V}) + 4f_{33}(\mathbf{U},\mathbf{V})f_{31}(\mathbf{U},\mathbf{V}) \\
&\quad - 12f_{32}(\mathbf{U},\mathbf{V})f_{31}^2(\mathbf{U},\mathbf{V}) + 3f_{32}^2(\mathbf{U},\mathbf{V}) + 6f_{31}^4(\mathbf{U},\mathbf{V})\,,
\end{aligned} \tag{2.107}$$

$$\begin{aligned}
f_{r5}(\mathbf{U},\mathbf{V}) &= R_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(2\mathbf{U}-\mathbf{V}) + R_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(2\mathbf{U}-\mathbf{V}) \\
&\quad + S_r(\mathbf{U}+\mathbf{V}) + 5f_{r4}(\mathbf{U},\mathbf{V})f_{r1}(\mathbf{U},\mathbf{V}) \\
&\quad - 20f_{r3}(\mathbf{U},\mathbf{V})f_{r1}^2(\mathbf{U},\mathbf{V}) + 60f_{r2}(\mathbf{U},\mathbf{V})f_{r1}^3(\mathbf{U},\mathbf{V}) \\
&\quad - 30f_{r2}^2(\mathbf{U},\mathbf{V})f_{r1}(\mathbf{U},\mathbf{V}) + 10f_{r3}(\mathbf{U},\mathbf{V})f_{r2}(\mathbf{U},\mathbf{V}) \\
&\quad - 24f_{r1}^5(\mathbf{U},\mathbf{V}) \qquad (r=1,2)\,, \\
f_{35}(\mathbf{U},\mathbf{V}) &= R_1(\mathbf{U}+\mathbf{V})\,\mathrm{Re}\,(\mathbf{V}-2\mathbf{U}) + R_2(\mathbf{U}+\mathbf{V})\,\mathrm{Im}\,(\mathbf{V}-2\mathbf{U}) \\
&\quad + S_1(\mathbf{U}+\mathbf{V}) + S_2(\mathbf{U}+\mathbf{V}) + 5f_{34}(\mathbf{U},\mathbf{V})f_{31}(\mathbf{U},\mathbf{V}) \\
&\quad - 20f_{33}(\mathbf{U},\mathbf{V})f_{31}^2(\mathbf{U},\mathbf{V}) + 60f_{32}(\mathbf{U},\mathbf{V})f_{31}^3(\mathbf{U},\mathbf{V}) \\
&\quad - 30f_{32}^2(\mathbf{U},\mathbf{V})f_{31}(\mathbf{U},\mathbf{V}) + 10f_{33}(\mathbf{U},\mathbf{V})f_{32}(\mathbf{U},\mathbf{V}) \\
&\quad - 24f_{31}^5(\mathbf{U},\mathbf{V})\,,
\end{aligned} \tag{2.108}$$

where $F_r,\ L_r,\ M_r,\ P_r,\ R_r : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathbf{C}),\ G_r,\ H_r,\ K_r,\ Q_r,\ S_r :\ \mathcal{V} \to \mathbf{C}$ $(r = 1, 2)$ *are arbitrary complex functions.*

Proof. The general continuous solution of the first equation of the system (2.103) according to [15] is given by the functions (2.104).

Now, if we use the first equation of the system (2.103), then the second equation takes on the following form

$$\begin{aligned}
&[f_{12}(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3) - f_{11}^2(\mathbf{Z}_1, \mathbf{Z}_2+\mathbf{Z}_3)] + [f_{22}(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1) \\
&-f_{21}^2(\mathbf{Z}_2, \mathbf{Z}_3+\mathbf{Z}_1)] = [f_{32}(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2) - f_{31}^2(\mathbf{Z}_3, \mathbf{Z}_1+\mathbf{Z}_2)]\,.
\end{aligned} \tag{2.109}$$

This equation has a form like the first equation of the system (2.103).

If we take into account the previous two equations, then the third equation of the system (2.103) becomes

$$\begin{aligned}
&[f_{13}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)-3f_{12}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad+2f_{11}^3(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)]+[f_{23}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad-3f_{22}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)f_{21}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)+2f_{21}^3(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)]\\
&=[f_{33}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)-3f_{32}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)f_{31}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)\\
&\quad+2f_{31}^3(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)]\,.
\end{aligned}\tag{2.110}$$

The equation (2.110) has also a form like the first equation of the system (2.103).

If we apply the above two equations, then the fourth equation of the system (2.103) takes on the form

$$\begin{aligned}
&[f_{14}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)-4f_{13}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad+12f_{12}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}^3(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)-3f_{12}^2(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad-6f_{11}^4(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)]+[f_{24}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad-4f_{23}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)f_{21}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad+12f_{22}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)f_{21}^2(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)-3f_{22}^2(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad-6f_{21}^4(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)]\\
&=[f_{34}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)-4f_{33}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)f_{31}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)\\
&\quad+12f_{32}(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)f_{31}^2(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)\\
&\quad-3f_{32}^2(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)-6f_{31}^4(\mathbf{Z}_3,\mathbf{Z}_1+\mathbf{Z}_2)]\,.
\end{aligned}\tag{2.111}$$

Thus also the equation (2.111) has a form like the first equation of the system (2.103).

Similarly, the last equation of the system (2.103) is reduced to the following equation

$$\begin{aligned}
&[f_{15}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)-5f_{14}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad+20f_{13}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}^2(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad-60f_{12}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}^3(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad+30f_{12}^2(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{11}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad-10f_{13}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)f_{12}(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)\\
&\quad+24f_{11}^5(\mathbf{Z}_1,\mathbf{Z}_2+\mathbf{Z}_3)]+[f_{25}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad-5f_{24}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)f_{21}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)\\
&\quad+20f_{23}(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)f_{21}^2(\mathbf{Z}_2,\mathbf{Z}_3+\mathbf{Z}_1)
\end{aligned}$$

$$
\begin{aligned}
&- 60 f_{22}(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1) f_{21}^2(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1)\\
&+ 30 f_{22}^2(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1) f_{21}(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1)\\
&- 10 f_{23}(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1) f_{22}(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1) + 24 f_{21}^5(\mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_1)]\\
= &[f_{35}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) - 5 f_{34}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) f_{31}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2)\\
&+ 20 f_{33}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) f_{31}^2(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2)\\
&- 60 f_{32}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) f_{31}^3(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2)\\
&+ 30 f_{32}^2(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) f_{31}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2)\\
&- 10 f_{33}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) f_{32}(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2) + f_{31}^5(\mathbf{Z}_3, \mathbf{Z}_1 + \mathbf{Z}_2)],
\end{aligned}
\tag{2.112}
$$

which has a form like the first equation of the system (2.103).

Therefore, we may summarize that the system of partial functional equations (2.103) considered, with five functional equations and fifteen unknown functions $f_{rj} : \mathcal{V}^2 \to \mathbf{C}, \quad (r = 1, 2, 3;\ j = 1, 2, 3, 4, 5)$ is equivalent to the system obtained which is formed by the first equation of the system (2.103) and the equations (2.109), (2.110), (2.111) and (2.112), whose general continuous solution is given by the formulas (2.104), (2.105), (2.106), (2.107) and (2.108). □

The vector systems of partial nonlinear functional equations considered here have not been taken into account in the references [2,3,5,10,11].

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