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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

R. KOPLATADZE, N. PARTSVANIA AND I. P. STAVROULAKIS

Abstract. Sufficient conditions are established for the oscillation of proper solutions of the system

$$u_1'(t) = p(t)u_2(\sigma(t))\,,$$
$$u_2'(t) = -q(t)u_1(\tau(t))\,,$$

where p, $q : R_+ \to R_+$ are locally summable functions, while τ and $\sigma : R_+ \to R_+$ are continuous and continuously differentiable functions, respectively, and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$.

1. Statement of the problem and the formulation of the main results

Consider the differential system

$$\begin{aligned} u_1'(t) &= p(t)u_2(\sigma(t)), \\ u_2'(t) &= -q(t)u_1(\tau(t))\,, \end{aligned} \tag{1.1}$$

where p, $q : R_+ \to R_+$ are locally summable functions, $\tau : R_+ \to R_+$ is a continuous function, and $\sigma : R_+ \to R_+$ is a continuously differentiable function. Throughout the paper we will assume that

$$\sigma'(t) \geq 0 \quad \text{for} \quad t \in R_+\,, \lim_{t\to+\infty} \tau(t) = +\infty, \quad \lim_{t\to+\infty} \sigma(t) = +\infty\,.$$

In the present paper, new sufficient conditions are established for the oscillation of system (1.1) (see Definition 1.3 below) as well as conditions for system (1.1) to have at least one proper solution. Analogous problems for second order ordinary

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differential equations and systems and for higher order functional differential equations are studied in [1, 2, 4, 9–12] and [6], respectively. For second order differential equations with deviating arguments the problem of oscillation is investigated in [5, 7, 8, 13] (see also the references therein).

Definition 1.1. Let $t_0 \in R_+$ and $a_0 = \min\big\{\inf\limits_{t\ge t_0}\tau(t);\ \inf\limits_{t\ge t_0}\sigma(t)\big\}$. A continuous vector function (u_1,u_2) defined on $[a_0,+\infty)$ is said to be a proper solution of system (1.1) in $[t_0,+\infty)$ if it is absolutely continuous on each finite segment contained in $[t_0,+\infty)$, satisfies (1.1) almost everywhere on $[t_0,+\infty)$, and $\sup\big\{|u_1(s)|+|u_2(s)|:\ s\ge t\big\}>0$ for $t\ge t_0$.

Definition 1.2. A proper solution (u_1,u_2) of system (1.1) is said to be oscillatory if both u_1 and u_2 have sequences of zeros tending to infinity; otherwise it is said to be nonoscillatory.

Definition 1.3. System (1.1) is said to be oscillatory if every its proper solution is oscillatory.

Let $\mu : R_+ \to R_+$ be a continuously differentiable function satisfying the following conditions

$$\mu'(t)\ge 0 \quad \text{for} \quad t\in R_+\,, \qquad \lim_{t\to+\infty}\mu(t)=+\infty\,. \tag{1.2}$$

In the sequel, we will use the notation

$$h(t)=\int\limits_0^t p(s)\,ds \quad \text{for} \quad t\ge 0\,, \tag{1.3}$$

$$\rho(t)=\begin{cases}1 & \text{for} \quad \tau(t)\ge\mu(t)\,,\\ \dfrac{h(\tau(t))}{h(\mu(t))} & \text{for} \quad \tau(t)<\mu(t)\,,\end{cases} \tag{1.4}$$

$$g(t,\lambda)=h^{-\lambda}(\mu(t))\int\limits_1^t p(\mu(s))\mu'(s)\int\limits_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\,d\xi\,ds \quad \text{for} \quad t\ge 1,\ \ \lambda\in(0,1), \tag{1.5}$$

$$g_*(\lambda)=\liminf_{t\to+\infty} g(t,\lambda), \qquad g^*(\lambda)=\limsup_{t\to+\infty} g(t,\lambda) \qquad \text{for } \lambda\in(0,1)\,. \tag{1.6}$$

It is easy to show that if $\int\limits^{+\infty} h(\tau(t))q(t)\,dt < +\infty$, then system (1.1) has a proper nonoscillatory solution. Therefore it will be assumed that

$$\int\limits^{+\infty} h(\tau(t))q(t)\,dt=+\infty\,. \tag{1.7}$$

Moreover, below we will assume that

$$\limsup_{t\to+\infty} h(\mu(t))\int\limits_t^{+\infty} q(s)\rho(s)\,ds<+\infty\,. \tag{1.8}$$

Note that condition (1.8) is not an essential restriction in the sense that if $\limsup_{t\to+\infty} h(\mu(t))/h(t) < +\infty$ and $\limsup_{t\to+\infty} h(\mu(t)) \int_t^{+\infty} q(s)\rho(s)\,ds = +\infty$, then, as it is easy to prove, system (1.1) is oscillatory.

Remark 1.1. Without loss of generality it will be assumed that

$$p(t) \neq 0 \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad \mu(1) > 0\,, \tag{1.9}$$

since the alternation of coefficients of the system in a finite interval has no influence on oscillatory properties of that system.

Theorem 1.1. *Let*

$$\lim_{t\to+\infty} h(t) = +\infty\,, \tag{1.10}$$

and let there exist a continuously differentiable function $\mu : R_+ \to R_+$ *such that conditions* (1.2), (1.8) *are fulfilled and for sufficiently large* t,

$$\sigma(\mu(t)) \leq t\,. \tag{1.11}$$

If, moreover, for some $\lambda \in (0,1)$,

$$g^*(\lambda) > \min\left\{\frac{c_0}{\lambda} + \frac{\lambda}{4(1-\lambda)}\,,\ \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\right\}, \tag{1.12}$$

where $h(t)$ *and* $g^*(\lambda)$ *are defined by* (1.3) *and* (1.4) - (1.6), *respectively, and*

$$\begin{aligned} c_0 &= \limsup_{t\to+\infty} h(\mu(t))\Big/\Big(h(\mu_0(t)) + \int_0^{\mu_0(t)} q(s)h(s)\,ds\Big)\,, \\ \mu_0(t) &= \begin{cases} \mu(t) & \text{for} \quad \mu(t) \leq t\,, \\ t & \text{for} \quad \mu(t) > t\,, \end{cases} \end{aligned} \tag{1.13}$$

then system (1.1) *is oscillatory.*

Corollary 1.1. *Let condition* (1.10) *hold, and let there exist a continuously differentiable function* $\mu : R_+ \to R_+$ *such that conditions* (1.2), (1.8), (1.11) *are fulfilled and* $\mu(t) \leq t$ *for sufficiently large* t. *If, moreover, for some* $\lambda \in (0,1)$,

$$g^*(\lambda) > \min\left\{\frac{(\lambda-2)^2}{4\lambda(1-\lambda)}\,,\ \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\right\}, \tag{1.14}$$

where $g^*(\lambda)$ *is defined by* (1.4)–(1.6), *then system* (1.1) *is oscillatory.*

Theorem 1.2. *Let conditions* (1.2), (1.8), (1.10), (1.11) *hold, and let*

$$\lim_{\lambda\to 1-} (1-\lambda)g^*(\lambda) > \frac{1}{4}\,, \tag{1.15}$$

where $g^*(\lambda)$ *is defined by* (1.4)–(1.6). *Then system* (1.1) *is oscillatory.*

Theorem 1.3. *Let conditions* (1.2)*,* (1.8)*,* (1.10)*,* (1.11) *be fulfilled, and let for some* $\lambda_0 \in (0,1)$*,*

$$g_*(\lambda_0) > \frac{1}{4\lambda_0(1-\lambda_0)}\,, \tag{1.16}$$

where $g_*(\lambda_0)$ *is defined by* (1.4)–(1.6)*. Then system* (1.1) *is oscillatory.*

Corollary 1.2. *If conditions* (1.2)*,* (1.8)*,* (1.10)*,* (1.11) *are fulfilled and for some* $\lambda_0 \in (0,1)$*,*

$$\liminf_{t\to+\infty} h^{1-\lambda_0}(\mu(t)) \int_t^{+\infty} q(s)\rho(s)h^{\lambda_0}(\mu(s))\,ds > \frac{1}{4(1-\lambda_0)}\,, \tag{1.17}$$

where $h(t)$ *and* $\rho(t)$ *are defined by* (1.3) *and* (1.4)*, then system* (1.1) *is oscillatory.*

Corollary 1.3. *If conditions* (1.2)*,* (1.8)*,* (1.10)*,* (1.11) *hold and for some* $\lambda_0 \in (0,1)$*,*

$$\liminf_{t\to+\infty} h^{-\lambda_0}(\mu(t)) \int_1^{t} q(s)\rho(s)h^{1+\lambda_0}(\mu(s))\,ds > \frac{1}{4\lambda_0}\,, \tag{1.18}$$

where $h(t)$ *and* $\rho(t)$ *are defined by* (1.3) *and* (1.4)*, then system* (1.1) *is oscillatory.*

Theorem 1.4. *Let condition* (1.10) *be fulfilled and let for some* $\lambda \in (0,1)$*,*

$$\limsup_{t\to+\infty} h^{-\lambda}(t) \int_0^{t} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)h^{\lambda}(\tau(\xi))\,d\xi\,ds < 1\,, \tag{1.19}$$

where $h(t)$ *is defined by* (1.3)*. Then system* (1.1) *has a proper nonoscillatory solution.*

Now consider the second order linear differential equation

$$u''(t) + q(t)u(\tau(t)) = 0\,, \tag{1.20}$$

where $q : R_+ \to R_+$ is a locally summable function, and $\tau : R_+ \to R_+$ is a continuous function such that $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. For equation (1.20), Theorem 1.3 and Corollaries 1.2 and 1.3 have the following form.

Theorem 1.3′. *Let*

$$\tau(t) \ge \alpha t \quad \text{for} \quad t \in R_+\,, \tag{1.21}$$

and let for some $\lambda \in (0,1)$*,*

$$\liminf_{t\to+\infty} t^{-\lambda} \int_1^{t} \int_s^{+\infty} \xi^{\lambda} q(\xi)\,d\xi\,ds > \frac{1}{4\alpha\lambda(1-\lambda)}\,, \tag{1.22}$$

where $\alpha \in (0,+\infty)$*. Then equation* (1.20) *is oscillatory.*

Corollary 1.2′. *If condition* (1.21) *holds and for some* $\lambda \in (0,1)$*,*

$$\liminf_{t\to+\infty} t^{1-\lambda} \int\limits_t^{+\infty} s^{\lambda} q(s)\, ds > \frac{1}{4\alpha(1-\lambda)}\,, \tag{1.23}$$

where $\alpha \in (0,+\infty)$*, then equation* (1.20) *is oscillatory.*

Corollary 1.3′. *If condition* (1.21) *is fulfilled and for some* $\lambda \in (0,1)$*,*

$$\liminf_{t\to+\infty} t^{-\lambda} \int\limits_1^{t} s^{1+\lambda} q(s)\, ds > \frac{1}{4\alpha\lambda}\,, \tag{1.24}$$

where $\alpha \in (0,+\infty)$*, then equation* (1.20) *is oscillatory.*

Remark 1.2. For the case where equation (1.20) is without delay (i.e, $\tau(t) \equiv t$; $\alpha = 1$) Corollaries 1.2′ and 1.3′ lead to the results by Nehari [12] and Lomtatidze [9], respectively. So, Theorem 1.3′ is important even for equations without delay, since the above mentioned results by Nehari and Lomtatidze are particular cases of that theorem. Moreover, it is possible to construct examples showing that conditions (1.23) and (1.24) are violated but condition (1.22) is satisfied.

2. Auxiliary statements

Lemma 2.1. *Let condition* (1.10) *be fulfilled,* $q(t) \not\equiv 0$ *in any neighbourhood of* $+\infty$*, and let* $(u_1(t), u_2(t))$ *be a proper nonoscillatory solution of system* (1.1)*. Then there exists* $t_* \in R_+$ *such that*

$$u_1(t)u_2(t) > 0 \quad \textit{for} \quad t \geq t_*\,. \tag{2.1}$$

For the proof of Lemma 2.1 see [8, Lemma 2.1].

Lemma 2.2. *Let condition* (1.10) *hold,* $q(t) \not\equiv 0$ *in any neighbourhood of* $+\infty$*, and let* $(u_1(t), u_2(t))$ *be a proper nonoscillatory solution of system* (1.1)*. Then there exists* $t_0 \in R_+$ *such that either*

$$h(t)u_2(\sigma(t)) - u_1(t) \geq 0 \quad \textit{for} \quad t \geq t_0 \tag{2.2}$$

or

$$h(t)u_2(\sigma(t)) - u_1(t) < 0 \quad \textit{for} \quad t \geq t_0\,, \tag{2.3}$$

where $h(t)$ *is defined by* (1.3)*.*

Proof. By Lemma 2.1 there exists $t_* \in R_+$ such that inequality (2.1) holds for $t \geq t_*$. Without loss of generality we can assume that $u_1(t) > 0$ and $u_2(t) > 0$ for $t \geq t_*$. Therefore, in view of (1.1), we find

$$\begin{aligned}\big(h(t)u_2(\sigma(t)) - u_1(t)\big)' &= p(t)u_2(\sigma(t)) + h(t)u_2'(\sigma(t))\sigma'(t) - u_1'(t) \\ &= h(t)u_2'(\sigma(t))\sigma'(t) \leq 0 \quad \text{for} \quad t \geq t_1\,,\end{aligned}$$

where $t_1 > t_*$ is a sufficiently large number. Consequently, since $h(t)u_2(\sigma(t)) - u_1(t)$ is a nonincreasing function, there exists $t_0 > t_1$ such that either condition (2.2) or condition (2.3) is fulfilled. □

Lemma 2.3. *If conditions* (1.2), (1.8)–(1.10) *are fulfilled, then for any* $\lambda \in (0,1)$,

$$\limsup_{t\to+\infty} h^{1-\lambda}(\mu(t)) \int_t^{+\infty} q(s)\rho(s)h^{\lambda}(\mu(s))\,ds < +\infty \tag{2.4}$$

and

$$\int_0^{+\infty} p(\mu(s))h^{-2}(\mu(s))\mu'(s) \int_0^{s} q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\,d\xi\,ds < +\infty\,, \tag{2.5}$$

where $h(t)$ *and* $\rho(t)$ *are defined by* (1.3) *and* (1.4).

Proof. First we show the validity of (2.4). Due to (1.8) there exist $M > 0$ and $t_0 \in R_+$ such that

$$h(\mu(t)) \int_t^{+\infty} q(s)\rho(s)\,ds \le M \quad \text{for} \quad t \ge t_0\,. \tag{2.6}$$

Note that according to (2.6) for any $\lambda \in (0,1)$,

$$\int^{+\infty} q(s)\rho(s)h^{\lambda}(\mu(s))\,ds < +\infty\,.$$

Thus, by (1.10) and (2.6), we have

$$\begin{aligned}
&h^{1-\lambda}(\mu(t)) \int_t^{+\infty} q(s)\rho(s)h^{\lambda}(\mu(s))\,ds = -h^{1-\lambda}(\mu(t)) \\
&\times \int_t^{+\infty} h^{\lambda}(\mu(s))\,d\int_s^{+\infty} q(\xi)\rho(\xi)\,d\xi = h(\mu(t)) \int_t^{+\infty} q(s)\rho(s)\,ds^{\,1)} \\
&+ \lambda h^{1-\lambda}(\mu(t)) \int_t^{+\infty} p(\mu(s))h^{\lambda-1}(\mu(s))\mu'(s) \int_s^{+\infty} q(\xi)\rho(\xi)\,d\xi\,ds \\
\le\ & M + \lambda M h^{1-\lambda}(\mu(t)) \int_t^{+\infty} p(\mu(s))h^{\lambda-2}(\mu(s))\mu'(s)\,ds \\
=\ & M + \frac{\lambda}{1-\lambda} M = \frac{M}{1-\lambda} \quad \text{for} \quad t \ge t_0\,.
\end{aligned}$$

Consequently inequality (2.4) is valid.

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1) Due to (1.8), it is obvious that for any $\lambda \in (0,1)$, $\lim\limits_{t\to+\infty} h^{\lambda}(\mu(t)) \int_t^{+\infty} q(s)\rho(s)\,ds = 0$.

Now show the validity of (2.5). Taking into account conditions (1.10), (2.6) and Remark 1.1, we get

$$\begin{aligned}
&\int_1^{+\infty} p(\mu(s))h^{-2}(\mu(s))\mu'(s)\int_0^s q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\,d\xi\,ds\\
&= -\int_1^{+\infty} p(\mu(s))h^{-2}(\mu(s))\mu'(s)\int_0^s h^{1+\lambda}(\mu(\xi))\,d\int_\xi^{+\infty} q(\xi_1)\rho(\xi_1)\,d\xi_1\,ds\\
&= -\int_1^{+\infty} p(\mu(s))\mu'(s)h^{\lambda-1}(\mu(s))\int_s^{+\infty} q(\xi)\rho(\xi)\,d\xi\,ds\\
&\quad + h^{1+\lambda}(\mu(0))\int_0^{+\infty} q(s)\rho(s)\,ds\int_1^{+\infty} p(\mu(s))\mu'(s)h^{-2}(\mu(s))\,ds + (1+\lambda)\\
&\quad \times\int_1^{+\infty} p(\mu(s))\mu'(s)h^{-2}(\mu(s))\int_0^s p(\mu(\xi))\mu'(\xi)h^{\lambda}(\mu(\xi))\int_\xi^{+\infty} q(\xi_1)\rho(\xi_1)\,d\xi_1\,d\xi\,ds\\
&\le h^{1+\lambda}(\mu(0))h^{-1}(\mu(1))\int_0^{+\infty} q(s)\rho(s)\,ds + \frac{(1+\lambda)M}{\lambda(1-\lambda)}\,h^{\lambda-1}(\mu(1))\,.
\end{aligned}$$

Therefore inequality (2.5) is fulfilled. □

Lemma 2.4. *Let conditions* (1.8) *and* (1.10) *be fulfilled. Then for any* $\lambda \in (0,1)$ *the function* $g(t,\lambda)$*, which is defined by* (1.5)*, admits the representation*

$$\begin{aligned}
g(t,\lambda) = h^{1-\lambda}(\mu(t))&\int_t^{+\infty} p(\mu(s))h^{-2}(\mu(s))\mu'(s)\\
&\times\int_0^s q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\,d\xi\,ds + O\big(h^{-\lambda}(\mu(t))\big)\,,
\end{aligned}\tag{2.7}$$

where $\mu(t)$ *satisfies* (1.2)*, and* $h(t)$*,* $\rho(t)$ *are given by* (1.3)*,* (1.4)*.*

Proof. First we show that

$$\lim_{t\to+\infty} h^{-1}(\mu(t))\int_0^t q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds = 0\,.\tag{2.8}$$

Let ε be an arbitrary positive number. By virtue of (1.8), we can choose $T > 0$ such that

$$\int_T^{+\infty} q(s)\rho(s)h^{\lambda}(\mu(s))\,ds < \varepsilon\,.\tag{2.9}$$

On the other hand,

$$\begin{aligned} & h^{-1}(\mu(t)) \int_0^t q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds \\ &= h^{-1}(\mu(t)) \int_0^T q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds + h^{-1}(\mu(t)) \int_T^t q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds \\ &\le h^{-1}(\mu(t)) \int_0^T q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds + \int_T^{+\infty} q(s)\rho(s)h^{\lambda}(\mu(s))\,ds\,. \end{aligned}$$

Hence, by (1.2), (1.10) and (2.9), we obtain

$$\begin{aligned} & \limsup_{t\to+\infty} h^{-1}(\mu(t)) \int_0^t q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds \\ \le\ & \limsup_{t\to+\infty} h^{-1}(\mu(t)) \int_0^T q(s)\rho(s)h^{1+\lambda}(\mu(s))\,ds + \varepsilon = \varepsilon\,. \end{aligned}$$

Therefore, taking into account the arbitrariness of ε, the last inequality yields (2.8).

In view of (1.8) and (2.8), for any $\lambda \in (0,1)$ we have

$$\begin{aligned} g(t,\lambda) &= h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\,d\xi\,ds \\ &= h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} h^{-1}(\mu(\xi))\,d\int_0^{\xi} q(\xi_1)\rho(\xi_1)h^{1+\lambda}(\mu(\xi_1))\,d\xi_1\,ds \\ &= -h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s)h^{-1}(\mu(s)) \int_0^s q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\,d\xi\,ds \\ &\quad + h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} p(\mu(\xi))\mu'(\xi)h^{-2}(\mu(\xi)) \\ &\quad \times \int_0^{\xi} q(\xi_1)\rho(\xi_1)h^{1+\lambda}(\mu(\xi_1))\,d\xi_1\,d\xi\,ds = h^{-\lambda}(\mu(t)) \\ &\quad \times \int_1^t h(\mu(s))\,d \int_s^{+\infty} p(\mu(\xi))\mu'(\xi)h^{-2}(\mu(\xi)) \int_0^{\xi} q(\xi_1)\rho(\xi_1)h^{1+\lambda}(\mu(\xi_1))\,d\xi_1\,d\xi \end{aligned}$$

$$+ h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} p(\mu(\xi))\mu'(\xi)h^{-2}(\mu(\xi))$$

$$\times \int_0^\xi q(\xi_1)\rho(\xi_1)h^{1+\lambda}(\mu(\xi_1))\, d\xi_1\, d\xi\, ds = h^{1-\lambda}(\mu(t))$$

$$\times \int_t^{+\infty} p(\mu(s))\mu'(s)h^{-2}(\mu(s)) \int_0^s q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\, d\xi\, ds - Ah^{-\lambda}(\mu(t))\,,$$

where

$$A = h(\mu(1)) \int_1^{+\infty} p(\mu(s))\mu'(s)h^{-2}(\mu(s)) \int_0^s q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\, d\xi\, ds\,,$$

and due to (2.5) (see Lemma 2.3), $A < +\infty$. Consequently (2.7) is valid. □

Lemma 2.5. *For any* λ, $\lambda_0 \in (0,1)$ $(\lambda \neq \lambda_0)$ *the following representation*

$$\begin{aligned} g(t,\lambda) = g(t,\lambda_0) &- 2(\lambda-\lambda_0)h^{-\lambda}(\mu(t)) \int_1^t \mu'(s)p(\mu(s))h^{\lambda-1}(\mu(s))g(s,\lambda_0)\, ds \\ &- (\lambda-\lambda_0)(\lambda-\lambda_0-1)h^{-\lambda}(\mu(t)) \\ &\times \int_1^t \mu'(s)p(\mu(s)) \int_s^{+\infty} \mu'(\xi)p(\mu(\xi))h^{\lambda-2}(\mu(\xi))g(\xi,\lambda_0)\, d\xi\, ds \end{aligned} \tag{2.10}$$

is valid, where $h(t)$ *and* $g(t,\lambda)$ *are defined by* (1.3) *and* (1.4), (1.5), *and* $\mu(t)$ *satisfies* (1.2).

Proof. For any λ, $\lambda_0 \in (0,1)$ we have

$$\begin{aligned} g(t,\lambda) &= h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\, d\xi\, ds = -h^{-\lambda}(\mu(t)) \\ &\times \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} h^{\lambda-\lambda_0}(\mu(\xi))\, d\int_\xi^{+\infty} q(\xi_1)\rho(\xi_1)h^{\lambda_0}(\mu(\xi_1))\, d\xi_1\, ds \\ &= h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s)h^{\lambda-\lambda_0}(\mu(s)) \int_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda_0}(\mu(\xi))\, d\xi\, ds \\ &+ (\lambda-\lambda_0)h^{-\lambda}(\mu(t)) \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} h^{\lambda-\lambda_0-1}(\mu(\xi))p(\mu(\xi))\mu'(\xi) \end{aligned}$$

$$
\begin{aligned}
&\quad \times \int\limits_{\xi}^{+\infty} q(\xi_1)\rho(\xi_1)h^{\lambda_0}(\mu(\xi_1))\,d\xi_1\,d\xi\,ds \\
&= h^{-\lambda}(\mu(t))\int\limits_{1}^{t} h^{\lambda-\lambda_0}(\mu(s))\,d\int\limits_{1}^{s}\mu'(\xi)p(\mu(\xi))\int\limits_{\xi}^{+\infty} q(\xi_1)\rho(\xi_1)h^{\lambda_0}(\mu(\xi_1))\,d\xi_1\,d\xi \\
&\quad + (\lambda-\lambda_0)h^{-\lambda}(\mu(t))\int\limits_{1}^{t}\mu'(s)p(\mu(s))\int\limits_{s}^{+\infty} h^{\lambda-\lambda_0-1}(\mu(\xi))\,d\int\limits_{1}^{\xi}\mu'(\xi_1)p(\mu(\xi_1)) \\
&\quad \times \int\limits_{\xi_1}^{+\infty} q(\xi_2)\rho(\xi_2)h^{\lambda_0}(\mu(\xi_2))\,d\xi_2\,d\xi_1\,ds \\
&= h^{-\lambda_0}(\mu(t))\int\limits_{1}^{t}\mu'(s)p(\mu(s))\int\limits_{s}^{+\infty} q(\xi)\rho(\xi)h^{\lambda_0}(\mu(\xi))\,d\xi\,ds \\
&\quad - (\lambda-\lambda_0)h^{-\lambda}(\mu(t))\int\limits_{1}^{t} p(\mu(s))\mu'(s)h^{\lambda-\lambda_0-1}(\mu(s))\int\limits_{1}^{s}\mu'(\xi)p(\mu(\xi)) \\
&\quad \times \int\limits_{\xi}^{+\infty} q(\xi_1)\rho(\xi_1)h^{\lambda_0}(\mu(\xi_1))\,d\xi_1\,d\xi\,ds \\
&\quad - (\lambda-\lambda_0)h^{-\lambda}(\mu(t))\int\limits_{1}^{t} p(\mu(s))\mu'(s)h^{\lambda-\lambda_0-1}(\mu(s))\int\limits_{1}^{s}\mu'(\xi)p(\mu(\xi)) \\
&\quad \times \int\limits_{\xi}^{+\infty} q(\xi_1)\rho(\xi_1)h^{\lambda_0}(\mu(\xi_1))\,d\xi_1\,d\xi\,ds - (\lambda-\lambda_0)(\lambda-\lambda_0-1)h^{-\lambda}(\mu(t)) \\
&\quad \times \int\limits_{1}^{t} p(\mu(s))\mu'(s)\int\limits_{s}^{+\infty} h^{\lambda-\lambda_0-2}(\mu(\xi))p(\mu(\xi))\mu'(\xi) \\
&\quad \times \int\limits_{1}^{\xi} p(\mu(\xi_1))\mu'(\xi_1)\int\limits_{\xi_1}^{+\infty} q(\xi_2)\rho(\xi_2)h^{\lambda_0}(\mu(\xi_2))\,d\xi_2\,d\xi_1\,d\xi\,ds \\
&= g(t,\lambda_0) - 2(\lambda-\lambda_0)h^{-\lambda}(\mu(t))\int\limits_{1}^{t} p(\mu(s))\mu'(s)h^{\lambda-1}(\mu(s))g(s,\lambda_0)\,ds \\
&\quad - (\lambda-\lambda_0)(\lambda-\lambda_0-1)h^{-\lambda}(\mu(t))
\end{aligned}
$$

$$\times \int_1^t p(\mu(s))\mu'(s) \int_s^{+\infty} p(\mu(\xi))\mu'(\xi)h^{\lambda-2}(\mu(\xi))(\xi,\lambda_0)\,d\xi\,ds\,.$$

Therefore (2.10) is valid. □

Lemma 2.6. *For any λ, $\lambda_0 \in (0,1)$ $(\lambda \neq \lambda_0)$ the following representation*

$$g(t,\lambda) = g(t,\lambda_0) + 2(\lambda-\lambda_0)h^{1-\lambda}(\mu(t)) \int_t^{+\infty} \mu'(s)p(\mu(s))h^{\lambda-2}(\mu(s))g(s,\lambda_0)\,ds$$

$$- (\lambda-\lambda_0)(\lambda-\lambda_0+1)h^{1-\lambda}(\mu(t)) \int_t^{+\infty} \mu'(s)p(\mu(s))h^{-2}(\mu(s))$$

$$\times \int_1^s \mu'(\xi)p(\mu(\xi))h^{\lambda-1}(\mu(\xi))g(\xi,\lambda_0)\,d\xi\,ds + O\big(h^{-\lambda}(\mu(t))\big) \tag{2.11}$$

is valid, where $h(t)$ and $g(t,\lambda)$ are defined by (1.3) *and* (1.4), (1.5), *and $\mu(t)$ satisfies* (1.2).

Lemma 2.6 can be proved analogously to Lemma 2.5 if we take into consideration Lemma 2.4.

Lemma 2.7. *Let conditions* (1.2), (1.8) *and* (1.10) *hold. Then $g_*(\lambda)$, $g^*(\lambda) \in C((0,1))$* [2]*. Moreover,*

$$\lim_{\lambda\to 0+} \lambda g^*(\lambda) = \lim_{\lambda\to 1-} (1-\lambda)g^*(\lambda), \tag{2.12}$$

$$\lim_{\lambda\to 0+} \lambda g_*(\lambda) = \lim_{\lambda\to 1-} (1-\lambda)g_*(\lambda), \tag{2.13}$$

and for any $\lambda_0 \in (0,1)$,

$$\lim_{\lambda\to 1-} (1-\lambda)g^*(\lambda) \le \lambda_0(1-\lambda_0)g^*(\lambda_0), \tag{2.14}$$

$$\lim_{\lambda\to 1-} (1-\lambda)g_*(\lambda) \ge \lambda_0(1-\lambda_0)g_*(\lambda_0), \tag{2.15}$$

where $g_*(\lambda)$, $g^*(\lambda)$ are defined by (1.4)–(1.6).

Proof. First we show that $g^*(\lambda) \in C((0,1))$. For any $\lambda_0 \in (0,1)$ we have

$$g^*(\lambda_0) < +\infty\,. \tag{2.16}$$

Indeed, according to conditions (1.2), (1.8), (1.10) and Lemma 2.3, condition (2.4) is satisfied for any $\lambda \in (0,1)$. Thus there exist a positive number $\gamma(\lambda_0)$ and $t_* \in R_+$ such that for any $\lambda_0 \in (0,1)$,

$$h^{1-\lambda_0}(\mu(t)) \int_t^{+\infty} q(s)\rho(s)h^{\lambda_0}(\mu(s))\,ds \le \gamma(\lambda_0) \quad \text{for} \quad t \ge t_*\,.$$

---

[2] By $C((a,b))$ we denote the set of continuous functions defined on (a,b).

Then

$$g(t,\lambda_0)=h^{-\lambda_0}(\mu(t))\int\limits_1^t p(\mu(s))\mu'(s)\int\limits_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda_0}(\mu(\xi))\,d\xi\,ds$$
$$\leq \gamma(\lambda_0)h^{-\lambda_0}(\mu(t))\int\limits_1^t p(\mu(s))\mu'(s)h^{\lambda_0-1}(\mu(s))\,ds\leq\frac{\gamma(\lambda_0)}{\lambda_0}\quad\text{for}\quad t\geq t_*\,.$$

Therefore (2.16) is valid.

Let $\lambda_0\in(0,1)$ and let ε be a positive number. Choose $t_0\in R_+$ such that

$$g(t,\lambda_0)\leq g^*(\lambda_0)+\varepsilon\quad\text{for}\quad t\geq t_0\,. \tag{2.17}$$

By (1.10) and (2.17), from (2.10) (see Lemma 2.5) we find

$$\begin{aligned}g^*(\lambda)&\leq g^*(\lambda_0)+2|\lambda-\lambda_0|\big(g^*(\lambda_0)+\varepsilon\big)\limsup_{t\to+\infty}h^{-\lambda}(\mu(t))\\&\quad\times\int\limits_{t_0}^t\mu'(s)p(\mu(s))h^{\lambda-1}(\mu(s))\,ds+|\lambda-\lambda_0|\,|\lambda-\lambda_0-1|\big(g^*(\lambda_0)+\varepsilon\big)\\&\quad\times\limsup_{t\to+\infty}h^{-\lambda}(\mu(t))\int\limits_{t_0}^t\mu'(s)p(\mu(s))\int\limits_s^{+\infty}\mu'(\xi)p(\mu(\xi))h^{\lambda-2}(\mu(\xi))\,d\xi\,ds\\&=g^*(\lambda_0)+\frac{2|\lambda-\lambda_0|}{\lambda}\big(g^*(\lambda_0)+\varepsilon\big)+\frac{|\lambda-\lambda_0|\,|\lambda-\lambda_0-1|}{\lambda(1-\lambda)}\big(g^*(\lambda_0)+\varepsilon\big)\,.\end{aligned}\tag{2.18}$$

Analogously to this we can show that

$$\begin{aligned}g^*(\lambda)\geq g^*(\lambda_0)&-\frac{2|\lambda-\lambda_0|}{\lambda}\big(g^*(\lambda_0)+\varepsilon\big)\\&-\frac{|\lambda-\lambda_0|\,|\lambda-\lambda_0-1|}{\lambda(1-\lambda)}\big(g^*(\lambda_0)+\varepsilon\big)\,.\end{aligned}\tag{2.19}$$

Due to (2.16), (2.18) and (2.19), it is clear that $g^*(\lambda)\in C((0,1))$. On the other hand, in view of the arbitrariness of ε, (2.18) implies

$$\limsup_{\lambda\to1-}(1-\lambda)g^*(\lambda)\leq\lambda_0(1-\lambda_0)g^*(\lambda_0)\,.$$

Since the last inequality is satisfied for any $\lambda_0\in(0,1)$, it is evident that there exists $\lim\limits_{\lambda\to1-}(1-\lambda)g^*(\lambda)$. Consequently (2.14) is fulfilled for any $\lambda_0\in(0,1)$.

Now we show the validity of (2.12). By (1.10) and (2.17), from (2.11) (see Lemma 2.6) we get

$$\begin{aligned}
g^*(\lambda) = \limsup_{t\to+\infty} \Big[& g(t,\lambda_0) + 2(\lambda-\lambda_0)h^{1-\lambda}(\mu(t)) \\
& \times \int_t^{+\infty} \mu'(s)p(\mu(s))h^{\lambda-2}(\mu(s))g(s,\lambda_0)\,ds - (\lambda-\lambda_0)(\lambda-\lambda_0+1)h^{1-\lambda}(\mu(t)) \\
& \times \int_t^{+\infty} \mu'(s)p(\mu(s))h^{-2}(\mu(s)) \int_0^s \mu'(\xi)p(\mu(\xi))h^{\lambda-1}(\mu(\xi))g(\xi,\lambda_0)\,d\xi\,ds \\
& + O\big(h^{-\lambda}(\mu(t))\big)\Big] \le g^*(\lambda_0) + \frac{2|\lambda-\lambda_0|}{1-\lambda}\big(g^*(\lambda_0)+\varepsilon\big) \\
& + |\lambda-\lambda_0|\,|\lambda-\lambda_0+1| \int_0^{t_0} \mu'(s)p(\mu(s))h^{\lambda-1}(\mu(s))g(s,\lambda_0)\,ds \\
& \times \limsup_{t\to+\infty} h^{-\lambda}(\mu(t)) + |\lambda-\lambda_0|\,|\lambda-\lambda_0+1|\big(g^*(\lambda_0)+\varepsilon\big) \\
& \times \limsup_{t\to+\infty} h^{1-\lambda}(\mu(t)) \int_t^{+\infty} \mu'(s)p(\mu(s))h^{-2}(\mu(s)) \int_{t_0}^s \mu'(\xi)p(\mu(\xi))h^{\lambda-1}(\mu(\xi))\,d\xi ds \\
= g^*(\lambda_0) + \frac{2|\lambda-\lambda_0|}{1-\lambda} & \big(g^*(\lambda_0)+\varepsilon\big) + \frac{|\lambda-\lambda_0|\,|\lambda-\lambda_0+1|}{\lambda(1-\lambda)}\big(g^*(\lambda_0)+\varepsilon\big).
\end{aligned}$$

Consequently,

$$\limsup_{\lambda\to0+} \lambda g^*(\lambda) \le \lambda_0(1-\lambda_0)\big(g^*(\lambda_0)+\varepsilon\big).$$

Since the last inequality is valid for any $\lambda_0 \in (0,1)$ and $\varepsilon > 0$, we conclude that there exists $\lim\limits_{\lambda\to0+} \lambda g^*(\lambda)$ and, moreover, for any $\lambda_0 \in (0,1)$,

$$\lim_{\lambda\to0+} \lambda g^*(\lambda) \le \lambda_0(1-\lambda_0)g^*(\lambda_0).$$

This inequality together with (2.14) results in (2.12).

Analogously to the above we can show that $g_*(\lambda) \in C((0,1))$ and (2.13), (2.15) are fulfilled. □

Lemma 2.8. *Let conditions* (1.2), (1.7), (1.8), (1.10), (1.11) *be fulfilled and let system* (1.1) *have a proper nonoscillatory solution. Then for any* $\lambda \in (0,1)$,

$$g^*(\lambda) \le \min\left\{\frac{c_0}{\lambda} + \frac{\lambda}{4(1-\lambda)}\,,\ \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\right\}, \tag{2.20}$$

where $g^*(\lambda)$ *and* c_0 *are defined by* (1.4) - (1.6) *and* (1.13).

Proof. Let $(u_1(t), u_2(t))$ be a proper nonoscillatory solution of system (1.1). Then by Lemma 2.1 there exists $t_* \in R_+$ such that (2.1) is fulfilled. Without loss of generality we can assume that $u_1(t) > 0$ and $u_2(t) > 0$ for $t \ge t_*$. On the other hand, by Lemma 2.2 there exists $t_0 \ge t_*$ such that either (2.2) or (2.3) is satisfied.

Suppose (2.2) holds. Then, by virtue of (1.1), we have

$$\left(\frac{u_1(t)}{h(t)}\right)' = \frac{1}{h^2(t)}\Big(h(t)u_1'(t) - p(t)u_1(t)\Big) = \frac{p(t)}{h^2(t)}\Big(h(t)u_2(\sigma(t)) - u_1(t)\Big) \ge 0 \quad \text{for} \quad t \ge t_1\,,$$

where $t_1 > t_0$ is a sufficiently large number. Thus there exist $a > 0$ and $\tilde{t} \ge t_1$ such that

$$u_1(\tau(t)) \ge a\, h(\tau(t)) \quad \text{for} \quad t \ge \tilde{t}\,.$$

Due to the last inequality, from system (1.1) we find

$$u_2(\tilde{t}) \ge \int_{\tilde{t}}^{+\infty} q(s)u_1(\tau(s))\,ds \ge a \int_{\tilde{t}}^{+\infty} q(s)h(\tau(s))\,ds\,.$$

This contradicts (1.7). The contradiction obtained proves that inequality (2.3) holds.

According to (2.3),

$$\left(\frac{u_1(t)}{h(t)}\right)' \le 0 \quad \text{for} \quad t \ge t_1\,. \tag{2.21}$$

If $\tau(t) \ge \mu(t)$, then, due to the fact that $u_1(t)$ is nondecreasing, we have

$$u_1(\tau(t)) \ge u_1(\mu(t)) \quad \text{for} \quad t \ge t_2\,, \tag{2.22}$$

and if $\tau(t) < \mu(t)$, then by (2.21),

$$u_1(\tau(t)) \ge \frac{h(\tau(t))}{h(\mu(t))}\,u_1(\mu(t)) \quad \text{for} \quad t \ge t_2\,, \tag{2.23}$$

where $t_2 > t_1$ is a sufficiently large number.

In view of (2.22) and (2.23), we have

$$u_1(\tau(t)) \ge \rho(t)u_1(\mu(t)) \quad \text{for} \quad t \ge t_2\,,$$

where the function $\rho(t)$ is defined by (1.4). Therefore from system (1.1), we obtain

$$u_2'(t) \le -q(t)\rho(t)u_1(\mu(t)) \quad \text{for} \quad t \ge t_2\,. \tag{2.24}$$

First we show that for any $\lambda \in (0,1)$,

$$g^*(\lambda) \le \frac{c_0}{\lambda} + \frac{\lambda}{4(1-\lambda)}\,. \tag{2.25}$$

Below we will assume that $c_0 < +\infty$; otherwise the validity of (2.25) is obvious.

Let $\lambda \in (0,1)$. Multiplying both sides of inequality (2.24) by $h^\lambda(\mu(t))/u_1(\mu(t))$ and integrating from t to $+\infty$, we get

$$\int_t^{+\infty} \frac{h^\lambda(\mu(s))u_2'(s)}{u_1(\mu(s))}\,ds \le -\int_t^{+\infty} q(s)\rho(s)h^\lambda(\mu(s))\,ds \quad \text{for} \quad t \ge t_2\,. \tag{2.26}$$

On account of (1.11) we have

$$\int_t^{+\infty} \frac{h^\lambda(\mu(s))u_2'(s)}{u_1(\mu(s))}\,ds = \int_t^{+\infty} h^\lambda(\mu(s))\,d\,\frac{u_2(s)}{u_1(\mu(s))} + \int_t^{+\infty} h^\lambda(\mu(s))\,\frac{u_2(s)u_1'(\mu(s))\mu'(s)}{u_1^2(\mu(s))}\,ds$$

$$\geq -\,h^\lambda(\mu(t))\,\frac{u_2(t)}{u_1(\mu(t))} - \lambda\int_t^{+\infty} h^{\lambda-1}(\mu(s))p(\mu(s))\mu'(s)\frac{u_2(s)}{u_1(\mu(s))}ds$$

$$+\int_t^{+\infty} h^\lambda(\mu(s))\frac{p(\mu(s))\mu'(s)u_2^2(s)}{u_1^2(\mu(s))}ds$$

$$=\int_t^{+\infty}\left[\frac{u_2(s)}{u_1(\mu(s))}h^{\frac{\lambda}{2}}(\mu(s))p^{\frac{1}{2}}(\mu(s))(\mu'(s))^{\frac{1}{2}} - \frac{\lambda}{2}h^{\frac{\lambda}{2}-1}(\mu(s))p^{\frac{1}{2}}(\mu(s))(\mu'(s))^{\frac{1}{2}}\right]^2 ds$$

$$-\,h^\lambda(\mu(t))\,\frac{u_2(t)}{u_1(\mu(t))} - \frac{\lambda^2}{4(1-\lambda)}\,h^{\lambda-1}(\mu(t))\,.$$

Thus, in view of (2.26), we find

$$\int_t^{+\infty} q(s)\rho(s)h^\lambda(\mu(s))ds \leq h^{\lambda-1}(\mu(t))\biggl(\frac{h(\mu(t))u_2(t)}{u_1(\mu(t))} + \frac{\lambda^2}{4(1-\lambda)}\biggr) \quad \text{for} \quad t\geq t_2\,. \tag{2.27}$$

On the other hand, since $\mu_0(t)\leq t$, by Lemma 2.3 in [8], we have

$$\limsup_{t\to+\infty}\frac{h(\mu(t))u_2(t)}{u_1(\mu(t))}\leq c_0\,,$$

where c_0 and $\mu_0(t)$ are defined by (1.13). Therefore, according to (2.27), for any $\varepsilon>0$ there is $t^*>t_2$ such that

$$\int_t^{+\infty} q(s)\rho(s)h^\lambda(\mu(s))\,ds \leq h^{\lambda-1}(\mu(t))\Bigl(c_0+\varepsilon+\frac{\lambda^2}{4(1-\lambda)}\Bigr) \quad \text{for} \quad t\geq t^*\,.$$

Multiplying both sides of this inequality by $p(\mu(t))\mu'(t)$ and integrating from t^* to t, we get

$$\int_{t^*}^{t} p(\mu(s))\mu'(s)\int_s^{+\infty} q(\xi)\rho(\xi)h^\lambda(\mu(\xi))\,d\xi\,ds$$

$$\leq \Bigl(\frac{c_0+\varepsilon}{\lambda}+\frac{\lambda}{4(1-\lambda)}\Bigr)\bigl(h^\lambda(\mu(t))-h^\lambda(\mu(t^*))\bigr) \quad \text{for} \quad t\geq t^*\,.$$

If we multiply both sides of this inequality by $h^{-\lambda}(\mu(t))$ and pass to the limit as $t \to +\infty$, then, taking into account the arbitrariness of ε, we obtain

$$\limsup_{t\to+\infty} h^{-\lambda}(\mu(t)) \int_{t^*}^{t} p(\mu(s))\mu'(s) \int_{s}^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\, d\xi\, ds \le \frac{c_0}{\lambda} + \frac{\lambda}{4(1-\lambda)}\,.$$

Therefore (2.25) is valid.

Now we show that for any $\lambda \in (0,1)$,

$$g^*(\lambda) \le \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\,. \tag{2.28}$$

Indeed, let $\lambda \in (0,1)$. If we multiply both sides of (2.24) by $h^{1+\lambda}(\mu(t))/u_1(\mu(t))$ and integrate from t_2 to t, then we find

$$\int_{t_2}^{t} \frac{u_2'(s)h^{1+\lambda}(\mu(s))}{u_1(\mu(s))}\, ds \le -\int_{t_2}^{t} q(s)\rho(s)h^{1+\lambda}(\mu(s))\, ds \quad \text{for} \quad t \ge t_2\,. \tag{2.29}$$

Analogously to the above reasoning we can obtain the following estimate

$$\int_{t_2}^{t} \frac{u_2'(s)h^{1+\lambda}(\mu(s))}{u_1(\mu(s))}\, ds \ge -\frac{(1+\lambda)^2}{4\lambda}\, h^{\lambda}(\mu(t)) - c \quad \text{for} \quad t \ge t_2\,,$$

where $c = h^{1+\lambda}(\mu(t_2))u_2(t_2)/u_1(\mu(t_2))$. Thus from (2.29) we have

$$\int_{t_2}^{t} q(s)\rho(s)h^{1+\lambda}(\mu(s))\, ds \le \frac{(1+\lambda)^2}{4\lambda}\, h^{\lambda}(\mu(t)) + c \quad \text{for} \quad t \ge t_2\,.$$

Multiplying both sides of the last inequality by $h^{-2}(\mu(t))p(\mu(t))\mu'(t)$ and integrating from t to $+\infty$, we get

$$\int_{t}^{+\infty} h^{-2}(\mu(s))p(\mu(s))\mu'(s) \int_{t_2}^{s} q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\, d\xi\, ds$$
$$\le \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\, h^{\lambda-1}(\mu(t)) + c\, h^{-1}(\mu(t)) \quad \text{for} \quad t \ge t_2\,.$$

Now multiplying both sides of this inequality by $h^{1-\lambda}(\mu(t))$ and passing to the limit as $t \to +\infty$, we obtain

$$\limsup_{t\to+\infty} h^{1-\lambda}(\mu(t)) \int_{t}^{+\infty} h^{-2}(\mu(s))p(\mu(s))\mu'(s) \int_{t_2}^{s} q(\xi)\rho(\xi)h^{1+\lambda}(\mu(\xi))\, d\xi\, ds$$
$$\le \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\,.$$

This, taking into account Lemma 2.4 (formula (2.7)), evidently results in (2.28). On the other hand, (2.25) and (2.28) imply (2.20). $\square$

The following lemma is a special case of the Schauder–Tikhonoff theorem (see, e.g., [3, p. 227]).

Lemma 2.9. *Let $t_0 \in R$, V be a closed bounded convex subset of $C([t_0, +\infty); R)$, and let $T : V \to V$ be a continuous mapping such that the set $T(V)$ is equicontinuous on every finite subsegment of $[t_0, +\infty)$. Then T has a fixed point.*

3. Proof of the main results

Proof of Theorem 1.1. First we show that from (1.12) it follows (1.7). Assume the contrary. Suppose

$$\int^{+\infty} h(\tau(t))q(t)\,dt < +\infty\,. \tag{3.1}$$

Since $\rho(t)h(\mu(t)) \le h(\tau(t))$, (3.1) yields

$$\int^{+\infty} q(t)\rho(t)h(\mu(t))\,dt < +\infty\,.$$

Let ε be an arbitrary positive number. We choose $T > 0$ such that

$$\int_T^{+\infty} q(s)\rho(s)h(\mu(s))\,ds < \varepsilon\,.$$

This together with (1.5) implies

$$\begin{aligned} g(t,\lambda) &\le h^{-\lambda}(\mu(t)) \int_1^T p(\mu(s))\mu'(s) \int_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\;d\xi\,ds \\ &\quad + \varepsilon\, h^{-\lambda}(\mu(t)) \int_T^t p(\mu(s))\mu'(s)h^{\lambda-1}(\mu(s))\,ds \\ &\le h^{-\lambda}(\mu(t)) \int_1^T p(\mu(s))\mu'(s) \int_s^{+\infty} q(\xi)\rho(\xi)h^{\lambda}(\mu(\xi))\;d\xi\,ds + \frac{\varepsilon}{\lambda}\,. \end{aligned}$$

Hence, by virtue of (1.2) and (1.10), passing to the limit as $t \to +\infty$, we find

$$g^*(\lambda) = \limsup_{t\to+\infty} g(t,\lambda) \le \frac{\varepsilon}{\lambda}\,.$$

In view of the arbitrariness of ε we have $g^*(\lambda) = 0$, which contradicts (1.12). The contradiction obtained proves that condition (1.7) is satisfied.

Now we assume that the theorem is not valid. Suppose system (1.1) has a proper nonoscillatory solution. Then all the conditions of Lemma 2.8 are fulfilled. Therefore inequality (2.20) holds. But this contradicts condition (1.12). The contradiction obtained proves the validity of the theorem. □

Proof of Corollary 1.1. It suffices to note that since $\mu(t) \le t$, we have $c_0 \le 1$ (c_0 is defined by (1.13)), and therefore (1.14) implies (1.12). $\square$

Proof of Theorem 1.2. By virtue of (2.14) (see Lemma 2.7) and (1.15) there exists $\varepsilon > 0$ such that for any $\lambda \in (0,1)$,

$$g^*(\lambda) \ge \frac{1+\varepsilon}{4\lambda(1-\lambda)}\,. \tag{3.2}$$

We choose $\lambda \in (0,1)$ so that $(1+\lambda)^2 < 1+\varepsilon$. Then from (3.2) we have

$$g^*(\lambda) > \frac{(1+\lambda)^2}{4\lambda(1-\lambda)}\,,$$

which results in (1.12). Therefore all the conditions of Theorem 1.1 are fulfilled. Thus the theorem is proved. $\square$

Proof of Theorem 1.3. According to (1.16) and (2.15) (see Lemma 2.7), we have

$$\lim_{\lambda\to 1-}(1-\lambda)g^*(\lambda) \ge \lim_{\lambda\to 1-}(1-\lambda)g_*(\lambda) > \frac{1}{4}\,.$$

Therefore all the conditions of Theorem 1.2 are satisfied. Thus the theorem is proved. $\square$

To prove Corollaries 1.2 and 1.3, it suffices to note that the fulfilment of each of conditions (1.17) and (1.18) guarantees the fulfilment of condition (1.16).

Proof of Theorem 1.4. According to (1.10) and (1.19) it is clear that

$$\limsup_{t\to+\infty} h^{-\lambda}(t)\left(1+\int_0^t p(s)\int_{\sigma(s)}^{+\infty} q(\xi)h^{\lambda}(\tau(\xi))\,d\xi\,ds\right) < 1\,.$$

Thus there exists $t_0 \in R_+$ such that

$$1+\int_{t_0}^t p(s)\int_{\sigma(s)}^{+\infty} q(\xi)h^{\lambda}(\tau(\xi))\,d\xi\,ds < h^{\lambda}(t) \quad \text{for} \quad t \ge t_0\,. \tag{3.3}$$

Let V be the set of all $v \in C([\sigma(\tau(t_0)),+\infty);R)$ satisfying the conditions

$$v(t)=1 \quad \text{for} \quad t\in[\sigma(\tau(t_0)),t_0] \quad \text{and} \quad 1\le v(t)\le h^{\lambda}(t) \quad \text{for} \quad t \ge t_0{}^{3)}\,. \tag{3.4}$$

---

[3] Here t_0 is chosen so large that $h(t_0) \ge 1$.

Define

$$T(v)(t)=\begin{cases}1+\int\limits_{t_0}^{t}p(s)\int\limits_{\sigma(s)}^{+\infty}q(\xi)v(\tau(\xi))\,d\xi\,ds & \text{for}\quad t\geq t_0{}^{4)}\,,\\ 1 & \text{for}\quad t\in[\sigma(\tau(t_0)),t_0)\,.\end{cases}\tag{3.5}$$

On account of (3.3) and (3.4) it is evident that $T(V)\subset V$. Moreover, the set $T(V)$ is equicontinuous on every finite subsegment of $[\sigma(\tau(t_0)),+\infty)$. Since V is closed and convex, by Lemma 2.9 there exists $v_0\in V$ such that $v_0=T(v_0)$. According to (3.5) it is obvious that the vector function $(u_1(t),u_2(t))$, the components of which are defined by the equalities

$$u_1(t)=1+\int\limits_{t_0}^{t}p(s)\int\limits_{\sigma(s)}^{+\infty}q(\xi)v_0(\tau(\xi))\,d\xi\,ds,\quad u_2(t)=\int\limits_{t}^{+\infty}q(s)v_0(\tau(s))\,ds\,,$$

is a proper nonoscillatory solution of system (1.1). □

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4) It is assumed that $\int\limits_t^{+\infty}q(s)\,ds>0$ for every $t\in R_+$; otherwise system (1.1) obviously has a proper nonoscillatory solution.

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A. Razmadze Mathematical Institute of the Georgian Academy of Sciences
1 M. Aleksidze St., Tbilisi 0193, Georgia
E-mail: roman@rmi.acnet.ge

A. Razmadze Mathematical Institute of the Georgian Academy of Sciences
1 M. Aleksidze St., Tbilisi 0193, Georgia
E-mail: ninopa@rmi.acnet.ge

Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
E-mail: ipstav@cc.uoi.gr