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PROJECTIVE REPARAMETRIZATION OF HOMOGENEOUS CURVES

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Abstract. We study the conditions when locally homogeneous curves in homogeneous spaces admit a natural projective parameter. In particular, we prove that this is always the case for trajectories of homogeneous nilpotent elements in parabolic spaces. On algebraic level this corresponds to the generalization of Morozov–Jacobson theorem to graded semisimple Lie algebras.

1. Symmetry algebras of orbits of virtual subgroups

Let $M = G/G_0$ be a real smooth homogeneous space, $o = eG_0$ an origin in M and $(\mathfrak{g}, \mathfrak{g}_0)$ the corresponding pair of Lie algebras. In the sequel we always assume that the homogeneous space G/G_0 is *locally effective*, i.e. the set of all elements of G acting trivially on M forms a discrete subgroup of G.

We can identify $\mathfrak{g}$ with the Lie algebra of vector fields on M generated by actions of one-parameter subgroups in G. Then the subalgebra $\mathfrak{g}_0$ consists precisely of those elements of $\mathfrak{g}$ that vanish at the origin.

Let L be any submanifold in M. Then the symmetry algebra $\operatorname{sym}(L)$ is defined as the set of all vector fields of $\mathfrak{g}$ that are tangent to L:

$$\operatorname{sym}(L) = \{X \in \mathfrak{g} \mid X_p \in T_pL \text{ for all } p \in L\}\,.$$

We say that L is *locally homogeneous* if the restriction of $\operatorname{sym}(L)$ to L is transitive, i.e. if the space $\operatorname{sym}(L)_p = \{X_p \mid X \in \operatorname{sym}(L)\}$ coincides with T_pL for all $p \in L$. The class of connected locally homogeneous submanifolds coincides with open connected subsets of orbits of virtual subgroups in G [4].

Consider any virtual subgroup $H \subset G$ and its orbit $L = H.o$ through the origin. It is clear that L is a homogeneous submanifold in M, but its complete symmetry algebra can be larger then the subalgebra $\mathfrak{h}$ of the subgroup H. The following result from [3, 4] describes the complete symmetry algebra of L.

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Theorem 1. 1. *The subalgebra* $\mathrm{sym}(L)$ *is the largest among all subalgebras* $\mathfrak{a}$ *such that* $\mathfrak{h} \subset \mathfrak{a} \subset \mathfrak{h} + \mathfrak{g}_0$.

2. *Consider the decreasing sequence of subalgebras* $\mathfrak{a}_0 = \mathfrak{g}_0$, $\mathfrak{a}_{i+1} = \{x \in \mathfrak{a}_i \mid [x, \mathfrak{h}] \subset \mathfrak{h} + \mathfrak{a}_i\}$. *Let* r *be the smallest integer such that* $\mathfrak{a}_r = \mathfrak{a}_{r+1}$. *Then* $\mathrm{sym}(L) = \mathfrak{a}_r + \mathfrak{h}$.

Note that $\mathfrak{a}_r$ coincides with all vector fields in $\mathrm{sym}(L)$ that vanish at the origin. Hence, the pair of Lie algebras $(\mathfrak{a}_r + \mathfrak{h}, \mathfrak{a}_r)$ determines the action of $\mathrm{sym}(L)$ on L up to local equivalence. In particular, $\dim(\mathfrak{a}_r + \mathfrak{h})/\mathfrak{a}_r = \dim L$. However, the pair $(\mathfrak{a}_r + \mathfrak{h}, \mathfrak{a}_r)$ in general does not need to be effective. Its *non-effectiveness ideal*, i.e. the maximal ideal of $\mathfrak{a}_r + \mathfrak{h}$ lying in $\mathfrak{a}_r$, consists of all vector fields of $\mathfrak{g}$ that vanish at all points of L.

2. Parameterization of homogeneous curves

Let, as above, $M = G/G_0$ be a homogeneous space of the Lie group G and let L be a trajectory of some one-parameter transformation group $\exp(tx) \subset G$ generated by a *nilpotent* element in $x \in \mathfrak{g}$ (i.e., the endomorphism $\mathrm{ad}_{\mathfrak{g}}\, x$ is nilpotent). We will assume that L contains the origin o, that is $L = \{\exp(tx)G_0\}$. In the previous section we have shown how to compute the complete symmetry algebra of L. The aim of this section is to determine when the restriction of $\mathrm{sym}(L)$ to L gives us the three-dimensional Lie algebra, which is locally equivalent to the algebra of all projective transformations of the line.

As above, let $\mathfrak{h}$ be a one-dimensional subalgebra generated by the element $x \in \mathfrak{g}$ and let $\{\mathfrak{a}_i\}$ be the decreasing sequence of subalgebras constructed in Theorem 1. Then the pair $(\mathfrak{h} + \mathfrak{a}_r, \mathfrak{a}_r)$ has codimension one. Denote by $\mathfrak{m}$ its ideal of non-effectiveness. Then the pair $((\mathfrak{h} + \mathfrak{a}_r)/\mathfrak{m}, \mathfrak{a}_r/\mathfrak{m})$ corresponds to some one-dimensional real homogeneous space, and it is well-known (see, for example the original Sophus Lie proof [8] or its modern versions in [6, 5]) that this pair is isomorphic to one of the following:

1. $(\mathbb{R}, \{0\})$;
2. $(\langle h, e\rangle, \langle h\rangle)$, where $[h, e] = e$;
3. $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{st}_2(\mathbb{R}))$, where $\mathfrak{st}_2(\mathbb{R})$ is the subalgebra consisting of all upper triangular matrices in $\mathfrak{sl}_2(\mathbb{R})$.

In the first two cases we shall say that the homogeneous curve L admits an *affine reparametrization*, while in the the third case we shall say that it admits a *projective reparametrization* (see [5]).

The main result of the paper connects the existence of projective parameterizations with the notion of $\mathfrak{sl}_2$-triples in Lie algebras. Recall that an $\mathfrak{sl}_2$-*triple* is the triple of elements $\{x, h, y\}$ in a Lie algebra $\mathfrak{g}$ forming a canonical basis of a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ (i.e., $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$).

Theorem 2. *Let* x *be an arbitrary element of the Lie algebra* $\mathfrak{g}$ *not contained in* $\mathfrak{g}_0$. *Suppose there exists an element* z *in* $\mathfrak{g}_0$ *such that* $[x, z] \in \mathfrak{g}_0$ *and* $[x, [x, z]] = x$. *Then*

(i) *there exist elements* $h, y \in \mathfrak{g}_0$ *that complete* x *to* $\mathfrak{sl}_2$*-triple;*

(ii) *the trajectory* $L = \exp(tx).o$ *admits a projective reparametrization.*

Proof. Let $h = -2[x, z]$. Then we have $[h, x] = 2x$. Hence, the subalgebra $\mathfrak{g}' = \langle x, h\rangle$ is solvable and $x \in [\mathfrak{g}', \mathfrak{g}']$. From Lie theorem applied to the $\mathfrak{g}'$-module $\mathfrak{g}$ we see that the element x is nilpotent.

Denote by y' the element $-2z$, so that we have $[x, y'] = h$. Consider the kernel $\mathfrak{n}$ of $\operatorname{ad}_{\mathfrak{g}} x$ and its subspace $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$. Let us show that $[h, y'] + 2y' \in \mathfrak{n}_0$. Indeed, we have

$$[x, [h, y'] + 2y'] = [[x, h], y'] + [h, [x, y']] + 2[x, y'] = -2[x, y'] + 2[x, y'] = 0\,,$$

so that $[h, y'] + 2y' \in \mathfrak{n}$. On the other hand, it is clear, that $[h, y']$ and y' lie in $\mathfrak{g}_0$.

Next, since $[\operatorname{ad}_{\mathfrak{g}} h, \operatorname{ad}_{\mathfrak{g}} x] = 2 \operatorname{ad}_{\mathfrak{g}} x$, we see that both subspaces $\mathfrak{n}$ and $\mathfrak{n}_0$ are stable with respect to $\operatorname{ad}_{\mathfrak{g}} h$. Let us show that the restriction of $\operatorname{ad}_{\mathfrak{g}} h + 2$ to $\mathfrak{n}_0$ is nondegenerate.

Let $M_n = (\operatorname{ad}_{\mathfrak{g}} x)^n \mathfrak{g}$ for any $n \geq 0$. As in [1, Ch. VIII, §11, Lemma 6], we see that for any $n > 0$

$$[\operatorname{ad}_{\mathfrak{g}} y', (\operatorname{ad}_{\mathfrak{g}} x)^n] = n((\operatorname{ad}_{\mathfrak{g}} h) - n + 1)(\operatorname{ad}_{\mathfrak{g}} x)^{n-1}.$$

Hence, for any $u \in M_{n-1}$ we have

$$n((\operatorname{ad}_{\mathfrak{g}} h) - n + 1)u \in (\operatorname{ad}_{\mathfrak{g}} y')(\operatorname{ad}_{\mathfrak{g}} x)u + M_n\,.$$

Since $\mathfrak{n}$ is stable with respect to $\operatorname{ad}_{\mathfrak{g}} h$, we see that

$$((\operatorname{ad}_{\mathfrak{g}} h) - n + 1)(\mathfrak{n} \cap M_{n-1}) \subset \mathfrak{n} \cap M_n\,.$$

Since the endomorphism $\operatorname{ad}_{\mathfrak{g}} x$ is nilpotent, we have $M_n = \{0\}$ for sufficiently large n. Hence, all eigenvalues of $\operatorname{ad}_{\mathfrak{g}} h|_{\mathfrak{n}}$ and $\operatorname{ad}_{\mathfrak{g}} h|_{\mathfrak{n}_0}$ are integral and non-negative. Therefore, the restriction of $\operatorname{ad}_{\mathfrak{g}} h + 2$ to $\mathfrak{n}_0$ is non-degenerate. In particular, there exists an element $y'' \in \mathfrak{n}_0$ such that

$$[h, y'] + 2y' = [h, y''] + 2y''\,.$$

Let $y = y' - y''$. Then $y \in \mathfrak{g}_0$, $[h, y] = -2y$ and $[x, y] = [x, y'] = h$. This completes the proof of item (i) of the theorem.

Let $\{x, h, y\}$ be an $\mathfrak{sl}_2$-triple existing by (i). As above, denote by $\{\mathfrak{a}_i\}$ the decreasing sequence of subalgebras constructed by the subalgebra $\mathfrak{h} = \langle x\rangle$ in $\mathfrak{g}$. Let us prove by induction that h and y lie in $\mathfrak{a}_i$ for all $i \geq 0$. For $i = 0$ this is a part of the theorem assumption. Since $[x, h] = -2x \in \mathfrak{h} \subset \mathfrak{h} + \mathfrak{a}_i$ then by definition of $\mathfrak{a}_{i+1}$ we have $h \in \mathfrak{a}_{i+1}$. Similarly, since $h \in \mathfrak{a}_i$ and $[x, y] = h \in \mathfrak{a}_i \subset \mathfrak{h} + \mathfrak{a}_i$, we see that $y \in \mathfrak{a}_{i+1}$.

This implies that h and y lie also in the symmetry algebra $\mathfrak{s} = \operatorname{sym}(L)$ of L. Let $\mathfrak{s}_0$ be the subalgebra of all vector fields in $\mathfrak{s}$ vanishing at the origin and let $\mathfrak{m}$ be the non-effectiveness ideal of the pair $(\mathfrak{s}, \mathfrak{s}_0)$. Then the pair $(\mathfrak{s}/\mathfrak{m}, \mathfrak{s}_0/\mathfrak{m})$ is effective and is isomorphic to one of the three pairs of codimension one listed above.

Let us prove that $(\operatorname{ad}_{\mathfrak{g}} x)^2_{\mathfrak{s}/\mathfrak{m}} \neq 0$, which is only possible if $\mathfrak{s}/\mathfrak{m}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Indeed, we have

$$(\operatorname{ad}_{\mathfrak{g}} x)^2_{\mathfrak{s}/\mathfrak{m}}(y + \mathfrak{m}) = -2x + \mathfrak{m} \neq 0\,,$$

which completes the proof. □

Example 1. Let us show that the statement (ii) of Theorem 2 can not be inverted. Indeed, consider the homogeneous space $M = \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ with respect to the action of $G = PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. Then we have $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R})$, where $\mathfrak{st}(2,\mathbb{R})$ is a subalgebra of all upper triangular matrices in $\mathfrak{sl}(2,\mathbb{R})$.

Consider the nilpotent element $x = (\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)) \in \mathfrak{g}$. Then the trajectory of $\exp(tx)$ through the point $(o,o) \in M$, $o = [1:0] \in \mathbb{R}\mathrm{P}^1$, coincides with $\mathbb{R}\mathrm{P}^1 \times \{o\}$ and has the symmetry algebra equal to $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R})$. It is clear that it admits a projective reparametrization. Yet it is easy to see that x does not lie in $(\operatorname{ad}_{\mathfrak{g}} x)^2\mathfrak{g}_0$ and, hence, there is no such element $z \in \mathfrak{g}_0$ that $[x,[x,z]] = x$.

The well-known Jacobson-Morozov theorem [7, Ch. III, Th. 17] states that any nilpotent element x in a semisimple Lie algebra over the field of zero characteristic can be included into an $\mathfrak{sl}_2$-triple $\{x,h,y\}$. This allows us to prove the following general result concerning the reparametrizations of distinguished curves in the parabolic geometries (cf. [2]).

Theorem 3. *Let $M = G/P$ be a parabolic homogeneous space corresponding to the real or complex graded semisimple Lie algebra $\mathfrak{g} = \sum_{j=-k}^{k} \mathfrak{g}_j$. Suppose x is an arbitrary nonzero element in $\mathfrak{g}_{-i}$, $i > 0$. Then*

(i) *there exist $h \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_i$ such that $\{x,h,y\}$ is an $\mathfrak{sl}_2$-triple;*
(ii) *the trajectory $\exp(tx).o$ admits a projective reparametrization.*

Remark. Item (i) of this theorem for the field of complex numbers is proved in [9]. Our proof holds true for any semisimple graded Lie algebra over any field of zero characteristic.

Proof. Indeed, x is a nilpotent element in the semisimple Lie algebra $\mathfrak{g}$. Hence, it can be included into some $\mathfrak{sl}_2$-triple $\{x,h',y'\}$, where each of the elements h' and y' decomposes into the sum of homogeneous elements: $h' = \sum_j h'_j$, $y' = \sum_j y'_j$ ($h'_j, y'_j \in \mathfrak{g}_j$). Then the elements $h = h'_0$ and $y'' = y'_i$ satisfy the relations $[h,x] = 2x$ and $[x,y''] = h$.

Similarly to the proof of Theorem 2, we can consider the kernel $\mathfrak{n}$ of $\operatorname{ad}_{\mathfrak{g}} x$ and prove that the restriction of $\operatorname{ad}_{\mathfrak{g}} h + 2$ on $\mathfrak{n}$ is nondegenerate. We can decompose $\mathfrak{n}$ as $\mathfrak{n} = \sum_j \mathfrak{n}_j$, $\mathfrak{n}_j = \mathfrak{n} \cap \mathfrak{g}_i$. Then since $h \in \mathfrak{g}_0$ and $[h,x] = 2x$, we see that each subspace $\mathfrak{n}_j$ is stable with respect to $\operatorname{ad}_{\mathfrak{g}} h$. Again, as in the proof of Theorem 2 we see that $[h,y''] + 2y''$ lies in $\mathfrak{n}_i$. Hence, there exists an element $y''' \in \mathfrak{n}_i$ such that

$$[h,y''] + 2y'' = [h,y'''] + 2y'''.$$

Thus, the element $y = y'' - y'''$ lies in $\mathfrak{g}_i$ and, together with x and h, forms an $\mathfrak{sl}_2$-triple.

The second part of the theorem follows immediately from Theorem 2(ii). □

Example 2. Consider the flag manifold $M = F_{1,2}(\mathbb{R}^3)$ of all flags $\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3$, $\dim V_i = i$, which is a homogeneous space with respect to the natural action of the Lie group $G = SL(3,\mathbb{R})$. Then M is a parabolic homogeneous space, and $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R}) = \sum_{j=-2}^{2} \mathfrak{g}_j$ is a graded Lie algebra, such that the subalgebra $\sum_{j\geq 0} \mathfrak{g}_j$ coincides with $\mathfrak{st}(3,\mathbb{R})$. Take $x = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, which is a nilpotent element in $\mathfrak{g}$, but is not homogeneous with respect to this grading. The direct computation shows that in this case $\mathfrak{a}_3 = \{0\}$ and the symmetry algebra of the trajectory $\exp(tx).o$ coincides with $\mathbb{R}x$. Hence, this trajectory does not admit a projective reparametrization. So, we see that the homogeneity condition of the nilpotent element x can not be dropped in Theorem 3.

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