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FROM EULER-LAGRANGE EQUATIONS TO CANONICAL NONLINEAR CONNECTIONS

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Abstract. The aim of this paper is to construct a canonical nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on the 1-jet space $J^1(T,M)$ from the Euler-Lagrange equations of the quadratic multi-time Lagrangian function

$$L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x)\,.$$

1. Kronecker h-regularity

We start our study considering a smooth multi-time Lagrangian function $L : E \to \mathbb{R}$, expressed locally by

$$E \ni (t^\alpha, x^i, x^i_\alpha) \to L(t^\alpha, x^i, x^i_\alpha) \in \mathbb{R}\,, \tag{1.1}$$

whose *fundamental vertical metrical d-tensor* is defined by

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}\,. \tag{1.2}$$

In the sequel, let us fix $h = (h_{\alpha\beta})$ a semi-Riemannian metric on the temporal manifold T and let $g_{ij}(t^\gamma, x^k, x^k_\gamma)$ be a symmetric d-tensor on $E = J^1(T,M)$, of rank n and having a constant signature.

Definition 1.1. A multi-time Lagrangian function $L : E \to \mathbb{R}$, having the fundamental vertical metrical d-tensor of the form

$$G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x^k_\gamma)\,, \tag{1.3}$$

is called a **Kronecker h-regular multi-time Lagrangian function**.

In this context, we can introduce the following important concept:

Definition 1.2. A pair $ML^n_p = (J^1(T,M), L)$, $p = \dim T$, $n = \dim M$, consisting of the 1-jet fibre bundle and a Kronecker h-regular multi-time Lagrangian function $L : J^1(T,M) \to \mathbb{R}$, is called a **multi-time Lagrange space**.

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Remark 1.3. i) In the particular case $(T,h) = (\mathbb{R},\delta)$, a multi-time Lagrange space is called a **relativistic rheonomic Lagrange space** and is denoted by

$$RL^n = \left(J^1(\mathbb{R},M), L\right).$$

For more details about the relativistic rheonomic Lagrangian geometry, the reader may consult [14].

ii) If the temporal manifold T is 1-dimensional one, then, via o temporal reparametrization, we have

$$J^1(T,M) \equiv J^1(\mathbb{R},M).$$

In other words, a multi-time Lagrange space, having $\dim T = 1$, is a reparametrized relativistic rheonomic *Lagrange space*.

Example 1.4. Let us suppose that the spatial manifold M is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the multi-time Lagrangian function

$$L_1 : E \to \mathbb{R}, \quad L_1 = h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta \tag{1.4}$$

is a Kronecker h-regular one. It follows that the pair

$$\mathcal{BSM}L^n_p = \left(J^1(T,M), L_1\right)$$

is a multi-time Lagrange space. It is important to note that the multi-time Lagrangian $\mathcal{L}_1 = L_1\sqrt{|h|}$ is exactly the "energy" Lagrangian, whose extremals are the harmonic maps between the semi-Riemannian manifolds (T,h) and (M,g) [4]. At the same time, the multi-time Lagrangian that governs the physical theory of bosonic strings is of kind of the Lagrangian $\mathcal{L}_1$ [6].

Example 1.5. In the above notations, taking $U^{(\alpha)}_{(i)}(t,x)$ a d-tensor field on E and $F : T\times M \to \mathbb{R}$ a smooth function, the more general multi-time Lagrangian function

$$L_2 : E \to \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x), \tag{1.5}$$

is also a Kronecker h-regular one. The multi-time Lagrange space

$$\mathcal{EDM}L^n_p = \left(J^1(T,M), L_2\right)$$

is called the **autonomous multi-time Lagrange space of electrodynamics**. This is because, in the particular case $(T,h) = (\mathbb{R},\delta)$, the space $\mathcal{EDM}L^n_1$ naturally generalizes the clasical Lagrange space of electrodynamics [10], that governs the movement law of a particle placed concomitently into a gravitational field and an electromagnetic one. In a such context, from a physical point of view, the semi-Riemannian metric $h_{\alpha\beta}(t)$ (resp. $g_{ij}(x)$) represents the **gravitational potentials** of the manifold T (resp. M), the d-tensor $U^{(\alpha)}_{(i)}(t,x)$ play the role of the **electromagnetic potentials**, and F is a **potential function**. The non-dynamical character of the spatial gravitational potentials $g_{ij}(x)$ motivates us to use the term *"autonomous"*.

Example 1.6. More general, if we consider the symmetrical d-tensor $g_{ij}(t,x)$ on E, of rank n and having a constant signature on E, we can define the Kronecker h-regular multi-time Lagrangian function

$$L_3 : E \to \mathbb{R}\,, \quad L_3 = h^{\alpha\beta}(t) g_{ij}(t,x) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x) x^i_\alpha + F(t,x)\,. \tag{1.6}$$

The multi-time Lagrange space

$$\mathcal{NEDML}^n_p = \left(J^1(T,M), L_3\right)$$

is called the **non-autonomous multi-time Lagrange space of electrodynamics**. From a physical point of view, we remark that the spatial gravitational potentials $g_{ij}(t,x)$ are dependent of the temporal coordinates t^γ. For that reason, we use the term "non-autonomous", in order to emphasize the dynamical character of $g_{ij}(t,x)$.

2. The characterization theorem of multi-time Lagrange spaces

An important role and, at the same time, an obstruction in the subsequent development of the theory of the multi-time Lagrange spaces, is played by

Theorem 2.1 (of characterization of multi-time Lagrange spaces). *If $p = \dim T \geq 2$, then the following statements are equivalent:*

i) *L is a Kronecker h-regular Lagrangian function on $J^1(T,M)$.*

ii) *The multi-time Lagrangian function L reduces to a multi-time Lagrangian function of non-autonomous electrodynamic kind, that is*

$$L = h^{\alpha\beta}(t) g_{ij}(t,x) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x) x^i_\alpha + F(t,x)\,.$$

Proof 1. ii) $\Rightarrow$ i) It is obvious.

i)$\Rightarrow$ ii) Let us suppose that L is a Kronecker h-regular multi-time Lagrangian function, that is

$$\frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k, x^k_\gamma)\,.$$

For the beginning, let us suppose that there are two distinct indices α and β from the set $\{1,\ldots,p\}$, such that $h^{\alpha\beta} \neq 0$. Let k (resp. γ) be an arbitrary element of the set $\{1,\ldots,n\}$ (resp. $\{1,\ldots,p\}$). Deriving the above relation with respect to the variable x^k_γ and using the Schwartz theorem, we obtain the equalities

$$\frac{\partial g_{ij}}{\partial x^k_\gamma} h^{\alpha\beta} = \frac{\partial g_{jk}}{\partial x^i_\alpha} h^{\beta\gamma} = \frac{\partial g_{ik}}{\partial x^j_\beta} h^{\gamma\alpha}, \quad \forall\, \alpha,\beta,\gamma \in \{1,\ldots,p\}\,, \quad \forall\, i,j,k \in \{1,\ldots,n\}\,.$$

Contracting now with $h_{\gamma\mu}$, we deduce

$$\frac{\partial g_{ij}}{\partial x^k_\gamma} h^{\alpha\beta} h_{\gamma\mu} = 0\,, \quad \forall\, \mu \in \{1,\ldots,p\}\,.$$

In these conditions, the supposing $h^{\alpha\beta} \neq 0$ implies that $\dfrac{\partial g_{ij}}{\partial x^k_\gamma} = 0$ for all two arbitrary indices k and γ. Consequently, we have $g_{ij} = g_{ij}(t^\mu, x^m)$.

Supposing now that $h^{\alpha\beta} = 0$, $\forall\ \alpha \neq \beta \in \{1, \ldots, p\}$, it follows that we have $h^{\alpha\beta} = h^{\alpha}\delta^{\alpha}_{\beta}$, $\forall\ \alpha, \beta \in \{1, \ldots, p\}$. In other words, we use an ortogonal system of coordinates on the manifold T. In these conditions, the relations

$$\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = 0\,, \quad \forall\ \alpha \neq \beta \in \{1, \ldots, p\}, \quad \forall\ i, j \in \{1, \ldots, n\}\,,$$

$$\frac{1}{2h^{\alpha}(t)} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\alpha} = g_{ij}(t^{\mu}, x^m, x^m_{\mu})\,, \quad \forall\ \alpha \in \{1, \ldots, p\}\,, \quad \forall\ i, j \in \{1, \ldots, n\}$$

hold good. If we fix now an indice α in the set $\{1, \ldots, p\}$, from the first relation we deduce that the local functions $\dfrac{\partial L}{\partial x^i_\alpha}$ depend only by the coordinates $(t^{\mu}, x^m, x^m_{\alpha})$. Considering $\beta \neq \alpha$ in the set $\{1, \ldots, p\}$, the second relation implies

$$\frac{1}{2h^{\alpha}(t)} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\alpha} = \frac{1}{2h^{\beta}(t)} \frac{\partial^2 L}{\partial x^i_\beta \partial x^j_\beta} = g_{ij}(t^{\mu}, x^m, x^m_{\mu})\,, \quad \forall\ i, j \in \{1, \ldots, n\}\,.$$

Because the first term of the above equality depends by $(t^{\mu}, x^m, x^m_{\alpha})$, while the second term is dependent only by the coordinates $(t^{\mu}, x^m, x^m_{\beta})$, and because we have $\alpha \neq \beta$, we conclude that $g_{ij} = g_{ij}(t^{\mu}, x^m)$.

Finally, the equality

$$\frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = h^{\alpha\beta}(t^{\gamma}) g_{ij}(t^{\gamma}, x^k)\,, \quad \forall\ \alpha, \beta \in \{1, \ldots, p\}, \quad \forall\ i, j \in \{1, \ldots, n\}$$

implies without difficulties that the multi-time Lagrangian function L is one of non-autonomous electrodynamic kind. □

Corollary 2.2. *The fundamental vertical metrical d-tensor of an arbitrary Kronecker h-regular multi-time Lagrangian function L is of the form*

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \begin{cases} h^{11}(t) g_{ij}(t, x^k, y^k), & p = \dim T = 1 \\ h^{\alpha\beta}(t^{\gamma}) g_{ij}(t^{\gamma}, x^k), & p = \dim T \geq 2. \end{cases} \tag{2.1}$$

Remark 2.3. i) It is obvious that the preceding theorem is an obstruction in the development of a fertile geometrical theory for the multi-time Lagrange spaces. This obstruction will be surpassed in the paper [12], when we will introduce the more general notion of a **generalized multi-time Lagrange space**. The generalized multi-time Riemann-Lagrange geometry on $J^1(T, M)$ will be constructed using only a Kronecker h-regular vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ and a nonlinear connection Γ, *"a priori"* given on the 1-jet space $J^1(T, M)$.

ii) In the case $p = \dim T \geq 2$, the preceding theorem obliges us to continue our geometrical study of the multi-time Lagrange spaces, sewering our attention upon the non-autonomous multi-time Lagrange spaces of electrodynamics.

3. Canonical nonlinear connection Γ

Let $ML_p^n = (J^1(T,M),L)$, where $\dim T = p$, $\dim M = n$, be a multi-time Lagrange space whose fundamental vertical metrical d-tensor metric is

$$G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2}\frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t)g_{ij}(t,x^k,y^k), & p=1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma,x^k), & p\geq 2. \end{cases}$$

Supposing that the semi-Riemannian temporal manifold (T,h) is compact and orientable, by integration on the manifold T, we can define the *energy functional* associated to the multi-time Lagrange function L, taking

$$\mathcal{E}_L : C^\infty(T,M) \to \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x_\alpha^i)\sqrt{|h|}\, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p\,,$$

where the smooth map f is locally expressed by $(t^\alpha) \to (x^i(t^\alpha))$ and $x_\alpha^i = \dfrac{\partial x^i}{\partial t^\alpha}$.

It is obvious that, for each index $i \in \{1,2,\ldots,n\}$, the extremals of the energy functional $\mathcal{E}_L$ verify the Euler-Lagrange equations

$$2G_{(i)(j)}^{(\alpha)(\beta)} x_{\alpha\beta}^j + \frac{\partial^2 L}{\partial x^j \partial x_\alpha^i} x_\alpha^j - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x_\alpha^i} + \frac{\partial L}{\partial x_\alpha^i} H_{\alpha\gamma}^\gamma = 0\,, \tag{3.1}$$

where $x_{\alpha\beta}^j = \dfrac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}$ and $H_{\alpha\beta}^\gamma$ are the Christoffel symbols of the semi-Riemannian temporal metric $h_{\alpha\beta}$.

Taking into account the Kronecker h-regularity of the Lagrangian function L, it is possible to rearrange the Euler-Lagrange equations of the Lagrangian $\mathcal{L} = L\sqrt{|h|}$ in the following *generalized Poisson form*:

$$\Delta_h x^k + 2\mathcal{G}^k(t^\mu, x^m, x_\mu^m) = 0\,, \tag{3.2}$$

where

$$\Delta_h x^k = h^{\alpha\beta}\{x_{\alpha\beta}^k - H_{\alpha\beta}^\gamma x_\gamma^k\}\,,$$

$$2\mathcal{G}^k = \frac{g^{ki}}{2}\Big\{\frac{\partial^2 L}{\partial x^j \partial x_\alpha^i} x_\alpha^j - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x_\alpha^i} + \frac{\partial L}{\partial x_\alpha^i} H_{\alpha\gamma}^\gamma + 2g_{ij}h^{\alpha\beta}H_{\alpha\beta}^\gamma x_\gamma^j\Big\}.$$

Proposition 3.1. i) *The geometrical object $\mathcal{G} = (\mathcal{G}^r)$ is a multi-time dependent spatial h-spray.*

ii) *Moreover, the spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ is the h-trace of a multi-time dependent spatial spray $G = (G_{(\alpha)\beta}^{(i)})$, that is $\mathcal{G}^l = h^{\alpha\beta}G_{(\alpha)\beta}^{(l)}$.*

Proof 2. i) By a direct calculation, we deduce the local geometrical entities

$$\begin{aligned} 2\mathcal{S}^k &= \frac{g^{ki}}{2}\Big\{\frac{\partial^2 L}{\partial x^j \partial x_\alpha^i} x_\alpha^j - \frac{\partial L}{\partial x^i}\Big\}\,, \\ 2\mathcal{H}^k &= \frac{g^{ki}}{2}\Big\{\frac{\partial^2 L}{\partial t^\alpha \partial x_\alpha^i} + \frac{\partial L}{\partial x_\alpha^i} H_{\alpha\gamma}^\gamma\Big\}\,, \\ 2\mathcal{J}^k &= h^{\alpha\beta}H_{\alpha\beta}^\gamma x_\gamma^j\,, \end{aligned} \tag{3.3}$$

verify the following transformation laws:

$$
\begin{aligned}
2\mathcal{S}^p &= 2\tilde{\mathcal{S}}^r \frac{\partial x^p}{\partial \tilde{x}^r} + h^{\alpha\mu}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{\partial \tilde{t}^\gamma}{\partial t^\mu}\frac{\partial \tilde{x}^l_\gamma}{\partial x^j}x^j_\alpha\,,\\
2\mathcal{H}^p &= 2\tilde{\mathcal{H}}^r \frac{\partial x^p}{\partial \tilde{x}^r} + h^{\alpha\mu}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{\partial \tilde{t}^\gamma}{\partial t^\mu}\frac{\partial \tilde{x}^l_\gamma}{\partial t^\alpha}\,,\\
2\mathcal{J}^p &= 2\tilde{\mathcal{J}}^r \frac{\partial x^p}{\partial \tilde{x}^r} - h^{\alpha\mu}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{\partial \tilde{t}^\gamma}{\partial t^\mu}\frac{\partial \tilde{x}^l_\gamma}{\partial t^\alpha}\,.
\end{aligned}
\tag{3.4}
$$

It follows that the local entities $2\mathcal{G}^p = 2\mathcal{S}^p + 2\mathcal{H}^p + 2\mathcal{J}^p$ modify by the transformation laws

$$
2\tilde{\mathcal{G}}^r = 2\mathcal{G}^p \frac{\partial \tilde{x}^r}{\partial x^p} - h^{\alpha\mu}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^r_\mu}{\partial x^p}\tilde{x}^j_\alpha\,,
\tag{3.5}
$$

that is what we were looking for.

ii) In the particular case $\dim T = 1$, any spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ is the h-trace of a spatial spray $G = (G^{(l)}_{(1)1})$, where $G^{(l)}_{(1)1} = h_{11}\mathcal{G}^l$. In other words, the equality $\mathcal{G}^l = h^{11}G^{(l)}_{(1)1}$ is true.

On the other hand, in the case $\dim T \geq 2$, the Theorem of characterization of the Kronecker h-regular Lagrangian functions ensures us that

$$
L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x)\,.
$$

In this particular situation, by computations, the expressions of the entities $\mathcal{S}^l$, $\mathcal{H}^l$ and $\mathcal{J}^l$ reduce to

$$
\begin{aligned}
2\mathcal{S}^l &= h^{\alpha\beta}\Gamma^l_{jk}x^j_\alpha x^k_\beta + \frac{g^{li}}{2}\left[U^{(\alpha)}_{(i)j}x^j_\alpha - \frac{\partial F}{\partial x^i}\right],\\
2\mathcal{H}^l &= -h^{\alpha\beta}H^\gamma_{\alpha\beta}x^l_\gamma + \frac{g^{li}}{2}\left[2h^{\alpha\beta}\frac{\partial g_{ij}}{\partial t^\alpha}x^j_\beta + \frac{\partial U^{(\alpha)}_{(i)}}{\partial t^\alpha} + U^{(\alpha)}_{(i)}H^\gamma_{\alpha\gamma}\right],\\
2\mathcal{J}^l &= h^{\alpha\beta}H^\gamma_{\alpha\beta}x^l_\gamma\,,
\end{aligned}
\tag{3.6}
$$

where

$$
\Gamma^l_{jk} = \frac{g^{li}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)
$$

are the *generalized Christoffel symbols of the multi-time dependent metric* g_{ij} and

$$
U^{(\alpha)}_{(i)j} = \frac{\partial U^{(\alpha)}_{(i)}}{\partial x^j} - \frac{\partial U^{(\alpha)}_{(j)}}{\partial x^i}\,.
$$

Consequently, the expression of the spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ becomes

$$
2\mathcal{G}^p = 2\mathcal{S}^p + 2\mathcal{H}^p + 2\mathcal{J}^p = h^{\alpha\beta}\Gamma^l_{jk}x^j_\alpha x^k_\beta + 2\mathcal{T}^l\,,
\tag{3.7}
$$

where the local components

$$
2\mathcal{T}^l = \frac{g^{li}}{2}\left[2h^{\alpha\beta}\frac{\partial g_{ij}}{\partial t^\alpha}x^j_\beta + U^{(\alpha)}_{(i)j}x^j_\alpha + \frac{\partial U^{(\alpha)}_{(i)}}{\partial t^\alpha} + U^{(\alpha)}_{(i)}H^\gamma_{\alpha\gamma} - \frac{\partial F}{\partial x^i}\right]
\tag{3.8}
$$

represent the components of a tensor d-field $\mathcal{T} = (\mathcal{T}^l)$ on $J^1(T, M)$. It follows that the d-tensor $\mathcal{T}$ can be written as the h-trace of the d-tensor

$$T^{(l)}_{(\alpha)\beta} = \frac{h_{\alpha\beta}}{p}\mathcal{T}^l ,$$

where $p = \dim T$. In other words, the relation $\mathcal{T}^l = h^{\alpha\beta}T^{(l)}_{(\alpha)\beta}$ is true. Obviously, this writing is not unique one but represents a natural extension of the case $\dim T = 1$.

Finally, we can conclude that the spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ is the h-trace of the spatial spray

$$G^{(l)}_{(\alpha)\beta} = \frac{1}{2}\Gamma^l_{jk}x^j_\alpha x^k_\beta + T^{(l)}_{(\alpha)\beta} , \tag{3.9}$$

that is the relation $\mathcal{G}^l = h^{\alpha\beta}G^{(l)}_{(\alpha)\beta}$ holds good. □

Following previous reasonings and the preceding result, we can regard the equations (3.2) as being the equations of the harmonic maps of a multi-time dependent spray.

Theorem 3.2. *The extremals of the energy functional $\mathcal{E}_L$ attached to the Kronecker h-regular Lagrangian function L are harmonic maps on $J^1(T, M)$ of the multi-time dependent spray (H, G) defined by the temporal components*

$$H^{(i)}_{(\alpha)\beta} = \begin{cases} -\dfrac{1}{2}H^1_{11}(t)y^i , & p = 1 \\ -\dfrac{1}{2}H^\gamma_{\alpha\beta}x^i_\gamma , & p \geq 2 \end{cases}$$

and the local spatial components $G^{(i)}_{(\alpha)\beta} =$

$$= \begin{cases} \dfrac{h_{11}g^{ik}}{4}\Big[\dfrac{\partial^2 L}{\partial x^j \partial y^k}y^j - \dfrac{\partial L}{\partial x^k} + \dfrac{\partial^2 L}{\partial t \partial y^k} + \dfrac{\partial L}{\partial x^k}H^1_{11} + 2h^{11}H^1_{11}g_{kl}y^l\Big] , & p = 1 \\ \dfrac{1}{2}\Gamma^i_{jk}x^j_\alpha x^k_\beta + T^{(i)}_{(\alpha)\beta} , & p \geq 2 , \end{cases}$$

where $p = \dim T$.

Definition 3.3. The multi-time dependent spray (H, G) constructed in the preceding Theorem is called the **canonical multi-time spray attached to the multi-time Lagrange space** ML^n_p.

In the sequel, by local computations, the canonical multi-time spray (H, G) of the multi-time Lagrange space ML^n_p induces naturally a nonlinear connection Γ on $J^1(T, M)$.

Theorem 3.4. ***The canonical nonlinear connection***

$$\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$$

of the multi-time Lagrange space *ML^n_p is defined by the temporal components*

$$M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = \begin{cases} -H^1_{11}y^i\,, & p=1 \\ -H^\gamma_{\alpha\beta}x^i_\gamma\,, & p\geq 2\,, \end{cases} \tag{3.10}$$

and the spatial components

$$N^{(i)}_{(\alpha)j} = \frac{\partial \mathcal{G}^i}{\partial x^j_\gamma}h_{\alpha\gamma} = \begin{cases} h_{11}\dfrac{\partial \mathcal{G}^i}{\partial y^j}\,, & p=1 \\ \Gamma^i_{jk}x^k_\alpha + \dfrac{g^{ik}}{2}\frac{\partial g_{jk}}{\partial t^\alpha} + \frac{g^{ik}}{4}h_{\alpha\gamma}U^{(\gamma)}_{(k)j}\,, & p\geq 2\,, \end{cases} \tag{3.11}$$

where $\mathcal{G}^i = h^{\alpha\beta}G^{(i)}_{(\alpha)\beta}$.

Remark 3.5. In the particular case $(T,h) = (\mathbb{R},\delta)$, the canonical nonlinear connection $\Gamma = (0, N^{(i)}_{(1)j})$ of the relativistic rheonomic Lagrange space

$$RL^n = (J^1(\mathbb{R},M),L)$$

generalizes naturally the canonical nonlinear connection of the classical rheonomic Lagrange space $L^n = (\mathbb{R}\times \mathbf{T}M, L)$ [10].

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