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ON UNICITY OF MEROMORPHIC FUNCTIONS DUE TO A RESULT OF YANG - HUA

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Abstract. This paper studies the unicity of meromorphic(resp. entire) functions of the form $f^n f'$ and obtains the following main result: Let f and g be two non-constant meromorphic (resp. entire) functions, and let $a \in \mathbb{C}\backslash\{0\}$ be a non-zero finite value. Then, the condition that $E_{3)}(a, f^n f') = E_{3)}(a, g^n g')$ implies that either $f = dg$ for some $(n+1)$-th root of unity d, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c, c_1 and c_2 with $(c_1c_2)^{n+1}c^2 = -a^2$ provided that $n \geq 11$ (resp. $n \geq 6$). It improves a result of C. C. Yang and X. H. Hua. Also, some other related problems are discussed.

1. Introduction and main results

In this paper, a meromorphic function will always mean meromorphic in the open complex plane $\mathbb{C}$. We adopt the standard notations in the *Nevanlinna's value distribution theory of meromorphic functions* such as *the characteristic function* $T(r, f)$, *the proximity function* $m(r, f)$ and *the counting function* $N(r, f)$ (reduced form $\bar{N}(r, f)$) of poles. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\big(T(r, f)\big)$, possibly outside a set of finite linear measure that is not necessarily the same at each occurrence. We refer the reader to Hayman [3], Yang and Yi [8] for more details.

Let f be a non-constant meromorphic function, let $a \in \mathbb{C}$ be a finite value, and let $k \in \mathbb{N} \cup \{+\infty\}$ be a positive integer or infinity. We denote by $E(a, f)$ the set of zeros of $f - a$ and count multiplicities, while by $\bar{E}(a, f)$ the set of zeros of $f - a$ but ignore multiplicities. Further, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities less than or equal to k (counting multiplicities). Obviously, $E(a, f) = E_{+\infty)}(a, f)$. Define $E(\infty, f) := E(0, 1/f)$ for the value ∞, and define $\bar{E}(\infty, f)$ and $E_{k)}(\infty, f)$ correspondingly. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{k)}\big(r, 1/(f - a)\big)$ the counting function corresponding to the set $E_{k)}(a, f)$, while by $N_{(k+1}\big(r, 1/(f - a)\big)$ the counting function corresponding to the set $E_{(k+1}(a, f) := E(a, f) - E_{k)}(a, f)$.

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Also, we denote by $\bar{N}_{k)}\big(r, 1/(f-a)\big)$ and $\bar{N}_{(k+1}\big(r, 1/(f-a)\big)$ the reduced forms of $N_{k)}\big(r, 1/(f-a)\big)$ and $N_{(k+1}\big(r, 1/(f-a)\big)$, respectively.

All those foregoing definitions and notations hold well for any small meromorphic function, say, α (i.e., whose characteristic function satisfies $T(r,\alpha) = S(r,f)$), of f.

Let f and g be two non-constant meromorphic functions, and let α be a common small meromorphic function of f and g. We say that f and g share α CM (resp. IM) provided that $E(\alpha, f) = E(\alpha, g)$ (resp. $\bar{E}(\alpha, f) = \bar{E}(\alpha, g)$).

W. K. Hayman proposed the following well-known conjecture in [4].

Hayman Conjecture. *If an entire function f satisfies $f^n f' \neq 1$ for all positive integers $n \in \mathbb{N}$, then f is a constant.*

It has been verified by Hayman himself in [5] for the cases $n > 1$ and Clunie in [1] for the cases $n \geq 1$, respectively.

It is well-known that if two non-constant meromorphic functions f and g share two values CM and other two values IM, then f is a Möbius transformation of g. In 1997, C. C. Yang and X. H. Hua studied the unicity of differential monomials of the form $f^n f'$ and obtained the following theorem in [7].

Theorem A. *Let f and g be two non-constant meromorphic (resp. entire) functions, let $n \geq 11$ (resp. $n \geq 6$) be an integer, and let $a \in \mathbb{C}\backslash\{0\}$ be a non-zero finite value. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$-th root of unity d, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c, c_1 and c_2 such that $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Remark 1. In fact, combining their original argumentations with a more precise calculation on equations (20) and (23) in [7, p.p. 403-404] could reduce the lower bound of the integer n from 7 to 6 [7, Remark 2] if f and g are entire.

In 2000, by using argumentations similar to those in [7], M. L. Fang and H. L. Qiu proved the following uniqueness theorem in [2].

Theorem B. *Let f and g be two non-constant meromorphic (resp. entire) functions, and let $n \geq 11$ (resp. $n \geq 6$) be an integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f = dg$ for some $(n+1)$-th root of unity d, or $f = c_1 e^{cz^2}$ and $g = c_2 e^{-cz^2}$ for three non-zero constants c, c_1 and c_2 such that $4(c_1 c_2)^{n+1} c^2 = -1$.*

In this paper, we shall weaken the assumption of sharing the non-zero finite value a CM (i.e., $E(a, f^n f') = E(a, g^n g')$) in Theorem A to $E_{3)}(a, f^n f') = E_{3)}(a, g^n g')$. In fact, we shall prove the following three uniqueness theorems.

Theorem 1. *Let f and g be two non-constant meromorphic (resp. entire) functions, let $n \geq 11$ (resp. $n \geq 6$) be an integer, and let $a \in \mathbb{C}\backslash\{0\}$ be a non-zero finite value. If $E_{3)}(a, f^n f') = E_{3)}(a, g^n g')$, then $f^n f'$ and $g^n g'$ share the value a CM.*

Theorem 2. *Let f and g be two non-constant meromorphic (resp. entire) functions, let $n \geq 15$ (resp. $n \geq 8$) be an integer, and let $a \in \mathbb{C}\backslash\{0\}$ be a non-zero* *finite value. If* $E_{2)}(a, f^n f') = E_{2)}(a, g^n g')$*, then* $f^n f'$ *and* $g^n g'$ *share the value* a *CM.*

Theorem 3. *Let* f *and* g *be two non-constant meromorphic (resp. entire) functions, let* $n \geq 19$ *(resp.* $n \geq 10$*) be an integer, and let* $a \in \mathbb{C}\backslash\{0\}$ *be a non-zero finite value. If* $E_{1)}(a, f^n f') = E_{1)}(a, g^n g')$*, then* $f^n f'$ *and* $g^n g'$ *share the value* a *CM.*

Remark 2. Obviously, Theorem 1 is an improvement of Theorem A.

2. Some lemmas

Lemma 1. *Let* f *and* g *be two non-constant meromorphic functions satisfying* $E_{k)}(1, f) = E_{k)}(1, g)$ *for some positive integer* $k \in \mathbb{N}$*. Define* H *as the following*

$$H := \Big(\frac{f''}{f'} - 2\frac{f'}{f-1}\Big) - \Big(\frac{g''}{g'} - 2\frac{g'}{g-1}\Big)\,. \tag{2.1}$$

If $H \not\equiv 0$*, then*

$$\begin{aligned} N(r, H) \leq \bar{N}_{(2}(r, f) + \bar{N}_{(2}\Big(r, \frac{1}{f}\Big) + \bar{N}_{(2}(r, g) + \bar{N}_{(2}\Big(r, \frac{1}{g}\Big) + \bar{N}_0\Big(r, \frac{1}{f'}\Big) + \bar{N}_0\Big(r, \frac{1}{g'}\Big) \\ + \bar{N}_{(k+1}\Big(r, \frac{1}{f-1}\Big) + \bar{N}_{(k+1}\Big(r, \frac{1}{g-1}\Big) + S(r, f) + S(r, g)\,, \end{aligned} \tag{2.2}$$

where $N_0\big(r, 1/f'\big)$ *denotes the counting function of zeros of* f' *but not the zeros of* $f(f-1)$*, and* $N_0\big(r, 1/g'\big)$ *is similarly defined.*

Proof. It is not difficult to see that simple poles of f is not poles of $\frac{f''}{f'} - \frac{2f'}{f-1}$ and simple poles of g is not poles of $\frac{g''}{g'} - \frac{2g'}{g-1}$. Then, the conclusion follows immediately since we assume $E_{k)}(1, f) = E_{k)}(1, g)$. □

Lemma 2 (see [7, p.p. 397])**.** *Under the condition of Lemma 1, we have*

$$N_{1)}\Big(r, \frac{1}{f-1}\Big) = N_{1)}\Big(r, \frac{1}{g-1}\Big) \leq N(r, H) + S(r, f) + S(r, g)\,. \tag{2.3}$$

Lemma 3 (see [7, p.p. 398] or [9])**.** *Let* f *be some non-constant meromorphic function on* $\mathbb{C}$*. Then,*

$$N\Big(r, \frac{1}{f'}\Big) \leq \bar{N}(r, f) + N\Big(r, \frac{1}{f}\Big) + S(r, f)\,. \tag{2.4}$$

Lemma 4 (see [8])**.** *Let* f *be a non-constant meromorphic function on* $\mathbb{C}$*, and let* $k \in \mathbb{N}$ *be a positive integer. Then,*

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leq T(r, f^{(k)}) - T(r, f) + N\Big(r, \frac{1}{f}\Big) + S(r, f)\,. \tag{2.5}$$

3. Proof of Theorem 1

Define $F := \frac{f^n f'}{a}$ and $F_1 := \frac{f^{n+1}}{a(n+1)}$. Then, $F_1' = F$. Similarly, define $G := \frac{g^n g'}{a}$ and $G_1 := \frac{g^{n+1}}{a(n+1)}$. Now, by equations (19)–(20) in [7, p.p. 403-404], we have

$$\bar{N}(r,F) = \bar{N}_{(2}(r,F) = \bar{N}(r,f)\,, \tag{3.1}$$

$$\bar{N}(r,G) = \bar{N}_{(2}(r,G) = \bar{N}(r,g)\,, \tag{3.2}$$

and

$$\bar{N}\Big(r,\frac{1}{F}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{F}\Big) \leq 2\bar{N}\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f'}\Big)\,, \tag{3.3}$$

$$\bar{N}\Big(r,\frac{1}{G}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{G}\Big) \leq 2\bar{N}\Big(r,\frac{1}{g}\Big) + N\Big(r,\frac{1}{g'}\Big)\,. \tag{3.4}$$

Then, by the conclusions of Lemma 4, we derive

$$\begin{aligned}(n+1)T(r,f) &= T(r,F_1) + O(1)\\ &\leq T(r,F) + N\Big(r,\frac{1}{F_1}\Big) - N\Big(r,\frac{1}{F}\Big) + S(r,f)\\ &\leq T(r,F) + N\Big(r,\frac{1}{f}\Big) - N\Big(r,\frac{1}{f'}\Big) + S(r,f)\,. \end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\begin{aligned}(n+1)T(r,g) &= T(r,G_1) + O(1)\\ &\leq T(r,G) + N\Big(r,\frac{1}{g}\Big) - N\Big(r,\frac{1}{g'}\Big) + S(r,g)\,. \end{aligned} \tag{3.6}$$

Firstly, we suppose that equation (2.1) is not identically zero, that is, $H \not\equiv 0$. Here, we replace the functions f and g in the statement of Lemma 1 by F and G, respectively. Combining the conclusions of Lemmas 1 and 2 with the assumption that $E_{3)}(1,F) = E_{3)}(1,G)$ yields

$$\begin{aligned}N_{1)}\Big(r,\frac{1}{F-1}\Big) &\leq \bar{N}_{(2}(r,F) + \bar{N}_{(2}(r,G) + \bar{N}_{(2}\Big(r,\frac{1}{F}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{G}\Big)\\ &\quad + N_0\Big(r,\frac{1}{F'}\Big) + N_0\Big(r,\frac{1}{G'}\Big) + \bar{N}_{(4}\Big(r,\frac{1}{F-1}\Big)\\ &\quad + \bar{N}_{(4}\Big(r,\frac{1}{G-1}\Big) + S(r,f) + S(r,g)\,. \end{aligned} \tag{3.7}$$

Applying *the second fundamental theorem* to the functions F and G with the values 0, 1 and ∞, respectively, to conclude that

$$
\begin{aligned}
T(r,F)+T(r,G) \le{} & \bar{N}(r,F)+\bar{N}\Big(r,\frac{1}{F}\Big)+\bar{N}\Big(r,\frac{1}{F-1}\Big)-N_0\Big(r,\frac{1}{F'}\Big)+S(r,f)\\
& +\bar{N}(r,G)+\bar{N}\Big(r,\frac{1}{G}\Big)+\bar{N}\Big(r,\frac{1}{G-1}\Big)-N_0\Big(r,\frac{1}{G'}\Big)+S(r,g)\\
\le{} & \bar{N}(r,F)+\bar{N}\Big(r,\frac{1}{F}\Big)+\bar{N}(r,G)+\bar{N}\Big(r,\frac{1}{G}\Big)+N_{1)}\Big(r,\frac{1}{F-1}\Big)\\
& +\bar{N}\Big(r,\frac{1}{F-1}\Big)+\bar{N}\Big(r,\frac{1}{G-1}\Big)-N_{1)}\Big(r,\frac{1}{F-1}\Big)\\
& -N_0\Big(r,\frac{1}{F'}\Big)-N_0\Big(r,\frac{1}{G'}\Big)+S(r,f)+S(r,g)\,. \qquad (3.8)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\bar{N}\Big(r,\frac{1}{F-1}\Big)-\frac{1}{2}N_{1)}\Big(r,\frac{1}{F-1}\Big)+\bar{N}_{(4}\Big(r,\frac{1}{F-1}\Big) &\le \frac{1}{2}N\Big(r,\frac{1}{F-1}\Big)\,,\\
\bar{N}\Big(r,\frac{1}{G-1}\Big)-\frac{1}{2}N_{1)}\Big(r,\frac{1}{G-1}\Big)+\bar{N}_{(4}\Big(r,\frac{1}{G-1}\Big) &\le \frac{1}{2}N\Big(r,\frac{1}{G-1}\Big)\,.
\end{aligned}
$$

Then, combining the above two equations with $E_{3)}(1,F)=E_{3)}(1,G)$ yields

$$
\begin{aligned}
&\bar{N}\Big(r,\frac{1}{F-1}\Big)+\bar{N}\Big(r,\frac{1}{G-1}\Big)+\bar{N}_{(4}\Big(r,\frac{1}{G-1}\Big)+\bar{N}_{(4}\Big(r,\frac{1}{F-1}\Big)\\
&\qquad -N_{1)}\Big(r,\frac{1}{F-1}\Big)\le\frac{1}{2}\big(T(r,F)+T(r,G)\big)+S(r,f)+S(r,g)\,. \qquad (3.9)
\end{aligned}
$$

Hence, equations (3.7) - (3.9) imply

$$
\begin{aligned}
T(r,F)+T(r,G) \le{} & 2\Big(N_2(r,F)+N_2(r,G)+N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)\Big)\\
& +S(r,f)+S(r,g)\,, \qquad (3.10)
\end{aligned}
$$

where $N_2(r,F):=\bar{N}(r,F)+N_{(2}(r,F)$ and $N_2\big(r,1/F\big):=\bar{N}\big(r,1/F\big)+\bar{N}_{(2}\big(r,1/F\big)$, and $N_2(r,G)$ and $N_2\big(r,1/G\big)$ are similarly defined.

From equations (3.1)–(3.6) and (3.10), and noting Lemma 3, we derive

$$
\begin{aligned}
(n+1)\big(T(r,f)+T(r,g)\big) \le{} & 2\left(N_2(r,F)+N_2(r,G)+N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)\right)\\
& +N\Big(r,\frac{1}{f}\Big)-N\Big(r,\frac{1}{f'}\Big)+N\Big(r,\frac{1}{g}\Big)-N\Big(r,\frac{1}{g'}\Big)\\
& +S(r,f)+S(r,g)\\
\le{} & 4\big(\bar{N}(r,f)+\bar{N}(r,g)\big)+5\left(N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\right)\\
& +N\Big(r,\frac{1}{f'}\Big)+N\Big(r,\frac{1}{g'}\Big)+S(r,f)+S(r,g)
\end{aligned}
$$

(3.11)

$$\begin{aligned}&\leq 5\big(\bar{N}(r,f)+\bar{N}(r,g)\big)+6\left(N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\right)\\&\quad+S(r,f)+S(r,g)\,,\end{aligned}\tag{3.12}$$

which implies $(n-10)\big(T(r,f)+T(r,g)\big)\leq S(r,f)+S(r,g)$, a contradiction against the assumption that $n\geq 11$.

In particular, if f and g are entire, equation (3.11) turns out to be

$$(n-5)\big(T(r,f)+T(r,g)\big)\leq S(r,f)+S(r,g)\,,\tag{3.13}$$

since both the terms $\bar{N}(r,f)$ and $\bar{N}(r,g)$ equal to $O(1)$ now. Obviously, it contradicts the assumption that $n\geq 6$.

Hence, $H\equiv 0$. Integrating the equation $H\equiv 0$ twice results in

$$\frac{F'}{F-1}=k_1\frac{G'}{G-1}+k_2\quad(k_1\in\mathbb{C}\backslash\{0\},\ k_2\in\mathbb{C}),$$

which implies that F and G share the value 1 CM.

This finishes the proof of Theorem 1. □

4. Proof of Theorem 2

From the condition that $E_{2)}(1,F)=E_{2)}(1,G)$, if we furthermore suppose that $H\not\equiv 0$, then similar to equation (3.7), we have

$$\begin{aligned}N_{1)}\Big(r,\frac{1}{F-1}\Big)&\leq \bar{N}_{(2}(r,F)+\bar{N}_{(2}(r,G)+\bar{N}_{(2}\Big(r,\frac{1}{F}\Big)+\bar{N}_{(2}\Big(r,\frac{1}{G}\Big)\\&\quad+N_0\Big(r,\frac{1}{F'}\Big)+N_0\Big(r,\frac{1}{G'}\Big)+\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)\\&\quad+\bar{N}_{(3}\Big(r,\frac{1}{G-1}\Big)+S(r,f)+S(r,g)\,.\end{aligned}\tag{4.1}$$

A routine calculation leads to

$$\bar{N}\Big(r,\frac{1}{F-1}\Big)-\frac{1}{2}N_{1)}\Big(r,\frac{1}{F-1}\Big)+\frac{1}{2}\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)\leq\frac{1}{2}N\Big(r,\frac{1}{F-1}\Big)\,,\tag{4.2}$$

$$\bar{N}\Big(r,\frac{1}{G-1}\Big)-\frac{1}{2}N_{1)}\Big(r,\frac{1}{G-1}\Big)+\frac{1}{2}\bar{N}_{(3}\Big(r,\frac{1}{G-1}\Big)\leq\frac{1}{2}N\Big(r,\frac{1}{G-1}\Big)\,.\tag{4.3}$$

Applying the conclusions of Lemma 3 to F and taking reduced forms of the counting functions on both sides of equation (2.4) to conclude

$$\begin{aligned}\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)&\leq\bar{N}_{(2}\Big(r,\frac{1}{F'}\Big)+S(r,f)\leq\bar{N}\Big(r,\frac{1}{F'}\Big)+S(r,f)\\&\leq\bar{N}(r,F)+\bar{N}\Big(r,\frac{1}{F}\Big)+S(r,f)\\&\leq\bar{N}(r,f)+\bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}\Big(r,\frac{1}{f'}\Big)+S(r,f)\\&\leq 2\bar{N}(r,f)+2\bar{N}\Big(r,\frac{1}{f}\Big)+S(r,f)\,,\end{aligned}\tag{4.4}$$

and similarly,

$$\bar{N}_{(3}\Big(r,\frac{1}{G-1}\Big) \le 2\bar{N}(r,g) + 2\bar{N}\Big(r,\frac{1}{g}\Big) + S(r,g)\,. \tag{4.5}$$

Hence, equations (4.2)–(4.5) yield

$$\begin{aligned}\bar{N}\Big(r,\frac{1}{F-1}\Big) + \bar{N}\Big(r,\frac{1}{G-1}\Big) + \bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big) + \bar{N}_{(3}\Big(r,\frac{1}{G-1}\Big)&\\ - N_{1)}\Big(r,\frac{1}{F-1}\Big) \le \frac{1}{2}\big(T(r,F)+T(r,G)\big)&\\ + \left(\bar{N}(r,f) + \bar{N}(r,g) + \bar{N}\Big(r,\frac{1}{f}\Big) + \bar{N}\Big(r,\frac{1}{g}\Big)\right) + S(r,f) + S(r,g)\,.&\end{aligned}$$

Analogous to equation (3.10), we have

$$\begin{aligned}T(r,F)+T(r,G) \le{}& 2\left(N_2(r,F) + N_2(r,G) + N_2\Big(r,\frac{1}{F}\Big) + N_2\Big(r,\frac{1}{G}\Big)\right)\\ &+ 2\left(\bar{N}(r,f) + \bar{N}(r,g) + \bar{N}\Big(r,\frac{1}{f}\Big) + \bar{N}\Big(r,\frac{1}{g}\Big)\right)\\ &+ S(r,f) + S(r,g)\,.\end{aligned}$$

Combining the above equation with equations (3.1)–(3.6) yields

$$\begin{aligned}(n+1)\big(T(r,f)+T(r,g)\big) \le{}& 7\big(\bar{N}(r,f)+\bar{N}(r,g)\big)\\ &+ 8\left(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\right) + S(r,f) + S(r,g)\,,\end{aligned} \tag{4.6}$$

which implies that $(n-14)\big(T(r,f)+T(r,g)\big) \le S(r,f)+S(r,g)$, a contradiction since we assume $n \ge 15$. In particular, if f and g are entire, then equation (4.6) turns into $(n-7)\big(T(r,f)+T(r,g)\big) \le S(r,f)+S(r,g)$. Obviously, it contradicts the assumption that $n \ge 8$.

Hence $H \equiv 0$, and F and G share the value 1 CM.

This finishes the proof of Theorem 2. □

5. Proof of Theorem 3

From the condition that $E_{1)}(1,F) = E_{1)}(1,G)$, if we furthermore assume that $H \not\equiv 0$, then similar to equation (3.7), we have

$$\begin{aligned}N_{1)}\Big(r,\frac{1}{F-1}\Big) \le{}& \bar{N}_{(2}(r,F) + \bar{N}_{(2}(r,G) + \bar{N}_{(2}\Big(r,\frac{1}{F}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{G}\Big)\\ &+ N_0\Big(r,\frac{1}{F'}\Big) + N_0\Big(r,\frac{1}{G'}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{F-1}\Big)\\ &+ \bar{N}_{(2}\Big(r,\frac{1}{G-1}\Big) + S(r,f) + S(r,g)\,.\end{aligned} \tag{5.1}$$

It is not difficult to see that

$$\bar{N}\Big(r, \frac{1}{F-1}\Big) - \frac{1}{2}N_{1)}\Big(r, \frac{1}{F-1}\Big) \le \frac{1}{2}N\Big(r, \frac{1}{F-1}\Big), \tag{5.2}$$

$$\bar{N}\Big(r, \frac{1}{G-1}\Big) - \frac{1}{2}N_{1)}\Big(r, \frac{1}{G-1}\Big) \le \frac{1}{2}N\Big(r, \frac{1}{G-1}\Big). \tag{5.3}$$

Also, as shown in inequality (4.4), we have

$$\begin{aligned} \bar{N}_{(2}\Big(r, \frac{1}{F-1}\Big) &\le \bar{N}\Big(r, \frac{1}{F'}\Big) + S(r,f) \\ &\le 2\bar{N}(r,f) + 2\bar{N}\Big(r, \frac{1}{f}\Big) + S(r,f) \qquad \text{and} \end{aligned} \tag{5.4}$$

$$\bar{N}_{(2}\Big(r, \frac{1}{G-1}\Big) \le 2\bar{N}(r,g) + 2\bar{N}\Big(r, \frac{1}{g}\Big) + S(r,g). \tag{5.5}$$

Hence, equations (5.2)–(5.5) yield

$$\begin{aligned} \bar{N}\Big(r, \frac{1}{F-1}\Big) &+ \bar{N}\Big(r, \frac{1}{G-1}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{F-1}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{G-1}\Big) \\ &- N_{1)}\Big(r, \frac{1}{F-1}\Big) \le \frac{1}{2}\big(T(r,F) + T(r,G)\big) \\ &+ 2\Big(\bar{N}(r,f) + \bar{N}(r,g) + \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}\Big(r, \frac{1}{g}\Big)\Big) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

Analogically, we have

$$\begin{aligned} T(r,F) + T(r,G) \le 2\Big(&N_2(r,F) + N_2(r,G) + N_2\Big(r, \frac{1}{F}\Big) + N_2\Big(r, \frac{1}{G}\Big)\Big) \\ &+ 4\Big(\bar{N}(r,f) + \bar{N}(r,g) + \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}\Big(r, \frac{1}{g}\Big)\Big) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

Hence,

$$\begin{aligned} (n+1)\big(T(r,f) + T(r,g)\big) &\le 9\big(\bar{N}(r,f) + \bar{N}(r,g)\big) \\ &+ 10\Big(N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{g}\Big)\Big) + S(r,f) + S(r,g), \end{aligned} \tag{5.6}$$

which implies that $(n-18)\big(T(r,f) + T(r,g)\big) \le S(r,f) + S(r,g)$, a contradiction since we assume $n \ge 19$. In particularly, if f and g are entire, then equation (5.6) turns into $(n-9)\big(T(r,f) + T(r,g)\big) \le S(r,f) + S(r,g)$. Obviously, it contradicts the assumption that $n \ge 10$.

Hence $H \equiv 0$, and F and G share the value 1 CM.

This finishes the proof of Theorem 3. □

6. Related results

Final Note 1. If we assume that f and g share the value ∞ CM (resp. IM) in the statement of Lemma 1 besides the assumption that $E_{k)}(1,f) = E_{k)}(1,g)$ for some positive integer $k \in \mathbb{N}$, then equation (2.2) becomes

$$\begin{aligned} N(r,H) \le\ & \bar{N}_{(2}\Big(r,\frac{1}{f}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{g}\Big) + \bar{N}_0\Big(r,\frac{1}{f'}\Big) + \bar{N}_0\Big(r,\frac{1}{g'}\Big) \\ & + \bar{N}_{(k+1}\Big(r,1\frac{1}{f-1}\Big) + \bar{N}_{(k+1}\Big(r,\frac{1}{g-1}\Big) + S(r,f) + S(r,g)\,, \end{aligned} \tag{2.2^a}$$

and respectively,

$$\begin{aligned} N(r,H) \le\ & \frac{1}{2}\bar{N}(r,f) + \frac{1}{2}\bar{N}(r,g) + \bar{N}_{(2}\Big(r,\frac{1}{f}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{g}\Big) + \bar{N}_0\Big(r,\frac{1}{f'}\Big) + \bar{N}_0\Big(r,\frac{1}{g'}\Big) \\ & + \bar{N}_{(k+1}\Big(r,1\frac{1}{f-1}\Big) + \bar{N}_{(k+1}\Big(r,\frac{1}{g-1}\Big) + S(r,f) + S(r,g)\,. \end{aligned} \tag{2.2^b}$$

Applying the argumentations used in our proofs with equation (2.2^a) (resp. (2.2^b)) could reduce the lower bounds of the integers n from $n \ge 11$, 15 and 19 in Theorems 1, 2 and 3 to $n \ge 9$, 13 and 17 (resp. $n \ge 10$, 14 and 18), respectively, provided that we assume furthermore that f and g, and thus F and G, share the value ∞ CM (resp. IM).

Final Note 2. Using similar argumentations as those in our proofs and replacing the notations F, F_1 (resp. G, G_1) in Section 3 by new ones $F = f^n f'/z$, $F_1 = f^{n+1}/(n+1)$ (resp. $G = g^n g'/z$, $G_1 = g^{n+1}/(n+1)$) (then, $F_1' = zF$ and $G_1' = zG$), we could weaken the assumption of sharing z CM (i.e., $E(z, f^n f') = E(z, g^n g')$) in the statement of Theorem C to $E_{k)}(z, f^n f') = E_{k)}(z, g^n g')$ for $k = 1$, 2 and 3.

In fact, if f and g are transcendental, our original proofs go well, while if f and g are rational functions (resp. polynomials), routine calculations on the term "$\log r$" would lead to analogous conclusions. However, in those cases we may have to increase the lower bounds of the integers n from $n \ge 11$, 15 and 19 (resp. $n \ge 6$, 8 and 10) to $n \ge 14$, 19 and 24 (resp. $n \ge 9$, 12 and 15), since now f and g have the same growth estimate as that of the function z, in other words, of $O(\log r)$. Below, we give an outline of the proof for those special cases.

Proof. First of all, according to the conclusion of [2, Theorem C], we know that f is rational whenever g is, and vice versa. Similarly, we have $\bar{N}(r,F) = \bar{N}(r,f) + \log r$ and $N_2(r,F) \le 2\bar{N}(r,f) + \log r$, and $\bar{N}(r,G) = \bar{N}(r,g) + \log r$ and $N_2(r,G) \le 2\bar{N}(r,g) + \log r$. Furthermore, we have

$$\begin{aligned} (n+1)T(r,f) &= T(r,F_1) + O(1) \le T(r,zF) + N\Big(r,\frac{1}{F_1}\Big) - N\Big(r,\frac{1}{zF}\Big) + O(1) \\ &\le T(r,F) + N\Big(r,\frac{1}{f}\Big) - N\Big(r,\frac{1}{f'}\Big) + \log r + O(1)\,, \end{aligned}$$

$$(n+1)T(r,g) = T(r,G_1) + O(1) \leq T(r,zG) + N\Big(r,\frac{1}{G_1}\Big) - N\Big(r,\frac{1}{zG}\Big) + O(1)$$
$$\leq T(r,G) + N\Big(r,\frac{1}{g}\Big) - N\Big(r,\frac{1}{g'}\Big) + \log r + O(1)\,.$$

If $E_{3)}(z,f^nf') = E_{3)}(z,g^ng')$, then analogous to equation (3.11), we derive

$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 5\big(\bar{N}(r,f)+\bar{N}(r,g)\big)$$
$$+ 6\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 6\log r + O(1)\,,$$

which implies that $(n-10)\big(T(r,f)+T(r,g)\big) \leq 6\log r + O(1)$.

Noting the discussions in [2, p.p. 437-438] fail here, we may have to suppose that $(n-10)\big(T(r,f)+T(r,g)\big) \geq (2n-20)\log r$, and hence $(2n-26)\log r \leq O(1)$, a contradiction since we assume $n \geq 14$.

If $E_{k)}(z,f^nf') = E_{k)}(z,g^ng')$ for $k = 1$, 2, then parallel to equations (4.4)–(4.5) and (5.4)–(5.5), we have

$$\bar{N}_{(k}\Big(r,\frac{1}{F-1}\Big) \leq 2\bar{N}(r,f) + 2\bar{N}\Big(r,\frac{1}{f}\Big) + \log r + O(1)\,,$$
$$\bar{N}_{(k}\Big(r,\frac{1}{G-1}\Big) \leq 2\bar{N}(r,g) + 2\bar{N}\Big(r,\frac{1}{g}\Big) + \log r + O(1)\,.$$

If $k = 2$, we have

$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 7\big(\bar{N}(r,f)+\bar{N}(r,g)\big)$$
$$+ 8\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 8\log r + O(1)\,,$$

which means $(2n-36)\log r \leq O(1)$, a contradiction since we assume $n \geq 19$.

If $k = 1$, we have

$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 9\big(\bar{N}(r,f)+\bar{N}(r,g)\big)$$
$$+ 10\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 10\log r + O(1)\,,$$

which shows $(2n-46)\log r \leq O(1)$, a contradiction since we assume $n \geq 24$.

If f and g are polynomials, then $N(r,F) = N(r,G) = \log r$, and hence $\bar{N}(r,F) = N_2(r,F) = \bar{N}(r,G) = N_2(r,G) = \log r$. Similarly, we derive

$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 6\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 6\log r + O(1) \qquad (k=3)\,,$$
$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 8\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 8\log r + O(1) \qquad (k=2)\,,$$
$$(n+1)\big(T(r,f)+T(r,g)\big) \leq 10\bigg(N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\bigg) + 10\log r + O(1) \quad (k=1)\,.$$

All the above three equations contradict the assumptions that $n \geq 9$ $(k = 3)$, $n \geq 12$ $(k = 2)$ and $n \geq 15$ $(k = 1)$, respectively. □

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