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DERIVATIONS OF THE SUBALGEBRAS INTERMEDIATE THE GENERAL LINEAR LIE ALGEBRA AND THE DIAGONAL SUBALGEBRA OVER COMMUTATIVE RINGS

Dengyin Wang and Xian Wang

Abstract. Let R be an arbitrary commutative ring with identity, $\mathrm{gl}(n,R)$ the general linear Lie algebra over R, $d(n,R)$ the diagonal subalgebra of $\mathrm{gl}(n,R)$. In case 2 is a unit of R, all subalgebras of $\mathrm{gl}(n,R)$ containing $d(n,R)$ are determined and their derivations are given. In case 2 is not a unit partial results are given.

1. Introduction

Let R be a commutative ring with identity, R^* the subset of R consisting of all invertible elements in R, $I(R)$ the set consisting of all ideals of R. Let $\mathrm{gl}(n,R)$ be the general linear Lie algebra consisting of all $n\times n$ matrices over R and with the bracket operation: $[x,y]=xy-yx$. We denote by $d(n,R)$ (resp., $t(n,R)$) the subset of $\mathrm{gl}(n,R)$ consisting of all $n\times n$ diagonal (resp., upper triangular) matrices over R. Let E be the identity matrix in $\mathrm{gl}(n,R)$, RE the set $\{rE \mid r\in R\}$ consisting of all scalar matrices, and $E_{i,j}$ the matrix in $\mathrm{gl}(n,R)$ whose sole nonzero entry 1 is in the (i,j) position. For $A\in \mathrm{gl}(n,R)$, we denote by A' the transpose of A.

For R-modules M and K, we denote by $\mathrm{Hom}_R(M,K)$ the set of all homomorphisms of R-modules from M to K. $\mathrm{Hom}_R(M,M)$ is abbreviated to $\mathrm{Hom}_R(M)$. For $1\le i\le n$, $\chi_i\colon d(n,R)\to R$, defined by $\chi_i(\mathrm{diag}(d_1,d_2,\dots,d_n))=d_i$, is a standard homomorphism from $d(n,R)$ to R.

Recently, significant work has been done in studying automorphisms and derivations of matrix Lie algebras (or sometimes matrix algebras) and their subalgebras (see [1]–[7]). Derivations of the parabolic subalgebras of $\mathrm{gl}(n,R)$ were described in [7]. Derivations of the subalgebras of $t(n,R)$ containing $d(n,R)$ were determined in [6]. In this article, when 2 is a unit of R, all subalgebras of $\mathrm{gl}(n,R)$ containing $d(n,R)$ are determined and their derivations are given. In case 2 is not a unit partial results are given.

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2. The subalgebras of gl(n, R) containing $d(n, R)$

Definition 2.1. Let $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i,\, j \leq n\}$ be a subset of $I(R)$ consisting of n^2 ideals of R. We call Φ a *flag* of ideals of R, if

(1) $A_{i,i} = R$, $i = 1, 2, \ldots, n$.

(2) $A_{i,k}A_{k,j} \subseteq A_{i,j}$ for any i, j, k $(1 \leq i, j, k \leq n)$.

Example 2.2. If $i \neq j$, let $A_{i,j}$ be 0, and let $A_{i,i} = R$ for $i = 1, 2, \ldots, n$. Then $\Phi = \{A_{i,j} \mid 1 \leq i,\, j \leq n\}$ is a flag of ideals of R.

Example 2.3. If all $A_{i,j}$ are taken to be R, then $\Phi = \{A_{i,j} \mid 1 \leq i,\ j \leq n\}$ is a flag of ideals of R.

Theorem 2.4. *If $\Phi = \{A_{i,j} \mid 1 \leq i,\, j \leq n\}$ is a flag of ideals of R, then $L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}$ is a subalgebra of* gl(n, R) *containing* $d(n, R)$.

Proof. Suppose that $\Phi = \{A_{i,j} \mid 1 \leq i,\, j \leq n\}$ is a flag of ideals of R and $L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}$. Let

$$x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}E_{i,j} \in L_\Phi\,, \qquad y = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}E_{i,j} \in L_\Phi\,,$$

where $a_{i,j}, b_{i,j} \in A_{i,j}$. It is obvious that $rx + sy \in L_\Phi$ for any $r, s \in R$. Notice that

$$[x, y] = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}E_{i,j}\,, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^{n}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\,.$$

By assumption (2) on Φ, we know that $(a_{i,k}b_{k,j} - b_{i,k}a_{k,j}) \in A_{i,j}$, forcing $c_{i,j} \in A_{i,j}$ and $[x, y] \in L_\Phi$. Hence L_Φ is a subalgebra of gl(n, R). Assumption (1) on Φ shows that L_Φ contains $d(n, R)$. □

The following result shows that these L_Φ nearly exhaust all subalgebras of gl(n, R) containing $d(n, R)$.

Theorem 2.5. *If L is a subalgebra of* gl(n, R) *containing $d(n, R)$, then there exists a flag $\Phi = \{A_{i,j} \mid 1 \leq i,\, j \leq n\}$ of ideals of R such that*

$$2L \subseteq L_\Phi \subseteq L\,.$$

Proof. Let L be a subalgebra of gl(n, R) containing $d(n, R)$. For $\forall i, j$ $(1 \leq i, j \leq n)$, define

$$A_{i,j} = \{a_{i,j} \in R \mid a_{i,j}E_{i,j} \in L\}\,,$$

and set

$$\Phi = \{A_{i,j} \mid 1 \leq i,\, j \leq n\}\,,$$

$$L_\Phi = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}E_{i,j}\,.$$

In the following, we will prove that Φ is a flag of ideals of R, and $2L \subseteq L_\Phi \subseteq L$. It's obvious that all $A_{i,j}$ are ideals of R and $A_{i,i} = R$ for $i = 1, 2, \cdots, n$. If $i \neq j$

and $a_{i,k} \in A_{i,k}$, $a_{k,j} \in A_{k,j}$, then by $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j} \in L$, we see that $a_{i,k}a_{k,j} \in A_{i,j}$, forcing $A_{i,k}A_{k,j} \subseteq A_{i,j}$. If $i = j$, since $A_{i,i} = R$, we also have that $A_{i,k}A_{k,j} \subseteq A_{i,j}$. Thus Φ is a flag of ideals of R. It is easy to see that $L_\Phi \subseteq L$. On the other hand, for $x = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}E_{i,j} \in L$, if $k \neq l$, then by

$$\big[E_{k,k}, [E_{l,l}, -x]\big] = a_{k,l}E_{k,l} + a_{l,k}E_{l,k} \in L\,,$$

$$[E_{k,k}, a_{k,l}E_{k,l} + a_{l,k}E_{l,k}] = a_{k,l}E_{k,l} - a_{l,k}E_{l,k} \in L\,,$$

we see that $2a_{k,l}E_{k,l} \in L$, $2a_{l,k}E_{l,k} \in L$. This shows that $2a_{k,l} \in A_{k,l}$, $2a_{l,k} \in A_{l,k}$, forcing $2x \in L_\Phi$. So $2L \subseteq L_\Phi$. □

Corollary 2.6. *Assume that $2 \in R^*$, then L is a subalgebra of $\mathrm{gl}(n,R)$ containing $d(n,R)$ if and only if there exists a flag $\Phi = \{A_{i,j} \mid 1 \le i,\, j \le n\}$ of ideals of R such that $L = L_\Phi$.*

Remark 2.7. Without the assumption $2 \in R^*$, Corollary 2.6 does not hold. The following is an example. Let R be $Z/2Z$ (Z is the ring of all integer numbers), then R has only two ideals: 0 and R. Set $L = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a,b,c \in Z/2Z \right\}$. Then L is a subalgebra of $\mathrm{gl}(2, Z/2Z)$ containing $d(2, Z/2Z)$, but $L \neq L_\Phi$ for any flag $\Phi = \{A_{i,j} \mid 1 \le i,j \le 2\}$ of ideals of R.

3. Construction of certain derivations of L_Φ

Let $L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}$ be a fixed subalgebra of $\mathrm{gl}(n,R)$ containing $d(n,R)$, with $\Phi = \{A_{i,j} \in I(R) \mid 1 \le i,\, j \le n\}$ a flag of ideals of R. We denote by Der L_Φ the set consisting of all derivations of L_Φ. We now construct certain derivations of L_Φ for building the derivation algebra *Der* L_Φ of L_Φ. For $A_{i,j} \in \Phi$, let $B_{i,j}$ denote the annihilator of $A_{i,j}$ in R, i.e., $B_{i,j} = \{r \in R \mid rA_{ij} = 0\}$.

(A) Inner derivations

Let $x \in L_\Phi$, then ad $x\colon L_\Phi \to L_\Phi$, $y \mapsto [x,y]$, is a derivation of L_Φ, called the *inner derivation* of L_Φ induced by x. Let ad L_Φ denote the set consisting of all ad x, $x \in L_\Phi$, which forms an ideal of Der L_Φ.

(B) Transpose derivations

Definition 3.3. Let $\Pi = \{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \le i,\, j \le n\}$ be a set consisting of n^2 homomorphisms of R-modules. We call Π *suitable for transpose derivations*, if the following conditions are satisfied for all i,j ($1 \le i,j \le n$):
(1) $\pi_{i,i} = 0$;
(2) $\pi_{i,j}(A_{i,k}A_{k,j}) = 0$ for all k which satisfies $k \neq i$ and $k \neq j$;
(3) $\pi_{i,j}(A_{i,j}) \subseteq B_{k,j}$ and $\pi_{i,j}(A_{i,j}) \subseteq B_{i,k}$ for all k which satisfies $k \neq i$ and $k \neq j$;
(4) $2\pi_{i,j}(A_{i,j}) = 0$.

Remark. In case 2 is a unit, (4) means that $\pi_{i,j}$ are necessarily zero maps.

Using the homomorphism $\Pi = \{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \le i,\, j \le n\}$ which is suitable for transpose derivations, we define $\phi_\Pi\colon L_\Phi \to L_\Phi$ by sending any $\sum_{i=1}^n \sum_{j=1}^n a_{i,j}E_{i,j} \in L_\Phi$ to $\sum_{i=1}^n \sum_{j=1}^n \pi_{i,j}(a_{i,j})E_{j,i}$.

Lemma 3.4. *The map ϕ_Π as defined above, is a derivation of L_Φ.*

Proof. Let

$$x = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} E_{i,j} \in L_\Phi\,, \qquad a_{i,j} \in A_{i,j}\,,$$

$$y = \sum_{i=1}^n \sum_{j=1}^n b_{i,j} E_{i,j} \in L_\Phi\,, \qquad b_{i,j} \in A_{i,j}\,.$$

Obviously, $\phi_\Pi(rx+sy) = r\phi_\Pi(x) + s\phi_\Pi(y)$ for $\forall r, s \in R$. Write

$$[x,y] = \sum_{i=1}^n \sum_{j=1}^n c_{i,j} E_{i,j}\,, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^n (a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\,.$$

Because Π is suitable for transpose derivations, we have that

$$\begin{aligned}\phi_\Pi\big([x,y]\big) &= \sum_{i=1}^n \sum_{j=1}^n \pi_{i,j}(c_{i,j})E_{j,i} = \sum_{i=1}^n \sum_{j=1}^n \pi_{i,j}\Big(\sum_{k=1}^n (a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\Big)E_{j,i} \\ &= \sum_{i=1}^n \sum_{j=1}^n \big[(a_{i,i}-a_{j,j})\pi_{i,j}(b_{i,j}) + (b_{j,j}-b_{i,i})\pi_{i,j}(a_{i,j})\big]E_{j,i} \\ &\quad \text{(by assumption (2))}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\big[\phi_\Pi(x),y\big] + \big[x,\phi_\Pi(y)\big] &= \sum_{i=1}^n \sum_{j=1}^n \Big[\sum_{k=1}^n \big(\pi_{k,j}(a_{k,j})b_{k,i} - b_{j,k}\pi_{i,k}(a_{i,k}) \\ &\qquad - \pi_{k,j}(b_{k,j})a_{k,i} + a_{j,k}\pi_{i,k}(b_{i,k})\big)\Big]E_{j,i} \\ &= \sum_{i=1}^n \sum_{j=1}^n \big[(a_{j,j}-a_{i,i})\pi_{i,j}(b_{i,j}) + (b_{i,i}-b_{j,j})\pi_{i,j}(a_{i,j})\big]E_{j,i} \\ &\quad \text{(by assumption (3))}.\end{aligned}$$

By assumption (4) on Π, we see that $\phi_\Pi\big([x,y]\big) = \big[\phi_\Pi(x),y\big] + \big[x,\phi_\Pi(y)\big]$. Hence ϕ_Π is a derivation of L_Φ. □

ϕ_Π is called a *transpose derivation* of L_Φ.

(C) **Ring derivations**

Definition 3.5. Let $\Sigma = \{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}),\, \sigma \in \mathrm{Hom}_R\big(d(n,R)\big) \mid 1 \le i,\, j \le n\}$ be a set consisting of n^2+1 endomorphisms of R-modules. We call Σ *suitable for ring derivations* if the following conditions are satisfied for $\forall\, i,j$ $(1 \le i,j \le n)$:

(1) $\chi_i\big(\sigma(D)\big) - \chi_j\big(\sigma(D)\big) \subseteq (B_{i,j} \cap B_{j,i})$ for $\forall\, D \in d(n,R)$;

(2) $\sigma\big(a_{i,j}a_{j,i}(E_{i,i}-E_{j,j})\big) = \big(\sigma_{i,j}(a_{i,j})a_{j,i} + a_{i,j}\sigma_{j,i}(a_{j,i})\big)(E_{i,i}-E_{j,j})$, $\forall\, a_{i,j} \in A_{i,j}, \forall\, a_{j,i} \in A_{j,i}$;

(3) $\sigma_{i,i} = 0$, $i = 1, 2, \dots n$

(4) When $i \neq j$, $\sigma_{i,j}(a_{i,k}a_{k,j}) = \sigma_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(a_{k,j})$ for $\forall\ k$ ($1 \leq k \leq n$), $\forall\ a_{i,k} \in A_{i,k}$ and $\forall\ a_{k,j} \in A_{k,j}$.

Using $\Sigma = \{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}),\ \sigma \in \mathrm{Hom}_R(d(n,R)) \mid 1 \leq i,\ j \leq n\}$ which is suitable for ring derivations, we define $\phi_\Sigma\colon L_\Phi \to L_\Phi$ by sending any $\sum_{i=1}^n \sum_{j=1}^n a_{i,j} E_{i,j} \in L_\Phi$ to $\sum_{1\leq i\neq j\leq n} \sigma_{i,j}(a_{i,j})E_{i,j} + \sigma\big(\sum_{k=1}^n a_{k,k}E_{k,k}\big)$.

Lemma 3.6. *The map ϕ_Σ, as defined above, is a derivation of L_Φ.*

Proof. Let $x = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}E_{i,j} \in L_\Phi$, $y = \sum_{i=1}^n \sum_{j=1}^n b_{i,j}E_{i,j} \in L_\Phi$, where $a_{i,j}, b_{i,j}$ lie in $A_{i,j}$. It is obvious that $\phi_\Sigma(rx+sy) = r\phi_\Sigma(x) + s\phi_\Sigma(y)$ for any $r, s \in R$. We know $[x,y] = \sum_{i=1}^n \sum_{j=1}^n c_{i,j}E_{i,j}$, where $c_{i,j} = \sum_{k=1}^n (a_{i,k}b_{k,j} - b_{i,k}a_{k,j})$. Because Σ is suitable for ring derivations, we have that

$$\begin{aligned}
\phi_\Sigma\big([x,y]\big) &= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,j}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\big)\Big]E_{i,j} \\
&\quad + \sigma\Big[\sum_{i=1}^n\sum_{k=1}^n (a_{i,k}b_{k,i} - b_{i,k}a_{k,i})E_{i,i}\Big] \\
&= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,j}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\big)\Big]E_{i,j} \\
&\quad + \sigma\Big(\sum_{i=1}^n\sum_{k=1}^n a_{i,k}b_{k,i}(E_{i,i} - E_{k,k})\Big)
\end{aligned}$$

$$\text{(note that}\quad \sum_{i=1}^n\sum_{k=1}^n (a_{i,k}b_{k,i} - b_{i,k}a_{k,i})E_{i,i} = \sum_{i=1}^n\sum_{k=1}^n a_{i,k}b_{k,i}(E_{i,i} - E_{k,k}))$$

$$\begin{aligned}
&= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,k}(a_{i,k})b_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j}) \\
&\quad - \sigma_{i,k}(b_{i,k})a_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j})\big)\Big]E_{i,j} \\
&\quad + \sum_{i=1}^n\sum_{k=1}^n \big[\sigma_{i,k}(a_{i,k})b_{k,i} + a_{i,k}\sigma_{k,i}(b_{k,i})\big](E_{i,i} - E_{k,k}),
\end{aligned}$$

(by assumption (2) and (4)).

On the other hand,

$$\begin{aligned}
\big[\phi_\Sigma(x), y\big] + \big[x, \phi_\Sigma(y)\big] &= \Big[\sum_{1\leq i\neq j\leq n} \sigma_{i,j}(a_{i,j})E_{i,j} + \sigma\Big(\sum_{i=1}^n a_{i,i}E_{i,i}\Big), y\Big] \\
&\quad + \Big[x, \sum_{1\leq i\neq j\leq n} \sigma_{i,j}(b_{i,j})E_{i,j} + \sigma\Big(\sum_{i=1}^n b_{i,i}E_{i,i}\Big)\Big]
\end{aligned}$$

$$= \Big[\sum_{1\le i\ne j\le n}\sigma_{i,j}(a_{i,j})E_{i,j}, y\Big] + \Big[x, \sum_{1\le i\ne j\le n}\sigma_{i,j}(b_{i,j})E_{i,j}\Big]$$
(by assumption (1))

$$= \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}(a_{i,j})E_{i,j}, y\Big] + \Big[x, \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}(b_{i,j})E_{i,j}\Big]$$
(by assumption (3))

$$\begin{aligned}
= &\sum_{1\le i\ne j\le n}\Big[\sum_{k=1}^{n}\sigma_{i,k}(a_{i,k})b_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j}) \\
&- \sigma_{i,k}(b_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j})\Big]E_{i,j} \\
&+ \sum_{i=1}^{n}\Big[\sum_{k=1}^{n}\sigma_{i,k}(a_{i,k})b_{k,i} + b_{k,i}\sigma_{i,k}(a_{i,k}) \\
&- \sigma_{i,k}(b_{i,k})a_{k,i} - a_{k,i}\sigma_{i,k}(b_{i,k})\Big]E_{i,i} \\
= &\sum_{1\le i\ne j\le n}\Big[\sum_{k=1}^{n}\sigma_{i,k}(a_{i,k})b_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j}) \\
&- \sigma_{i,k}(b_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j})\Big]E_{i,j} \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{n}\big[\sigma_{i,k}(a_{i,k})b_{k,i} + b_{k,i}\,\sigma_{i,k}(a_{i,k})\big](E_{i,i} - E_{k,k})\,.
\end{aligned}$$

We see that

$$\big[\phi_\Sigma(x), y\big] + \big[x, \phi_\Sigma(y)\big] = \phi_\Sigma\big([x,y]\big)\,.$$

Hence ϕ_Σ is a derivation of L_Φ. □

ϕ_Σ is called a *ring derivation* of L_Φ.

4. The derivation algebra of L_Φ

If $n > 1$, for each fixed k $(1 \le k \le n-1)$, we assume that $n = kq + p$ with q and p two non-negative integers and $p \le k-1$. Let $D_k = \operatorname{diag}\big(E_k, 2E_k, \dots, qE_k, (q+1)E_p\big) \in d(n,R)$, $k = 1, 2, \dots, n-1$ (where E_k denotes the $k\times k$ identity matrix). Let $\Phi = \big\{A_{i,j} \in I(R) \mid 1 \le i < j \le n\big\}$ be a flag of ideals of R, we denote $\sum_{1\le i\ne j\le n} A_{i,j}E_{i,j}$ by w.

Theorem 4.1. *Let R be an arbitrary commutative ring with identity, $n \ge 1$,*

$$L_\Phi = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}E_{i,j}$$

a subalgebra of $\mathrm{gl}(n,R)$ *containing* $d(n,R)$ *with* $\Phi = \{A_{i,j} \in I(R) \mid 1 \le i < j \le n\}$ *a flag of ideals of R. Then every derivation of L_Φ may be uniquely written as the* *sum of an inner derivation induced by an element in w, a transpose derivation and a ring derivation.*

Proof. If $n = 1$, then it's easy to determine Der L_Φ. From now on, we assume that $n > 1$. Let ϕ be a derivation of L_Φ. In the following we give the proof by steps.

Step 1: There exists $W_0 \in w$ such that $d(n, R)$ is stable under $\phi + \text{ad } W_0$.

For $k = 1, 2, \ldots, n$, we set $v_k = \sum_{i=k}^{n} \sum_{j=1}^{i-k+1} A_{i,j} E_{i,j}$. Denote $L_\Phi \cap t(n, R)$ by t. For any $H \in d(n, R)$, suppose that

$$\phi(H) \equiv (\sum_{1\le i<j\le n} a_{j,i}(H) E_{j,i})(\mod t),$$

where $a_{j,i}(H) \in A_{j,i}$ are relative to H. By $[D_1, H] = 0$, we have that

$$\left[H, \phi(D_1)\right] = \left[D_1, \phi(H)\right],$$

which follows that

$$\sum_{1\le i<j\le n} \big(\chi_j(H) - \chi_i(H)\big) a_{j,i}(D_1) E_{j,i} = \sum_{1\le i<j\le n} \big(\chi_j(D_1) - \chi_i(D_1)\big) a_{j,i}(H) E_{j,i}\,.$$

This yields that

$$\big(\chi_j(H) - \chi_i(H)\big) a_{j,i}(D_1) = \big(\chi_j(D_1) - \chi_i(D_1)\big) a_{j,i}(H)\,, \quad \forall\, i, j (1 \le i < j \le n-1)\,.$$

In particular, we have that

$$a_{i+1,i}(H) = \big(\chi_{i+1}(H) - \chi_i(H)\big) a_{i+1,i}(D_1)\,, \quad i = 1, 2, \ldots, n\,.$$

Let $X_1 = \sum_{i=1}^{n-1} a_{i+1,i}(D_1) E_{i+1,i} \in L_\Phi$, then $(\phi + \text{ad } X_1)\big(d(n, R)\big) \subseteq t + v_3$. If $n = 2$, this step is completed. If $n > 2$, for any $H \in d(n, R)$, we now suppose that

$$(\phi + \text{ad } X_1)(H) \equiv \Big(\sum_{1\le i<j\le n-1} b_{j+1,i}(H) E_{j+1,i}\Big)(\mod t)\,,$$

where $b_{j+1,i}(H) \in A_{j+1,i}$ are relative to H. By $[D_2, H] = 0$, we have that

$$\left[H, (\phi + \text{ad } X_1)(D_2)\right] = \left[D_2, (\phi + \text{ad } X_1)(H)\right],$$

which follows that

$$\sum_{1\le i<j\le n-1} (\chi_{j+1}(H) - \chi_i(H)) b_{j+1,i}(D_2) E_{j+1,i} = \sum_{1\le i<j\le n-1} (\chi_{j+1}(D_2) - \chi_i(D_2)) b_{j+1,i}(H) E_{j+1,i}\,.$$

This yields that

$$(\chi_{j+1}(H) - \chi_i(H)) b_{j+1,i}(D_2) = (\chi_{j+1}(D_2) - \chi_i(D_2)) b_{j+1,i}(H)\,,$$

for all $i, j (1 \le i < j \le n - 1)$. In particular, we have that

$$b_{i+2,i}(H) = \big(\chi_{i+2}(H) - \chi_i(H)\big) b_{i+2,i}(D_2)\,, \quad i = 1, 2, \ldots, n - 2\,.$$

Let $X_2 = \sum_{i=1}^{n-2} b_{i+2,i}(D_2) E_{i+2,i}$, then $(\phi + \text{ad } X_1 + \text{ad } X_2)\big(d(n, R)\big) \subseteq t + v_4$. If $n = 3$, this step is completed. If $n > 3$, we repeat above process. After $n - 2$ steps, we may assume that $\big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\big)\big(d(n,R)\big)\subseteq t+v_n$. For any $H\in d$, suppose that $\big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\big)(H)\equiv c_{n,1}(H)E_{n,1}(\mod t)$, where $c_{n,1}(H)\in A_{n,1}$ is relative to H. By $[D_{n-1},H]=0$, we have that

$$\Big[H,\Big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\Big)(D_{n-1})\Big]=\Big[D_{n-1},\Big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\Big)(H)\Big],$$

which follows that

$$\big(\chi_n(H)-\chi_1(H)\big)c_{n,1}(D_{n-1})=\big(\chi_n(D_{n-1})-\chi_1(D_{n-1})\big)c_{n,1}(H).$$

So we have that

$$c_{n,1}(H)=\big(\chi_n(H)-\chi_1(H)\big)c_{n,1}(D_{n-1}).$$

Let $X_{n-1}=c_{n,1}(D_{n-1})E_{n,1}$, then $\big(\phi+\sum_{i+1}^{n-1}\text{ad } X_i\big)\big(d(n,R)\big)\subseteq t$. If we choose $X_0=\sum_{i=1}^{n-1}X_i$, then $(\phi+\text{ad } X_0)\big(d(n,R)\big)\subseteq t$.

Similarly, we may further choose $Y_0\in\sum_{j=1}^{n}\sum_{i=1}^{j-1}A_{i,j}E_{i,j}$ (the process is omitted) such that $(\phi+\text{ad } X_0+\text{ad } Y_0)\big(d(n,R)\big)\subseteq d(n,R)$.

Thus we may choose $W_0=X_0+Y_0\in w$ such that $(\phi+\text{ad } W_0)\big(d(n,R)\big)\subseteq d(n,R)$. Denote $\phi+\text{ad } W_0$ by ϕ_1, then $\phi_1\big(d(n,R)\big)\subseteq d(n,R)$.

Step 2: If $k\neq l$, then $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under ϕ_1.

For any fixed $b_{k,l}\in A_{k,l}$, we suppose that $\phi_1(b_{k,l}E_{k,l})=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}$, where $a_{i,j}\in A_{i,j}$. By applying ϕ_1 to $[E_{k,k},b_{k,l}E_{k,l}]=b_{k,l}E_{k,l}$, we have that

$$\big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}\big]+\big[E_{k,k},\phi_1(b_{k,l}E_{k,l})\big]=\phi_1(b_{k,l}E_{k,l}).$$

This follows that

$$(*)\qquad \big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}\big]+\Big[E_{k,k},\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\Big]=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\dots$$

Note that $\phi_1(E_{k,k})\in d(n,R)$ (by Step 1), thus $\big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}\big]\in A_{k,l}E_{k,l}$. It is easy to see that $\big[E_{k,k},\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\big]=\sum_{j=1}^{n}a_{k,j}E_{k,j}-\sum_{i=1}^{n}a_{i,k}E_{i,k}$. By comparing the two sides of $(*)$, we see that $a_{i,j}=0$ when $i\neq k$ and $j\neq k$. For the same reason, we know that $a_{i,j}=0$ when $i\neq l$ and $j\neq l$. Hence $\phi_1(b_{k,l}E_{k,l})\in A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$, which leads to $\phi_1(A_{k,l}E_{k,l})\subseteq A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$. Similarly, $\phi_1(A_{l,k}E_{l,k})\subseteq A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$. So $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under ϕ_1.

Step 3: There exists a ring derivation ϕ_Σ such that each $A_{k,l}E_{k,l}$ $(k\neq l)$ is send by $\phi_1-\phi_\Sigma$ to $A_{l,k}E_{l,k}$ and $d(n,R)$ is send by it to 0.

We denote the the restriction of ϕ_1 to $d(n,R)$ by σ, and let $\sigma_{i,i}\colon A_{i,i}\to A_{i,i}$ be zero. By Step 2, we know that $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under ϕ_1 if $k\neq l$. Now for any k,l $(1\leq k,l\leq n)$ we define the map $\sigma_{k,l}$ from $A_{k,l}$ to itself according to the following rule:

(a) $\sigma_{k,l}=0$ when $k=l$;

(b) If $k\neq l$, define $\sigma_{k,l}\colon A_{k,l}\to A_{k,l}$ such that for any $a_{k,l}\in A_{k,l}$, $\sigma_{k,l}(a_{k,l})$ satisfies the condition: $\phi_1(a_{k,l}E_{k,l})\equiv\sigma_{k,l}(a_{k,l})E_{k,l}$ $(\mod A_{l,k}E_{l,k})$.

Then σ, $\sigma_{k,l}$ $(k \neq l)$ are all endomorphism of the R-modules. Set $\Sigma = \{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}), \sigma \mid 1 \leq i, j \leq n\}$. We intend to prove that Σ is suitable for ring derivations.

For all $D \in d(n,R)$, $a_{i,j} \in A_{i,j}$, by applying ϕ_1 to $[D, a_{i,j}E_{i,j}] = \big(\chi_i(D) - \chi_j(D)\big)a_{i,j}E_{i,j}$, we have that $a_{i,j}\big(\chi_i(\sigma(D)) - \chi_j(\sigma(D))\big) = 0$, leads to $\chi_i\big(\sigma(D)\big) - \chi_j(\sigma(D) \in B_{i,j}$. Similarly, we may prove that $\chi_i\big(\sigma(D)\big) - \chi_j\big(\sigma(D)\big) \in B_{j,i}$.

For all i,j $(1 \leq i,j \leq n)$, $\forall a_{i,j} \in A_{i,j}$, $a_{j,i} \in A_{j,i}$, by applying ϕ_1 to $[a_{i,j}E_{i,j}, a_{j,i}E_{j,i}] = a_{i,j}a_{j,i}(E_{i,i} - E_{j,j})$, we have that $\sigma\big(a_{i,j}a_{j,i}(E_{i,i} - E_{j,j})\big) = \big(\sigma_{i,j}(a_{i,j})a_{j,i} + a_{i,j}\sigma_{j,i}(a_{j,i})\big)(E_{i,i} - E_{j,j})$.

When $i \neq j$, for all $a_{i,k} \in A_{i,k}$, $a_{k,j} \in A_{k,j}$, by applying ϕ_1 to $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j}$, we have that

$$\big[\sigma_{i,k}(a_{i,k})E_{i,k}, a_{k,j}E_{k,j}\big] + \big[a_{i,k}E_{i,k}, \sigma_{k,j}(a_{k,j})E_{k,j}\big] = \sigma_{i,j}(a_{i,k}a_{k,j})E_{i,j}\,.$$

This shows that

$$\sigma_{i,j}(a_{i,k}a_{k,j}) = \sigma_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(a_{k,j})\,.$$

Now we see that Σ is suitable for ring derivations. Using Σ we construct the ring derivation ϕ_Σ as in Section 3, and denote $\phi_1 - \phi_\Sigma$ by ϕ_2. Then we see that $\phi_2(A_{k,l}E_{k,l}) \subseteq A_{l,k}E_{l,k}$ for all k, l satisfy $k \neq l$ and ϕ_2 sends $d(n,R)$ to 0.

Step 4: ϕ_2 exactly is a transpose derivation.

By Step 3, we know that $A_{k,l}E_{k,l}$ is send by ϕ_2 to $A_{l,k}E_{l,k}$ when $k \neq l$ and $d(n,R)$ is send by it to 0. Now for any k, l $(1 \leq k,\ l \leq n)$ we define the map $\pi_{k,l}$ from $A_{k,l}$ to $A_{l,k}$ according to the following rule:

(a) $\pi_{k,l} = 0$ when $k = l$;

(b) If $k \neq l$, define $\pi_{k,l} \colon A_{k,l} \to A_{l,k}$ such that for any $a_{k,l} \in A_{k,l}$, $\sigma_{k,l}(a_{k,l})$ satisfies the condition: $\phi_2(a_{k,l}E_{k,l}) = \pi_{k,l}(a_{k,l})E_{l,k}$.

Then $\sigma_{k,l}$ is an homomorphism from the R-module $A_{k,l}$ to $A_{l,k}$. Set $\Pi = \{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \leq i,j \leq n\}$. We intend to prove that Π is suitable for transpose derivations. If $i \neq j$, for $\forall a_{i,k} \in A_{i,k}$, $\forall a_{k,j} \in A_{k,j}$, by applying ϕ_2 to $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j}$, we have that

$$\big[\pi_{i,k}(a_{i,k})E_{k,i}, a_{k,j}E_{k,j}\big] + \big[a_{i,k}E_{i,k}, \pi_{k,j}(a_{k,j})E_{j,k}\big] = \pi_{i,j}(a_{i,k}a_{k,j})E_{j,i}\,.$$

If $k \neq i$, $k \neq j$, we see that the left side of above is 0, then $\pi_{i,j}(a_{i,k}a_{k,j}) = 0$, leads to $\pi_{i,j}(A_{i,k}A_{k,j}) = 0$.

If $i \neq k$, $i \neq j$, $\forall a_{i,k} \in A_{i,k}$, $\forall a_{i,j} \in A_{i,j}$, by applying ϕ_2 to $[a_{i,k}E_{i,k}, a_{i,j}E_{i,j}] = 0$, we see that

$$\big[\pi_{i,k}(a_{i,k})E_{k,i}, a_{i,j}E_{i,j}\big] + \big[a_{i,k}E_{i,k}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = 0\,.$$

This shows that

$$\pi_{i,k}(a_{i,k})a_{i,j}E_{k,j} - a_{i,k}\pi_{i,j}(a_{i,j})E_{j,k} = 0\,.$$

Thus $a_{i,k}\pi_{i,j}(a_{i,j}) = 0$, leads to $A_{i,k}\pi_{i,j}(A_{i,j}) = 0$ for $i \neq k$. Similarly, $A_{k,j}\pi_{i,j}(A_{i,j}) = 0$ for $k \neq j$.

For all $i \neq j$, $\forall a_{i,j} \in A_{i,j}$, by applying ϕ_2 to $[E_{i,i}, a_{i,j}E_{i,j}] = a_{i,j}E_{i,j}$, we have that

$$\big[E_{i,i}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = \pi_{i,j}(a_{i,j})E_{j,i}\,.$$

Since $\big[E_{i,i}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = -\pi_{i,j}(a_{i,j})E_{j,i}$, we see that $\pi_{i,j}(a_{i,j}) = -\pi_{i,j}(a_{i,j})E_{j,i}$. So $2\pi_{i,j}(A_{i,j}) = 0$ for $i \neq j$. Then $2\pi_{i,j}(A_{i,j}) = 0$ for $\forall i,\ j$.

Now we see that Π is suitable for transpose derivations. Using Π we construct the transpose derivation ϕ_Π as in Section 3, and denote $\phi_2 - \phi_\Pi$ by ϕ_3. Then we see that $\phi_3(A_{k,l}E_{k,l}) = 0$ for all k, l satisfy $k \neq l$ and $\phi_3(d(n,R)) = 0$. So $\phi_3 = 0$.

Thus $\phi = \phi_\Pi + \phi_\Sigma - \text{ad } W_0$, as desired.

For the uniqueness of the decomposition of ϕ, we first prove that if $\phi_\Pi + \phi_\Sigma + \text{ad } W_0 = 0$, then $\phi_\Pi = \phi_\Sigma = \text{ad } W_0 = 0$. Suppose that $\phi_\Pi + \phi_\Sigma + \text{ad } W_0 = 0$, where $W_0 \in w$ and ϕ_Π, ϕ_Σ are the the transpose and the ring derivation of L_Φ, respectively. By $(\phi_\Pi + \phi_\Sigma + \text{ad } W_0)(d(n,R)) = 0$, we easily see that $W_0 = 0$. Then we have that $\phi_\Pi + \phi_\Sigma = 0$. By applying $\phi_\Pi + \phi_\Sigma$ to $a_{i,j}E_{i,j}$ for $1 \leq i \neq j \leq n, a_{i,j} \in A_{i,j}$, we have that $\sigma_{i,j}(a_{i,j})E_{i,j} + \pi_{i,j}(a_{i,j})E_{j,i} = 0$, leads to $\sigma_{i,j}(a_{i,j}) = \pi_{i,j}(a_{i,j}) = 0$. This forces that $\phi_\Pi = \phi_\Sigma = 0$. Now suppose that

$$\phi = \phi_{\Pi_1} + \phi_{\Sigma_1} - \text{ad } W_1 = \phi_{\Pi_2} + \phi_{\Sigma_2} - \text{ad } W_2\,,$$

is two decompositions of ϕ. Then we have that

$$(\phi_{\Pi_1} - \phi_{\Pi_2}) + (\phi_{\Sigma_1} - \phi_{\Sigma_2}) + (\text{ad } W_2 - \text{ad } W_1) = 0\,.$$

Note that $\phi_{\Pi_1} - \phi_{\Pi_2}$ (resp., $\phi_{\Sigma_1} - \phi_{\Sigma_2}$) is also a transpose (resp., ring) derivation of L_Φ and $\text{ad } W_2 - \text{ad } W_1 = ad\ (W_2 - W_1)$. This implies that $\phi_{\Sigma_1} = \phi_{\Sigma_2}$, $\phi_{\Pi_1} = \phi_{\Pi_2}$ and $\text{ad } W_1 = \text{ad } W_2$. □

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Department of Mathematics
China University of Mining and Technology
Xuzhou, 221008, People's Republic of China
E-mail: wdengyin@126.com

Graduate School of Natural Science and Technology
Okayama University
Okayama 700-8530, Japan