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GENERAL IMPLICIT VARIATIONAL INCLUSION PROBLEMS INVOLVING *A*-MAXIMAL RELAXED ACCRETIVE MAPPINGS IN BANACH SPACES

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Abstract. A class of existence theorems in the context of solving a general class of nonlinear implicit inclusion problems are examined based on A-maximal relaxed accretive mappings in a real Banach space setting.

1. Introduction

We consider a real Banach space X with X^*, its dual space. Let $\|\cdot\|$ denote the norm on X and X^*, and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between X and X^*. We consider the implicit inclusion problem: determine a solution $u \in X$ such that

$$0 \in A(u) + M\big(g(u)\big)\,, \tag{1}$$

where A, $g\colon X \to X$ are single-valued mappings, and $M\colon X \to 2^X$ is a set-valued mapping on X such that $\operatorname{range}(g) \cap \operatorname{dom}(M) \neq \emptyset$.

Recently, Huang, Fang and Cho [4] applied a three-step algorithmic process to approximating the solution of a class of implicit variational inclusion problems of the form (1) in a Hilbert space. In their investigation, they used the resolvent operator of the form $J_\rho^M = (I + \rho M)^{-1}$ for $\rho > 0$, in a Hilbert space setting. Here we generalize the existence results to the case of A-maximal relaxed accretive mappings in a real uniformly smooth Banach space setting. As matter of fact, the obtained results generalize their investigation to the case of H-maximal accretive mappings as well. For more literature, we refer the reader to [2]–[20].

2. *A*-maximal relaxed accretiveness

In this section we discuss some basic properties and auxiliary results on A-maximal relaxed accretiveness. Let X be a real Banach space and X^* be the dual space of X. Let $\|\cdot\|$ denote the norm on X and X^* and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between X and X^*. Let $M\colon X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\big\{(x,y) : y \in M(x)\big\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and

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$M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$D(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}\,.$$

$D(M) = X$, shall denote the full domain of M, and the range of M is defined by

$$R(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}\,.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}\,.$$

As we prepare for basic notions, we start with the generalized duality mapping $J_q \colon X \to 2^{X^*}$, that is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\} \;\forall\, x \in X\,,$$

where $q > 1$. As a special case, J_2 is the normalized duality mapping, and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. Next, as we are heading to uniformly smooth Banach spaces, we define the modulus of smoothness $\rho_X \colon [0, \infty) \to [0, \infty)$ by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space X is uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0\,,$$

and X is $q-$uniformly smooth if there is a positive constant c such that

$$\rho_X(t) \leq c t^q\,, \quad q > 1\,.$$

Note that J_q is single-valued if X is uniformly smooth. In this context, we state the following Lemma from Xu [17].

Lemma 2.1 ([17]). *Let X be a uniformly smooth Banach space. Then X is q-uniformly smooth if there exists a positive constant c_q such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q \|y\|^q\,.$$

Lemma 2.2. *For any two nonnegative real numbers a and b, we have*

$$(a + b)^q \leq 2^q (a^q + b^q)\,.$$

Definition 2.1. Let $M \colon X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) $(r)-$ strongly accretive if there exists a positive constant r such that

$$\langle u^* - v^*, J_q(u - v)\rangle \geq r\|u - v\|^q \,\forall\, (u, u^*), (v, v^*) \in \operatorname{graph}(M)\,.$$

(ii) $(m)-$relaxed accretive if there exists a positive constant m such that

$$\langle u^* - v^*, J_q(u - v)\rangle \geq (-m)\|u - v\|^q \,\forall\, (u, u^*), (v, v^*) \in \operatorname{graph}(M)\,.$$

Definition 2.2 ([5]). Let $A\colon X \to X$ be a single-valued mapping. The map $M\colon X \to 2^X$ is said to be A- maximal (m)-relaxed accretive if:

(i) M is (m)-relaxed accretive for $m > 0$.

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.3 ([5]). Let $A\colon X \to X$ be an (r)-strongly accretive mapping and let $M\colon X \to 2^X$ be an A-maximal accretive mapping. Then the generalized resolvent operator $J^M_{\rho,A}\colon X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u)\,.$$

Definition 2.4 ([2]). Let $H\colon X \to X$ be (r)-strongly accretive. The map $M\colon X \to 2^X$ is said to be to H-maximal accretive if

(i) M is accretive,

(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Definition 2.5. Let $H\colon X \to X$ be an (r)-strongly accretive mapping and let $M\colon X \to 2^X$ be an H-accretive mapping. Then the generalized resolvent operator $J^M_{\rho,H}\colon X \to X$ is defined by

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u)\,.$$

Proposition 2.1 ([5]). *Let $A\colon X \to X$ be an (r)-strongly accretive single-valued mapping and let $M\colon X \to 2^X$ be an A-maximal (m)-relaxed accretive mapping. Then $(A + \rho M)$ is maximal accretive for $\rho > 0$.*

Proposition 2.2 ([5]). *Let $A\colon X \to X$ be an (r)-strongly accretive mapping and let $M\colon X \to 2^X$ be an A-maximal relaxed accretive mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

Proposition 2.3 ([2]). *Let $H\colon X \to X$ be a (r)-strongly accretive single-valued mapping and let $M\colon X \to 2^X$ be an H-maximal accretive mapping. Then $(H+\rho M)$ is maximal accretive for $\rho > 0$.*

Proposition 2.4 ([2]). *Let $H\colon X \to X$ be an (r)-strongly accretive mapping and let $M\colon X \to 2^X$ be an H-maximal accretive mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued.*

3. Existence theorems

This section deals with the existence theorems on solving the implicit inclusion problem (1) based on the $A-$ maximal relaxed accretiveness.

Lemma 3.1 ([5]). *Let X be a real Banach space, let $A\colon X \to X$ be (r)-strongly accretive, and let $M\colon X \to 2^X$ be A-maximal relaxed accretive. Then the generalized resolvent operator associated with M and defined by*

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u)\,\forall\, u \in X\,,$$

is $\left(\frac{1}{r-\rho m}\right)$-Lipschitz continuous for $r - \rho m > 0$.

Lemma 3.2. *Let X be a real Banach space, let $A\colon X \to X$ be (r)-strongly accretive, and let $M\colon X \to 2^X$ be A-maximal (m)-relaxed accretive. In addition, let $g\colon X \to X$ be a (β)-Lipschitz continuous mapping on X. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,A}^{M}(u) = (A + \rho M)^{-1}(u)\,\forall\, u \in X\,,$$

satisfies

$$\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\| \leq \frac{\beta}{r - \rho m}\|u - v\|\,,$$

where $r - \rho m > 0$.

Furthermore, we have

$$\langle J_q(J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))), g(u) - g(v)\rangle \geq (r - \rho m)\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\|^q\,,$$

where $r - \rho m > 0$.

Proof. For any elements $u, v \in X$ (and hence $g(u), g(v) \in X$), we have from the definition of the resolvent operator $J_{\rho,A}^{M}$ that

$$\frac{1}{\rho}\big[g(u) - A\big(J_{\rho,A}^{M}(g(u))\big)\big] \in M\big(J_{\rho,A}^{M}(g(u))\big)\,,$$

and

$$\frac{1}{\rho}\big[g(v) - A\big(J_{\rho,A}^{M}(g(v))\big)\big] \in M\big(J_{\rho,A}^{M}(g(v))\big)\,.$$

Since M is A-maximal (m)-relaxed accretive, it implies that

$$\begin{aligned}\big\langle g(u) - g(v) - \big[A\big(J_{\rho,A}^{M}(g(u))\big) - A\big(J_{\rho,A}^{M}(g(v))\big)\big], J_q\big(J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big)\big\rangle \\ \geq (-\rho m)\big\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big\|^q\,.\end{aligned} \tag{2}$$

Based on (2), using the (r)-strong accretiveness of A, we get

$$\begin{aligned}\big\langle g(u)\ &- g(v), J_q\big(J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big)\big\rangle \\ \geq&\big\langle A\big(J_{\rho,A}^{M}(g(u))\big) - A\big(J_{\rho,A}^{M}(g(v))\big), J_q\big(J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big)\big\rangle \\ &- \rho m\big\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big\|^q \\ \geq&(r - \rho m)\big\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big\|^q\,.\end{aligned}$$

Therefore,

$$\big\langle g(u) - g(v), J_q\big(J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big)\big\rangle \geq (r - \rho m)\big\|J_{\rho,A}^{M}(g(u)) - J_{\rho,A}^{M}(g(v))\big\|^q\,.$$

This completes the proof. □

Theorem 3.1. *Let X be a real Banach space, let $A\colon X \to X$ be (r)-strongly accretive, and let $M\colon X \to 2^X$ be A-maximal (m)-relaxed accretive. Let $g\colon X \to X$ be a map on X. Then the following statements are equivalent:*

(i) *An element $u \in X$ is a solution to* (1).

(ii) *For an $u \in X$, we have*

$$g(u) = J_{\rho,A}^{M}\big(A(g(u)) - \rho A(u)\big)\,,$$

where

$$J_{\rho,A}^{M}(u) = (A+\rho M)^{-1}(u)\,.$$

Proof. It follows from the definition of the resolvent operator $J_{\rho,A}^{M}$. □

Theorem 3.2. *Let X be a real Banach space, let $H\colon X \to X$ be (r)-strongly accretive, and let $M\colon X \to 2^X$ be H-maximal accretive. Let $g\colon X \to X$ be a map on X. Then the following statements are equivalent:*

(i) *An element $u \in X$ is a solution to* (1)*.*

(ii) *For an $u \in X$, we have*

$$g(u) = J_{\rho,H}^{M}\big(H(g(u)) - \rho H(u)\big)\,,$$

where

$$J_{\rho,H}^{M}(u) = (H+\rho M)^{-1}(u)\,.$$

Theorem 3.3. *Let X be a real q-uniformly smooth Banach space, let $A\colon X \to X$ be (r)-strongly accretive and (s)-Lipschitz continuous, and let $M\colon X \to 2^X$ be A-maximal (m)-relaxed accretive. Let $g\colon X \to X$ be (t)-strongly accretive and (β)-Lipschitz continuous. Then there exists a unique solution $x^* \in X$ to* (1) *for*

$$\begin{aligned}\theta =&\Big(1+\frac{1}{r-\rho m}\Big)\sqrt[q]{1-qt+c_q\beta^q} + \frac{1}{r-\rho m}\sqrt[q]{\beta^q - qrt^q + c_q s^q\beta^q}\\ &+\frac{1}{r-\rho m}\sqrt[q]{1-qr\rho + c_q\rho^q s^q} < 1\,,\end{aligned} \tag{3}$$

for $r-\rho m > 1$ and $c_q > 0$.

Proof. First we define a function $F\colon X \to X$ by

$$F(u) = u - g(u) + J_{\rho,A}^{M}\big(A(g(u)) - \rho A(u)\big)\,,$$

and then prove that F is contractive. Applying Lemma 3.1, we estimate

$$\begin{aligned}\|F(u)-F(v)\| =&\big\|u-v-(g(u)-g(v)) + J_{\rho,A}^{M}\big(A(g(u))-\rho A(u)\big)\\ &- J_{\rho,A}^{M}(A(g(v))-\rho A(v))\big\|\\ \le&\big\|u-v-(g(u)-g(v))\big\| + \frac{1}{r-\rho m}\big\|A(g(u))\\ &- A(g(v)) - \rho(A(u)-A(v))\big\|\\ \le&\Big(1+\frac{1}{r-\rho m}\Big)\big\|u-v-(g(u)-g(v))\big\|\\ &+\frac{1}{r-\rho m}\big\|A(g(u))-A(g(v))-(g(u)-g(v))\big\|\\ &+\frac{1}{r-\rho m}\big\|u-v-\rho(A(u)-A(v))\big\|\,.\end{aligned} \tag{4}$$

Since g is (t)-strongly accretive and (β)-Lipschitz continuous, we have

$$\begin{aligned}\big\|u-v-(g(u)-g(v))\big\|^q &= \|u-v\|^q - q\big\langle g(u)-g(v), J_q(u-v)\big\rangle + c_q\|g(u)-g(v)\|^q\\ &\le \|u-v\|^q - qt\|u-v\|^q + c_q\beta^q\|u-v\|^q\\ &= (1-qt+c_q\beta^q)\|u-v\|^q\,.\end{aligned}$$

Therefore, we have

$$\big\|u - v - (g(u) - g(v))\big\| \le \sqrt[q]{1 - qt + c_q\beta^q}\,. \tag{5}$$

Similarly, based on the strong accretiveness and Lipschitz continuity of A and g, we get

$$\big\|A(g(u)) - A(g(v)) - (g(u) - g(v))\big\| \le \sqrt[q]{\beta^q - qrt^q + c_q s^q \beta^q}\,, \tag{6}$$

and

$$\big\|u - v - \rho(A(u) - A(v))\big\| \le \sqrt[q]{1 - qr\rho + c_q\rho^q s^q}\,. \tag{7}$$

In light of above arguments, we have

$$\|F(u) - F(v)\| \le \theta\|u - v\|\,, \tag{8}$$

where

$$\begin{aligned} \theta = \Big(1 + \frac{1}{r - \rho m}\Big)\sqrt[q]{1 - qt + c_q\beta^q} + \frac{1}{r - \rho m}\sqrt[q]{\beta^q - qrt^q + c_q s^q\beta^q} \\ + \frac{1}{r - \rho m}\sqrt[q]{1 - qr\rho + c_q\rho^q s^q} < 1\,, \end{aligned} \tag{9}$$

for $r - \rho m > 1$. □

Corollary 3.1. *Let X be a real $q-$ uniformly smooth Banach space, let $H\colon X \to X$ be (r)- strongly accretive and (s)-Lipschitz continuous, and let $M\colon X \to 2^X$ be H-maximal accretive. Let $g\colon X \to X$ be (t)-strongly accretive and (β)-Lipschitz continuous. Then there exists a unique solution $x^* \in X$ to* (1) *for*

$$\begin{aligned} \theta = \Big(1 + \frac{1}{r}\Big)\sqrt[q]{1 - qt + c_q\beta^q} + \frac{1}{r}\sqrt[q]{\beta^q - qrt^q + c_q s^q\beta^q} \\ + \frac{1}{r}\sqrt[q]{1 - qr\rho + c_q\rho^q s^q} < 1\,, \end{aligned} \tag{10}$$

for $r > 1$.

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