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WEAK*-CONTINUOUS DERIVATIONS IN DUAL BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a dual Banach algebra. We investigate the first weak*-continuous cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{A}$. Hence, we obtain conditions on $\mathcal{A}$ for which

$$H^1_{w^*}(\mathcal{A},\mathcal{A}) = \{0\}\,.$$

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. The right and left actions of $\mathcal{A}$ on the dual space X^* of X can be defined as follows

$$\langle fa, b\rangle = \langle f, ab\rangle\,,\quad \langle af, b\rangle = \langle f, ba\rangle \qquad (a,b\in\mathcal{A},\ f\in X^*)\,.$$

Then X^* becomes a Banach $\mathcal{A}$-bimodule. For example, $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with respect to the product in $\mathcal{A}$. Then $\mathcal{A}^*$ is a Banach $\mathcal{A}$-bimodule.

The second dual space $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ admits a Banach algebra product known as the first (left) Arens product. We briefly recall the definition of this product.

By [1], for m, $n\in\mathcal{A}^{**}$, the first (left) Arens product indicated by mn is given by

$$\langle mn, f\rangle = \langle m, nf\rangle \qquad (f\in\mathcal{A}^*)\,,$$

where nf as an element of $\mathcal{A}^*$ is defined by

$$\langle nf, a\rangle = \langle n, fa\rangle \qquad (a\in\mathcal{A})\,.$$

A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = {\mathcal{A}_*}^*$. Let $\mathcal{A}$ be a dual Banach algebra. A dual Banach $\mathcal{A}$-bimodule X is called normal if, for every $x\in X$, the maps $a\longmapsto a\cdot x$ and $a\longmapsto x\cdot a$ are weak*-continuous from $\mathcal{A}$ into X. For example, if G is a locally compact topological group, then $M(G)$ is a dual Banach algebra with predual $C_0(G)$. Also, if $\mathcal{A}$ is an Arens regular Banach algebra, then $\mathcal{A}^{**}$ is a dual Banach algebra with predual $\mathcal{A}^*$.

If X is a Banach $\mathcal{A}$-bimodule then a derivation from $\mathcal{A}$ into X is a linear map D, such that for every $a,b\in\mathcal{A}$, $D(ab) = D(a)\cdot b + a\cdot D(b)$. If $x\in X$, and we define $\delta_x\colon\mathcal{A}\to X$ by $\delta_x(a) = a\cdot x - x\cdot a \quad (a\in\mathcal{A})$, then δ_x is a derivation. Derivations

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of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if every bounded derivation from $\mathcal{A}$ into dual of every Banach $\mathcal{A}$-bimodule X is inner; i.e., $H^1(\mathcal{A}, X^*) = \{0\}$, [10]. Let $n \in \mathbb{N}$, then a Banach algebra $\mathcal{A}$ is n-weakly amenable if every (bounded) derivation from $\mathcal{A}$ into n-th dual of $\mathcal{A}$ is inner; i.e., $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$ (see [4]). A dual Banach algebra $\mathcal{A}$ is Connes-amenable if every weak*-continuous derivation from $\mathcal{A}$ into each normal dual Banach $\mathcal{A}$-bimodule X is inner; i.e., $H^1_{w^*}(\mathcal{A}, X) = \{0\}$, this definition was introduced by V. Runde (see Section 4 of [15] or [6] and [7]). In this paper we study the weak*-continuous derivations from $\mathcal{A}$ into itself when $\mathcal{A}$ is a dual Banach algebra. Hence, we obtain conditions on $\mathcal{A}$ for which the following holds

$$(*) \qquad H^1_{w^*}(\mathcal{A}, \mathcal{A}) = \{0\}\,.$$

One can see that every Connes-amenable dual Banach algebra satisfies in $(*)$. We have already some examples to show that the condition $(*)$ does not imply Connes-amenability (see Corollary 2.3).

Example 1.1. Let $\mathcal{B}$ be a von-Neumann algebra. Then $H^1_{w^*}(\mathcal{B}, \mathcal{B}) \subseteq H^1(\mathcal{B}, \mathcal{B}) = \{0\}$ (Theorem 4.1.8 of [16]). Thus $\mathcal{B}$ satisfies $(*)$.

Example 1.2. Let $\mathcal{A}$ be a commutative semisimple dual Banach algebra, then by commutative Singer-Warmer theorem, (see for example [2, Section 18, Theorem 16]) we have $H^1(\mathcal{A}, \mathcal{A}) = \{0\}$, so $\mathcal{A}$ satisfies in $(*)$.

Let now $\mathcal{A}$ be a commutative Banach algebra which is Arens regular and let $\mathcal{A}^{**}$ be semisimple. Trivially $\mathcal{A}^{**}$ is commutative. Then $\mathcal{A}^{**}$ is a dual Banach algebra which satisfies $(*)$.

Let $\mathcal{A}$ be a Banach algebra. The Banach $\mathcal{A}$-submodule X of $\mathcal{A}^*$ is called left introverted if $\mathcal{A}^{**}X \subseteq X$ (i.e. $X^*X \subseteq X$). Let X be a left introverted Banach $\mathcal{A}-$ submodule of $\mathcal{A}^*$, then X^* by the following product is a Banach algebra:

$$\langle x'y', x\rangle = \langle x', y' \cdot x\rangle \qquad (x', y' \in X^*, x \in X)\,.$$

(See [1] for further details.) For each $y' \in X^*$, the mapping $x' \longmapsto x'y'$ is weak*-continuous. However, for certain x', the mapping $y' \longmapsto x'y'$ may fail to be weak*-continuous. Due to this lack of symmetry the topological center $Z_t(X^*)$ of X^* is defined by

$$Z_t(X^*) := \{x' \in X^* : y' \longmapsto x'\ y' : X^* \to X^* \quad \text{is weak}^*\text{-continuous}\}\,.$$

See [5] and [12] for further details. If $X = \mathcal{A}^*$, then $Z_t(X^*) = Z_t(\mathcal{A}^{**})$ is the left topological center of $\mathcal{A}^{**}$.

2. Main results

In this section we study the first weak*-continuous cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{A}$, when $\mathcal{A}$ is a dual Banach algebra. Indeed we show that an Arens regular Banach algebra $\mathcal{A}$ is 2-weakly amenable if and only if the second dual of $\mathcal{A}$ holds in $(*)$. So we prove that a dual Banach algebra $\mathcal{A}$ holds in $(*)$ if it is 2-weakly amenable.

We have the following lemma for the left introverted subspaces.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra and let X be a left introverted subspace of A^*. Then the followings are equivalent.*

(a) *X^* is a dual Banach algebra.*

(b) *$Z_t(X^*) = X^*$.*

(c) *$\widehat{X}$ the canonical image of X in its bidual, is a right X^*-submodule of X^{**}.*

Proof. $(a) \Longleftrightarrow (b)$ It follows from 4.4.1 of [15].

$(b) \to (c)$ Let $x \in X$, $x' \in X^*$ and let $y'_\alpha \xrightarrow{\text{weak}^*} y'$ in X^*. Then by (b), $x'y'_\alpha \xrightarrow{\text{weak}^*} x'y'$ in X^*. So we have

$$\langle \widehat{x}x', y'_\alpha \rangle = \langle \widehat{x}, x'y'_\alpha \rangle = \langle x'y'_\alpha, x \rangle \to \langle x'y', x \rangle = \langle \widehat{x}, x'y' \rangle = \langle \widehat{x}x', y' \rangle \,.$$

It follows that $\widehat{x}x' \colon X^* \to \mathbb{C}$ is weak*-continuous. Thus $\widehat{x}x' \in \widehat{X}$.

$(c) \Longrightarrow (b)$ Let $x' \in X^*$ and let $y'_\alpha \xrightarrow{\text{weak}^*} y'$ in X^*. Then for every $x \in X$, we have

$$\langle x'y'_\alpha, x \rangle = \langle y'_\alpha, xx' \rangle \to \langle y', xx' \rangle = \langle x'y', x \rangle \,.$$

Then (b) holds. □

Let G be a locally compact topological group, then the dual Banach algebra $M(G)$ is Connes-amenable if and only if $L^1(G)$ is amenable (see Section 4 of [15]). Also $L^1(G)$ is always weakly amenable (see [11] or [8]). In the following we show that $M(G)$ has condition $(*)$.

Theorem 2.2. *For every locally compact topological group G, $M(G)$ has the condition $(*)$.*

Proof. Let $D \colon M(G) \to M(G)$ be a weak*-continuous derivation, since $L^1(G)$ is a two sided ideal in $M(G)$, then for every $a, b \in L^1(G)$, we have $D(ab) = D(a) \cdot b + a \cdot D(b)$ belongs to $L^1(G)$. We know that for every (bounded) derivation $D \colon L^1(G) \to L^1(G)$, there is a $\mu \in M(G)$ such that for every $a \in L^1(G)$, $D(a) = a\mu - \mu a$, [13, Corollary 1.2]. On the other hand $L^1(G)$ is weak*-dense in $M(G)$, and D is weak*-continuous. Then $D(a) = a\mu - \mu a$ for all $a \in M(G)$. □

Corollary 2.3. *If G is a non-amenable group, then $M(G)$ is a dual Banach algebra satisfies in $(*)$, but is not Connes-amenable.*

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and let X be a left introverted $\mathcal{A}$-submodule of A^* such that $D^*|_X \colon X \to \mathcal{A}^*$ taking values in X for every derivation $D \colon \mathcal{A} \to X^*$. If $Z_t(X^*) = X^*$, then the followings are equivalent.*

(a) *X^* has the condition $(*)$.*

(b) *$H^1(\mathcal{A}, X^*) = \{0\}$.*

Proof. $(a) \Longrightarrow (b)$ Let $D \colon \mathcal{A} \to X*$ be a (bounded) derivation. Then, by Proposition 1.7 of [4], $D^{**} \colon \mathcal{A}^{**} \to (X^*)^{**}$ the second transpose of D is a derivation. We define $D_1 \colon X^* \to X^*$ by

$$\langle D_1(x'), x \rangle = \langle D^{**}(x'), \widehat{x} \rangle \qquad (x' \in X^*, x \in X) \,.$$

Since $Z_t(X^*) = X^*$, then by Lemma 2.1, $\widehat{X}$ is a X^*-submodule of X^{**}. Then for every $x', y' \in X^*$ and $x \in \mathcal{A}^*$, we have

$$\begin{aligned}\langle D_1(x'y'), x\rangle &= \langle D^{**}(x'y'), \widehat{x}\rangle = \langle D^{**}(x')y', \widehat{x}\rangle + \langle x'D^{**}(y'), \widehat{x}\rangle \\ &= \langle D^{**}(x'), y'\widehat{x}\rangle + \langle D^{**}(y'), \widehat{x}x'\rangle = \langle D^{**}(x'), \widehat{y'x}\rangle + \langle D^{**}(y'), \widehat{xx'}\rangle \\ &= \langle D_1(x'), y'x\rangle + \langle D_1(y'), xx'\rangle = \langle D_1(x')y', x\rangle + \langle x'D_1(y'), x\rangle\,.\end{aligned}$$

So D_1 is a derivation. Now let $x'_\alpha \xrightarrow{\text{weak}^*} x'$ in X^*. Since D^{**} is weak*-continuous, then for every $x \in X$, we have

$$\lim_\alpha \langle D_1(x'_\alpha), x\rangle = \lim_\alpha \langle D^{**}(x'_\alpha), \widehat{x}\rangle = \langle D^{**}(x'), \widehat{x}\rangle = \langle D_1(x'), x\rangle\,.$$

It follows that D_1 is weak*-weak*-continuous. Then there exists $x' \in X^*$ such that $D_1 = \delta_{x'}$, so $D = \delta_{x'}$.

$(b) \Longrightarrow (a)$ Let $D\colon X^* \to X^*$ be a weak*-continuous derivation, then $D\mid_{\mathcal{A}}\colon \mathcal{A} \to X^*$ is a bounded derivation. Thus, there is $x' \in X^*$ such that $D(\widehat{a}) = \widehat{a}x' - x'\widehat{a}$ for every $a \in \mathcal{A}$. Since X^* is a dual Banach algebra, then $\delta_{x'}\colon X^* \to X^*$ is weak*-continuous. On the other hand $\widehat{\mathcal{A}}$ is weak*-dense in X^*, and D is weak*-continuous, then we have $D = \delta_{x'}$. □

Corollary 2.5. *Let $\mathcal{A}$ be an Arens regular Banach algebra, then $\mathcal{A}^{**}$ has the condition $(*)$ if and only if $\mathcal{A}$ is 2-weakly amenable.*

Theorem 2.6. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is 2-weakly amenable, then $\mathcal{A}$ has the condition $(*)$.*

Proof. Let $\mathcal{A}$ be a dual algebra with predual $\mathcal{A}_*$, and let $D\colon \mathcal{A} \to \mathcal{A}$ be a weak*-continuous derivation, then D is bounded. In other wise, there exists a sequence $\{x_n\}$ in $\mathcal{A}$ such that $\lim_n \|x_n\| = 0$ and $\lim_n \|D(x_n)\| = \infty$. By uniform boundedness theorem, $D(x_n) \overset{\text{weak}^*}{\not\to} 0$. On the other hand, $\text{weak}^* - \lim_n x_n = 0$, therefore D is not weak*-continuous, which is a contradiction. The natural embedding $\widehat{}\colon \mathcal{A} \to \mathcal{A}^{**}$ is an $\mathcal{A}$-bimodule morphism, then $\widehat{}\, oD\colon \mathcal{A} \to \mathcal{A}^{**}$ is a bounded derivation. Since $\mathcal{A}$ is 2-weakly amenable, then there exists $a'' \in \mathcal{A}^{**}$ such that $\widehat{}\, oD = \delta_{a''}$. We have the following direct sum decomposition

$$\mathcal{A}^{**} = \mathcal{A} \oplus {\mathcal{A}_*}^{\perp}$$

as $\mathcal{A}$-bimodules, [9]. Let $\pi\colon \mathcal{A}^{**} \to \mathcal{A}$ be the projection map. Then π is an $\mathcal{A}$-bimodule morphism, so that $D = \delta_{\pi(a'')}$. □

In the following (example 1) we will show that the converse of Theorem 2.6 does not hold.

Examples

1 Let $\omega\colon \mathbb{Z} \to \mathbb{R}$ define by $\omega(n) = 1 + |n|$ and let

$$\ell^1(\mathbb{Z}, \omega) = \Big\{ \sum_{n\in\mathbb{Z}} f(n)\delta_n : \|f\|_\omega = \sum |f(n)|\,\omega(n) < \infty \Big\}\,.$$

Then $\ell^1(\mathbb{Z},\omega)$ is a Banach algebra with respect to the convolution product defined by the requirement that

$$\delta_m\delta_n = \delta_{mn} \qquad (m,n\in\mathbb{Z})\,.$$

We define

$$\ell^\infty\Big(\mathbb{Z},\frac{1}{\omega}\Big) = \Big\{\lambda = \sum_{n\in\mathbb{Z}}\lambda(n)\lambda_n : \sup\frac{|\lambda(n)|}{\omega(n)} < \infty\Big\}\,,$$

and

$$C_0(\mathbb{Z},\frac{1}{\omega}) = \{\lambda\in l^\infty(\mathbb{Z},\frac{1}{\omega}) : \frac{|\lambda|}{\omega(n)}\in C_0(\mathbb{Z})\}\,.$$

Then $\mathcal{A} = \ell^1(\mathbb{Z},\omega)$ is an Arens regular dual Banach algebra with predual $C_0(\mathbb{Z},\frac{1}{\omega})$ [5]. $\mathcal{A}$ is commutative and semisimple, then $\mathcal{A}$ has the condition $(*)$ (see Example 1.2). On the other hand, by [5], $\mathcal{A}$ is not 2-weakly amenable. It follows that $\mathcal{A}^{**}$ does not have the condition $(*)$.

2 The algebra $C^{(1)}(\mathbb{I})$ consists of the continuously differentiable functions on the unit interval $\mathbb{I}=[0,1]$; $C^{(1)}(\mathbb{I})$ is a Banach function algebra on $\mathbb{I}$ with respect to the norm $\|f\|_1 = \|f\|_{\mathbb{I}} + \|f'\|_{\mathbb{I}}$ $(f\in C^{(1)}(\mathbb{I}))$. By Proposition 3.3 of [4], $C^{(1)}(\mathbb{I})$ is Arens regular but it is not 2-weakly amenable. Thus by Corollary 2.5 above, $C^{(1)}(\mathbb{I})^{**}$ is a dual Banach algebra which does not have the condition $(*)$.

3 For a function $f\in L^1(\mathbb{T})$, the associated Fourier series is $(\hat{f}(n)\colon n\in\mathbb{Z})$. For $\alpha\in(0,1)$ the associated Beurling algebra $A_\alpha(\mathbb{T})$ on $\mathbb{T}$ consists of the continuous functions f on $\mathbb{T}$ such that $\|f\|_\alpha = \sum_{n\in\mathbb{Z}} \mid \hat{f}(n) \mid (1+\mid n\mid)^\alpha < \infty$. By Proposition 3.7 of [4], $A_\alpha(\mathbb{T})$ is Arens regular and 2-weakly amenable. Then by applying Corollary 3.5 above, $A_\alpha(\mathbb{T})^{**}$ has the condition $(*)$.

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References

[1] Arens, R., *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

[2] Bonsall, F. F., Duncan, J., *Complete normed algebras*, Springer, Berlin, 1973.

[3] Dales, H. G., *Banach algebras and automatic continuity*, Oxford, New York, 2000.

[4] Dales, H. G., Ghahramani, F., Grønbæk, N., *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), 19–54.

[5] Dales, H. G., Lau, A. T. M., *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **177** (836) (2005).

[6] Daws, M., *Connes–amenability of bidual algebras*, Math. Scand. **99** (2) (2006), 217–246.

[7] Daws, M., *Dual Banach algebras: representations and injectivity*, Studia Math. **178** (3) (2007), 231–275.

[8] Despic, M., Ghahramani, F., *Weak amenability of group algebras of locally compact groups*, Canad. Math. Bull. **37** (1994), 165–167.

[9] Ghahramani, F., Laali, J., *Amenability and topological centers of the second duals of Banach algebras*, Bull. Austral. Math. Soc. **65** (2002), 191–197.

[10] Johnson, B. E., *Cohomology in Banach algebras*, Mem. Amer. Math Soc. **127** (1972).

[11] Johnson, B. E., *Weak amenability of group algebras*, Bull. London Math. Soc. **23** (1991), 281–284.

[12] Lau, A. T. M., Ülger, A., *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. **348** (1996), 1191–1212.

[13] Losert, V., *The derivation problem for group algebras*, Ann. of Math. (2) **168** (1) (2008), 221–246.

[14] Runde, V., *Amenability for dual Banach algebras*, Studia Math. **148** (1) (2001), 47–66.

[15] Runde, V., *Lectures on amenability*, Springer Verlag, Berlin–Heidelberg–New York, 2002.

[16] Sakai, S., *C^*–algebras and W^*–algebras*, Springer, New York, 1971.

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