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INITIAL COEFFICIENTS FOR GENERALIZED SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED WITH SUBORDINATION

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Abstract. This paper is concerned with certain generalized subclasses of bi-univalent functions defined with subordination in the open unit disc $E = \{z :| z |< 1\}$. The bounds for the initial coefficients for the functions in these classes are studied. The earlier known results follow as special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f having Taylor-Maclaurin series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \,, \tag{1}$$

defined in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Further, the class of functions $f \in \mathcal{A}$ and univalent in E, is denoted by $\mathcal{S}$. By $\mathcal{U}$, we denote the class of Schwarz functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc E and satisfy the conditions $u(0) = 0$ and $|u(z)| < 1$.

For $\delta \geq 1$ and $f \in \mathcal{A}$, Al-Oboudi [2] introduced the following differential operator:

$$D_\delta^0 f(z) = f(z) \,,$$
$$D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z),$$

and in general,

$$D_\delta^n f(z) = D(D_\delta^{n-1} f(z)) = (1-\delta) D_\delta^{n-1} f(z) + \delta z (D_\delta^{n-1} f(z))', n \in \mathcal{N}$$

or equivalent to

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\} \,,$$

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with $D_\delta^n f(0) = 0$. For $\delta = 1$, the operator $D_\delta^n f(z)$ reduces to the Sãlãgean operator introduced in [13].

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarz function $u(z) \in \mathcal{U}$ such that $f(z) = g(u(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

It is obvious that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z(z \in E)$$

and

$$f(f^{-1}(w)) = w\Big(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\Big)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. The class of functions bi-univalent in E and given by (1) is denoted by Σ. Some examples of the functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. But, the well known Koebe function $f(z) = \frac{z}{(1-z)^2}$ is not a member of Σ.

Lewin [9] was the first, who investigated the class Σ and proved that $|a_2| < 1.51$. Subsequently, bounds for the initial coefficients of various sub-classes of bi-univalent functions were studied by various authors in [4, 5, 8, 10, 11] and more recently by Abirami et al. [1], Sivapalan et al. [18] and Singh et al. [15]–[17].

In the sequel, we lay down once and for all that $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $0 < \beta \leq 1$, $0 \leq \eta < 1$, $\delta \geq 1$, $-1 \leq B < A \leq 1$, $z \in E$, $w \in E$ and $g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{\lambda,\alpha,\beta}(A, B; s, t)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha\frac{(s-t)[(zf'(z))'\,]^\lambda}{(f(sz) - f(tz))'} \prec \left(\frac{1+Az}{1+Bz}\right)^\beta$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw) - g(tw)} + \alpha\frac{(s-t)[(wg'(w))'\,]^\lambda}{(g(sw) - g(tw))'} \prec \left(\frac{1+Aw}{1+Bw}\right)^\beta,$$

where s, $t \in \mathcal{C}$ with $s \neq t$, $|t| \leq 1$.

The following observations are obvious:

(i) $\mathcal{S}_\Sigma^{1,\alpha,\beta}(A, B; 1, -1) \equiv \mathcal{M}_\Sigma^s(\beta, \alpha; A, B)$, the class studied by Singh [14].

(ii) $\mathcal{S}_\Sigma^{\lambda,0,\beta}(1, -1; s, t) \equiv \mathcal{S}_\Sigma^{\lambda,\beta}(s, t)$, the class studied by Mazi and Opoola [12].

(iii) For $0 \le \gamma < 1$, $\mathcal{S}_\Sigma^{\lambda,0,1}(1-2\gamma,-1;s,t) \equiv \mathcal{S}_\Sigma^{\lambda}(\gamma,s,t)$, the class studied by Mazi and Opoola [12].

(iv) $\mathcal{S}_\Sigma^{\lambda,0,\beta}(1,-1;1,0) \equiv \mathcal{S}_\Sigma^{\lambda,\beta}$, the class studied by Joshi and Pawar [7].

(v) For $0 \le \gamma < 1$, $\mathcal{S}_\Sigma^{\lambda,0,1}(1-2\gamma,-1;1,0) \equiv \mathcal{S}_\Sigma^{\lambda}(\gamma)$, the class studied by Joshi and Pawar [7].

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{\lambda}(k,\beta;A,B)$ if the following conditions are satisfied:

$$\frac{z[(D^k f(z))']^\lambda}{D^k f(z)} \prec \Big(\frac{1+Az}{1+Bz}\Big)^\beta$$

and

$$\frac{w[(D^k g(w))']^\lambda}{D^k g(w)} \prec \Big(\frac{1+Aw}{1+Bw}\Big)^\beta.$$

Specifically,

(i) $\mathcal{S}_\Sigma^{\lambda}(k,\beta;1,-1) \equiv \mathcal{S}_\Sigma^{\lambda}(k,\beta)$, the class studied by Joshi et al. [6].

(ii) For $0 \le \gamma < 1$, $\mathcal{S}_\Sigma^{\lambda}(k,1;1-2\gamma,-1) \equiv \mathcal{S}_\Sigma^{\lambda}(k,\gamma)$, the class studied by Joshi et al. [6].

Definition 1.3. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz)-f(tz)} + \alpha\frac{(s-t)[(zf'(z))'']^\lambda}{(f(sz)-f(tz))'} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]z}{1+Bz}\Big)^\beta$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw)-g(tw)} + \alpha\frac{(s-t)[(wg'(w))'']^\lambda}{(g(sw)-g(tw))'} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]w}{1+Bw}\Big)^\beta,$$

where $s,t \in \mathcal{C}$ with $s \ne t$, $|t| \le 1$.
In particular, $\mathcal{S}_\Sigma^{\lambda,\alpha,\beta,0}(A,B;s,t) \equiv \mathcal{S}_\Sigma^{\lambda,\alpha,\beta}(A,B;s,t)$.

Definition 1.4. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{\lambda,\delta,\eta}(k,\beta;A,B)$ if the following conditions are satisfied:

$$\frac{z[(D_\delta^k f(z))']^\lambda}{D_\delta^k f(z)} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]z}{1+Bz}\Big)^\beta$$

and

$$\frac{w[(D_\delta^k g(w))']^\lambda}{D_\delta^k g(w)} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]w}{1+Bw}\Big)^\beta.$$

Particularly, $\mathcal{S}_\Sigma^{\lambda,1,0}(k,\beta;A,B) \equiv \mathcal{S}_\Sigma^{\lambda}(k,\beta;A,B)$.

For deriving our main results, we need to the following lemma

Lemma 1.1 ([3]). *If* $p(z) = \dfrac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)} = 1+\sum_{k=1}^{\infty} p_k z^k$, $u(z) \in \mathcal{U}$, *then*

$$|p_n| \leq (A-B)(1-\eta)\,, \quad n \geq 1\,.$$

2. The class $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$

Theorem 2.1. *If* $f \in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$, *then*

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)(1-\eta)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}} \tag{2}$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2(1-\eta)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}. \tag{3}$$

Proof. From Definition 1.3, by principle of subordination, we have

$$(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^{\lambda}}{(f(sz)-f(tz))'} = \left(\frac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)}\right)^{\beta} = [p(z)]^{\beta},\ u \in \mathcal{U} \tag{4}$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-g(tw)} + \alpha\frac{(s-t)[(wg'(w))']^{\lambda}}{(g(sw)-g(tw))'} = \left(\frac{1+[B+(A-B)(1-\eta)]v(w)}{1+Bv(w)}\right)^{\beta} = [q(w)]^{\beta},\ v \in \mathcal{U}\,, \tag{5}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \ldots$.

On expanding and equating the coefficients of z and z^2 in (4) and of w and w^2 in (5), we obtain

$$(1+\alpha)(2\lambda-s-t)a_2 = \beta p_1\,, \tag{6}$$

$$(1+3\alpha)[(s^2+2st+t^2)-2\lambda(s+t-\lambda+1)]a_2^2 + (1+2\alpha)(3\lambda-s^2-st-t^2)a_3 = \beta p_2 + \frac{\beta(\beta-1)p_1^2}{2} \tag{7}$$

and

$$-(1+\alpha)(2\lambda-s-t)a_2 = \beta q_1\,, \tag{8}$$

$$[(6\lambda-s^2-t^2)-2\lambda(s+t-\lambda+1)-\alpha(6\lambda(s+t-\lambda-1)+(s-t)^2)]a_2^2 - (1+2\alpha)(3\lambda-s^2-st-t^2)a_3 = \beta q_2 + \frac{\beta(\beta-1)q_1^2}{2}\,. \tag{9}$$

(6) and (8) together gives

$$p_1 = -q_1 \tag{10}$$

and

$$2(1+\alpha)^2(2\lambda - s - t)^2 a_2^2 = \beta^2(p_1^2 + q_1^2)\,. \tag{11}$$

Adding (7) and (9) and using (11), it yields

$$\begin{aligned}&[(2\lambda - 4\lambda(s+t-\lambda) + 2st) + 2\alpha((s^2+4st+t^2) - 6\lambda(s+t-\lambda))]a_2^2\\ &= \beta(p_2+q_2) + \frac{(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2 a_2^2}{\beta}\,.\end{aligned} \tag{12}$$

(12) gives

$$a_2^2 = \frac{\beta^2(p_2+q_2)}{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}. \tag{13}$$

On applying Lemma 1.1 to the coefficients p_2 and q_2, we can easily obtain (2). Now subtracting (9) from (7), we get

$$-2(1+2\alpha)(3\lambda-s^2-st-t^2)a_2^2+2(1+2\alpha)(3\lambda-s^2-t^2-st)a_3 = \beta(p_2-q_2)\,. \tag{14}$$

Using (10) and (11) in (14), using Lemma 1.1 and on applying triangle inequality, (3) can be easily obtained.

□

On putting $\eta = 0$, Theorem 2.1 gives the following result:

Corollary 2.1. *If* $f \in \mathcal{S}_\Sigma^{\lambda,\alpha,\beta}(A,B;s,t)$, *then*

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}}$$

and

$$|a_3| \leq \frac{\beta(A-B)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}\,.$$

For $\eta = 0, \lambda = 1, s = 1, t = -1$, Theorem 2.1 gives the following result due to Singh [14]:

Corollary 2.2. *If* $f \in \mathcal{M}_\Sigma^s(\beta,\alpha;A,B)$, *then*

$$|a_2| \leq \frac{\beta\sqrt{A-B}}{\sqrt{2((1+\alpha)^2-\beta\alpha^2)}}$$

and

$$|a_3| \leq \frac{\beta^2(A-B)^2}{4(1+\alpha)^2} + \frac{\beta(A-B)}{2(1+2\alpha)}.$$

3. The class $\mathcal{S}_{\Sigma}^{\lambda,\delta,\eta}(k,\beta;A,B)$

Theorem 3.1. *If $f \in \mathcal{S}_{\Sigma}^{\lambda,\delta,\eta}(k,\beta;A,B)$, then*

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)(1-\eta)}}{\sqrt{4\beta(3\lambda-1)(1+2\delta)^k + [4\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2(1+\delta)](1+\delta)^{2k}}} \tag{15}$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(3\lambda-1)(1+2\delta)^k} + \frac{2\beta^2(A-B)^2(1-\eta)^2}{(2\lambda-1)^2(1+\delta)^{2k+1}}. \tag{16}$$

Proof. From Definition 1.4, by principle of subordination, we have

$$\frac{z[(D_\delta^k f(z))']^\lambda}{D_\delta^k f(z)} = \left(\frac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)}\right)^\beta = [p(z)]^\beta, \ u \in \mathcal{U} \tag{17}$$

and

$$\frac{w[(D_\delta^k g(w))']^\lambda}{D_\delta^k g(w)} = \left(\frac{1+[B+(A-B)(1-\eta)]v(w)}{1+Bv(w)}\right)^\beta = [q(w)]^\beta, \ v \in \mathcal{U}, \tag{18}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \ldots$.

On expanding and equating the coefficients of z and z^2 in (17) and of w and w^2 in (18), we obtain

$$(2\lambda-1)(1+\delta)^k a_2 = \beta p_1, \tag{19}$$

$$(3\lambda-1)(1+2\delta)^k a_3 + (2\lambda^2-4\lambda+1)(1+\delta)^{2k} a_2^2 = \beta p_2 + \frac{\beta(\beta-1)p_1^2}{2} \tag{20}$$

and

$$-(2\lambda-1)(1+\delta)^k a_2 = \beta q_1, \tag{21}$$

$$[2(3\lambda-1)(1+2\delta))^k + (2\lambda^2-4\lambda+1)(1+\delta)^{2k}]a_2^2 - (3\lambda-1)(1+2\delta)^k a_3 = \beta q_2 + \frac{\beta(\beta-1)q_1^2}{2}. \tag{22}$$

(19) and (21) together give

$$p_1 = -q_1 \tag{23}$$

and

$$(2\lambda-1)^2(1+\delta)^{2k+1} a_2^2 = \beta^2(p_1^2+q_1^2). \tag{24}$$

Adding (20) and (22) and using (24), it yields

$$[2\beta(3\lambda-1)(1+2\delta)^k+\{2\beta(2\lambda^2-4\lambda+1)-\frac{(\beta-1)}{2}(2\lambda-1)^2(1+\delta)\}(1+\delta)^{2k}]a_2^2$$

$$= \beta^2(p_2+q_2)\,. \tag{25}$$

(25) gives

$$a_2^2 = \frac{2\beta^2(p_2+q_2)}{4\beta(3\lambda-1)(1+2\delta)^k+\{4\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2(1+\delta)\}(1+\delta)^{2k}}\,. \tag{26}$$

On applying Lemma 1.1 to the coefficients p_2 and q_2 in (26), we can easily obtain (15).

Now subtracting (22) from (20), we get

$$2(3\lambda-1)(1+2\delta)^k a_3-2(3\lambda-1)(1+2\delta)^k a_2^2=\beta(p_2-q_2)\,. \tag{27}$$

Using (e24), (e27) yields

$$a_3=\frac{\beta^2(p_1^2+q_1^2)}{(2\lambda-1)^2(1+\delta)^{2k+1}}+\frac{\beta(p_2-q_2)}{2(3\lambda-1)(1+2\delta)^k}\,. \tag{28}$$

Applying Lemma 1.1 to the coefficients p_2, q_2 and p_1 in (28), (16) is obvious. □

For $\delta=1$, $\eta=0$, the following result can be easily obtained from Theorem 3.1:

Corollary 3.1. *If $f\in\mathcal{S}_\Sigma^\lambda(k,\beta;A,B)$, then*

$$|a_2|\le\frac{\beta\sqrt{2(A-B)}}{\sqrt{2\beta(3\lambda-1)3^k+[2\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2]2^{2k}}}$$

and

$$|a_3|\le\frac{\beta(A-B)}{(3\lambda-1)3^k}+\frac{\beta^2(A-B)^2}{(2\lambda-1)^2 2^{2k}}\,.$$

For $\delta=1$, $\eta=0$, $A=1$, $B=-1$, Theorem 3.1 gives the following result due to Joshi et al. [6]:

Corollary 3.2. *If $f\in\mathcal{S}_\Sigma^\lambda(k,\beta;A,B)$, then*

$$|a_2|\le\frac{2\beta}{\sqrt{2\beta(3\lambda-1)3^k+\{2\beta(2\lambda^2-4\lambda-1)-(\beta-1)(2\lambda-1)^2\}2^{2k}}}$$

and

$$|a_3|\le\frac{2\beta}{(3\lambda-1)3^k}+\frac{4\beta^2}{(2\lambda-1)^2 2^{2k}}\,.$$

Putting $\delta=1$, $\eta=0$, $A=1-2\gamma$, $B=-1$ and $\beta=1$ in Theorem 3.1, we obtain the following result due to Joshi et al. [6]:

Corollary 3.3. *If* $f \in \mathcal{S}_{\Sigma}^{\lambda}(k,\gamma)$, *then*

$$|a_2| \le \frac{2\sqrt{1-\gamma}}{\sqrt{2(3\lambda-1)3^k + [(2\lambda-1)^2-(4\lambda-1)]2^{2k}}}$$

and

$$|a_3| \le \frac{4(1-\gamma)^2}{(2\lambda-1)^2 2^{2k}} + \frac{2(1-\gamma)}{(3\lambda-1)3^k}.$$

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References

[1] Abirami, C., Magesh, N., Gati, N.B., Yamini, J., *Horadam polynomial coefficient estimates for a class of λ-bi-pseudo-starlike functions*, J. Anal. **2020** (2020), 1–10, `https://doi.org/10.1007/s41478-020-00224-2`.

[2] Al-Oboudi, F.M., *On univalent functions defined by a generalized Sãlãgean operator*, Int. J. Math. Math. Sci. **27** (2004), 1429–1436.

[3] Aouf, M.K., *On a class of p-valent starlike functions of order α*, Int. J. Math. Math. Sci. **10** (4) (1987), 733–744.

[4] Brannan, D.A., Taha, T.S., *On some classes of bi-univalent functions*, Mathematical Analysis and its Applications, Kuwait, February 18–21, 1985 (Mazhar, S.M., Hamoni, A., Faour, N.S., eds.), KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53–60, 1985, See also Studia Univ. Babes-Bolyai Math., 1986, **31**(2), 70–77.

[5] Frasin, B.A., Aouf, M.K., *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.

[6] Joshi, S., Altinkya, S., Yalcin, S., *Coefficient estimates for Sãlãgean type λ-bi-pseudo-starlike functions*, Kyungpook Math. J. **57** (2017), 613–621.

[7] Joshi, S., Pawar, H., *On some subclasses of bi-univalent functions associated with pseudo-starlike functions*, J. Egyptian Math. Soc. **24** (2016), 522–525.

[8] Juma, A.R S., Aziz, F.S., *Applying Ruscheweyh derivative on two subclasses of bi-univalent functions*, Int. J. Basic Appl. Sci. **12** (6) (2012), 68–74.

[9] Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[10] Magesh, N., Bulut, S., *Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions*, Afrika Mat. **29** (2018), 203–209.

[11] Magesh, N., Rosy, T., Varma, S., *Coefficient estimate problem for a new subclass of bi-univalent functions*, J. Complex Anal. **2013** (2013), 3 pp., Article ID 474231.

[12] Mazi, E.P., Opoola, T.O., *On some subclasses of bi-univalent functions associating pseudo-starlike functions with Sakaguchi type functions*, Gen. Math. **25** (2017), 85–95.

[13] Sãlãgean, G.S., *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest **1** (1983), 362–372.

[14] Singh, Gagandeep, *Coefficient estimates for bi-univalent functions with respect to symmetric points*, J. Nonlinear Func. Anal. **1** (2013), 1–9, Article ID 2013.

[15] Singh, Gurmeet, Singh, Gagandeep, Singh, Gurcharanjit, *Certain subclasses of Sakaguchi-type bi-univalent functions*, Ganita **69** (2) (2019), 45–55.

[16] Singh, Gurmeet, Singh, Gagandeep, Singh, Gurcharanjit, *A generalized subclass of alpha-convex bi-univalent functions of complex order*, Jnanabha **50** (1) (2020), 65–71.

[17] Singh, Gurmeet, Singh, Gagandeep, Singh, Gurcharanjit, *Certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers*, Gen. Math. **28** (1) (2020), 1258–140.

[18] Sivapalan, J., Magesh, N., Murthy, S., *Coefficient estimates for bi-univalent functions with respect to symmetric conjugate points associated with Horadam Polynomials*, Malaya J. Mat. **8** (2) (2020), 565–569.

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