On the Connes operator in Hochschild homology

by

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Abstract. Let $\mathbb{K}$ be a field of characteristic $p \geq 0$ and X a topological space. If N^*X is the algebra of normalized singular cochains on X with coefficients in $\mathbb{K}$, then a product is defined on the negative Hochschild complex $\mathfrak{C}_*X := \mathfrak{C}_*N^*X$ ([2]) which induces a natural commutative graded algebra structure on $H\mathfrak{C}_*X := HH_*X$. We prove that if X is simply connected, the Connes operator $B : \mathfrak{C}_*X \to \mathfrak{C}_{*+1}X$ is an algebra derivation in homology.

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Introduction.

Throughout this note, $\mathbb{K}$ is a field of characteristic $p \geq 0$. The classical definition of the normalized negative Hochschild complex $\mathfrak{C}_*A$ of an algebra A naturally extends to a cochain algebra.

Let (A, d_A) be a cochain algebra over $\mathbb{K}$. The negative Hochschild complex of the cochain algebra (A, d_A) with coefficients in (A, d_A) is $\mathfrak{C}_*(A, d_A) = (A \otimes BA, D)$ where B denotes the reduced bar construction functor and D is defined in section 1. By definition $HH_*A = H_*\mathfrak{C}_*(A, d_A)$ is the negative Hochschild homology of (A, d_A) with coefficients in (A, d_A). If $(A, d_A) = N^*X$ the algebra of normalized cochains on the topological space X, then we note $\mathfrak{C}_*X$ (resp. HH_*X) instead of $\mathfrak{C}_*N^*X$ (resp. HH_*N^*X) and call it the negative normalized Hochschild complex (resp. the negative Hochschild homology) of X.

In [2], theorem 1, a natural product is defined on $\mathfrak{C}_*X$ such that HH_*X becomes a commutative graded algebra.

For any (cochain) algebra, we have (see [3] or [8]) the Connes operator $B : \mathfrak{C}_*A \to \mathfrak{C}_{*+1}A$ satisfying $DB + BD = 0$ and $B \circ B = 0$. Thus, in particular, it defines a linear map of lower degree 1 in homology, say $B_* : HH_*A \to HH_{*+1}A$. In this note, we prove the following result.

Theorem. *If X be a simply connected topological space, then*
1- $B_* : HH_*X \to HH_{*+1}X$ is an algebra derivation,
2- $B_* \circ B_* = 0$.
Note that property *2* is obvious and is true even if X is not simply connected.

We adopt the convention that negative lower degree is positive upper degree. The degree of an homogeneous element x is denoted $|x|$.

1- Hochschild homology.

1.1 Let **DA** and **DC** denote respectively the category of connected cochain algebras and the category of connected cochain coalgebras. That is, in particular, the differential is of upper degree *one*. The reduced bar and cobar constructions are a pair of adjoint functors $B : \mathbf{DA} \leftrightarrow \mathbf{DC} : \Omega$, [4]. This adjunction yields, for a cochain algebra A, a natural homomorphism $\alpha_A : \Omega BA \to A$ of cochain algebras which induces an isomorphism in homology [4]. The elements of BA (resp. ΩC) are denoted $[a_1|a_2|...|a_k] \in B_kA$ (resp. $\langle c_1|c_2|....|c_l\rangle \in \Omega_l C$ and $[] = 1 \in B_0A \simeq \mathbb{K}$ (resp. $\langle\rangle = 1 \in \Omega_0 C \simeq \mathbb{K}$).

The linear map $\iota_A : A \to \mathbb{K} \oplus \bar{A}$, $\iota_A(1) = 1$ and $\iota_A(a) = \langle[a]\rangle$, $a \in \bar{A}$ commutes with the differentials, but is not a morphism of cochain algebras. In any case, it satisfies $\alpha_A \circ \iota_A = id_A$ and $id_{\Omega BA} - \iota_A \circ \alpha_A = d_{\Omega BA} \circ h + h \circ d_{\Omega BA}$ for some chain homotopy $h : \Omega BA \to \Omega BA$ such that $\alpha_A \circ h = 0$, $h \circ \iota_A = 0$, $h^2 = 0$.

1.2 Let (A, d_A) be a cochain algebra. We denote by d_{BA} the differential of the reduced bar construction BA. The tensor product $(A, d_A) \otimes (BA, d_{BA})$ is then a differential module whose differential is denoted $d_{A\otimes BA}$. The Hochshild differential, denoted by D, is defined by :

$$Da_0[a_1|....|a_n] = (d_0 - d_n)a_0[a_1|....|a_n] + d_{A\otimes BA}a_0[a_1|....|a_n]$$

where $d_0a_0[a_1|....|a_n] = (-1)^{|a_0|}a_0a_1[a_2|...|..|a_n]$ and

$d_na_0[a_1|....|a_n] = (-1)^{(|a_n|+1)(|a_0|+...+|a_{n-1}|+n-1)}a_na_0[a_1|...|..|a_{n-1}]$.

By definition,

$$\mathfrak{C}_*A = (A \otimes BA, D)$$

is the normalized negative Hochschild complex of (A, d_A) with coefficients in (A, d_A) and $HH_*A = H\mathfrak{C}_*A$ is the *negative Hochschild homology* of the cochain algebra (A, d_A) with coefficients in (A, d_A) . It is clear that $\mathfrak{C}_*A$ is concentrated in non-negative total upper degrees. Hence, so is HH_*A.

If $(A, d_A) = N^*X$ is the algebra of normalized singular cochains on the topological space X, then $\mathfrak{C}_*N^*X := \mathfrak{C}_*X$ is the normalized negative Hochschild complex of X and $HH_*N^*X := HH_*X$ is the negative Hochschild homology of X.

1.3 For the cochain algebra (A, d_A), the Connes operator is the linear map

$$B : \mathfrak{C}_*A \to \mathfrak{C}_{*+1}A$$

defined by $Ba_0[a_1|....|a_n] = \sum_{i=0}^{n}(-1)^{\epsilon_i}1[a_i|..|a_n|a_0|....|a_{i-1}]$ where $\epsilon_i = |a_0| + (|a_0| + |a_1| + .. + |a_{i-1}| + i)(|a_i| + .. + |a_n| + n - i + 1)$.

The Connes operator satisfies:

$B^2 = 0$ and $BD + DB = 0$. So it induces a linear map of lower degree 1,

$B_* : HH_*A \to HH_{*+1}A$.

2- Proof of the theorem.

2.1 Let X and Y be topological spaces. One has the Eilenberg-Zilber homomorphism of complexes $EZ : C_*X \otimes C_*Y \to C_*(X \times Y)$ and the Alexander-Whitney homomorphism of complexes $AW : C_*(X \times Y) \to C_*X \otimes C_*Y$. The diagonal map $\triangle_{top} : X \to X \times X$ and the Alexander-Whitney homomorphism define a diagonal $\triangle = AW \circ C_*\triangle_{top} : C_*X \to C_*S^1 \otimes C_*X$, and C_*X is a graded coalgebra. Thus C_*S^1 is a graded coalgebra.

Recall that the set of continous maps from the unit circle S^1 to X endowed with the compact-open topology is the free loop space of X and is denoted LX. There is an action of S^1 on LX given by the rotation of the loops.

The action $\bar{\nu} : S^1 \times LX \to LX$ yields an action $\nu = C_*\bar{\nu} \circ EZ : C_*S^1 \otimes C_*LX \to C_*LX$.

2.2 Let $z \in C_*S^1$ be a representative of the fundamental class of S^1 and consider the map $I : C^nLX \to C^{n-1}LX, n \geq 1$ defined in [7], section 4, by

$\langle I(x), \sigma\rangle = (-1)^{|x|}\langle x, \nu(z \otimes \sigma)\rangle, \quad \sigma \in C_{n-1}LX.$

The map I satisfies $I^2 = 0$ and $dI = -Id$ where d is the differential of C^*LX.

If $\triangle_{C_*S^1}$ denotes the diagonal on C_*S^1, for degree reasons, we have $\triangle z = z \otimes 1 + 1 \otimes z$. That is, z is primitive. The following result is the cornerstone in the proof of the theorem stated.

2.3 Lemma *The map I is a derivation when C^*LX is endowed with the usual cup product.*

Proof. The diagonal map on $C_*S^1 \otimes C_*LX$, say $\triangle_{C_*S^1 \otimes C_*LX}$ is given by

$$\triangle_{C_*S^1 \otimes C_*LX} : C_*S^1 \otimes C_*LX \overset{\triangle_{C_*S^1} \otimes \triangle_{C_*LX}}{\longrightarrow} (C_*S^1)^{\otimes 2} \otimes (C_*LX)^{\otimes 2} \overset{id \otimes T \otimes id}{\longrightarrow} (C_*S^1 \otimes C_*LX)^{\otimes 2}$$

where $T : A \otimes B \to B \otimes A$ is the interchange isomorphism defined by $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$, id is the identity map and $M^{\otimes 2} = M \otimes M$.

Let $j : C_*LX \to C_*S^1 \otimes C_*LX$ denote the map defined by $j(\sigma) = z \otimes \sigma$. It is clear that $I = (\nu \circ j)^\vee = j^\vee \circ \nu^\vee$ where $^\vee$ denotes the dual of a map between the dual vector spaces.

Recall that $\nu^\vee = (EZ)^\vee \circ C^*\bar{\nu}$ is a homomorphism of graded algebras.

To finish the proof, it is now sufficient to prove that $j^\vee$ is a derivation of algebras or equivalently that j is a coderivation of coalgebras.

From the definition of $\triangle_{C_*S^1 \otimes C_*LX}$, it is obvious that $\triangle_{C_*S^1 \otimes C_*LX} \circ j = (j \otimes Id) \circ \triangle_{C_*LX} + (Id \otimes j) \circ \triangle_{C_*LX}$.

2.4 In [7] (lemma 5.5 and proofs of theorems A and B), J. D.S. Jones has constructed a natural chain map $\Psi : \mathfrak{C}_*X \to C^{-*}LX$ such that $I\Psi - \Psi B$ is homotopic to 0 ([7], theorem 4.1). Since $dI = -Id$, I induces $I^* : H^*(LX; \mathbb{K}) \to H^{*-1}(LX; \mathbb{K})$ and $H_*(\Psi) \circ B_* = I^{-*} \circ H^{-*}(\Psi)$.

2.5 In [2] theorem 1 and Part II 1.1, it is proved that there exists a natural product on $\mathfrak{C}_*X$ such that Ψ induces an isomorphism of graded algebras in homology when C^*LX is endowed with the usual cup product.

Apply lemma 2.3 and 2.4 to end the proof of the theorem.

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