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THE FACTORIZATION OF ABELIAN GROUPS

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Abstract. If G is a finite abelian group and $n > 1$ is an integer, we say that G is n-good, if from each factorization $G = A_1 A_2 \cdots A_n$ of G into direct product of subsets, it follows that at least one of the subsets A_i is periodic, in the sense that there exists $x \in G - \{e\}$ such that $xA_i = A_i$. In this paper, we shall study some 3-groups with respect to this property.

0. Notations and definitions

Throughout this paper, G will denote a finite abelian group, e its identity element, and if a is an element of G, then $|a|$ denotes its order. Furthermore, for a subset A of G, $|A|$ will denote the number of the elements in A. G is said to be of type $(p_1^{\alpha_1}, p_2^{\alpha_2}, p_3^{\alpha_3}, \ldots, p_1^{\alpha_S})$ if it is the direct product of cyclic groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, p_3^{\alpha_3}, \ldots, p_1^{\alpha_S}$, where p_i are primes. $G = A_1 \cdots A_n$ is said to be a factorization of G if every element a of G has a unique representation of the form $a = a_1 \cdots a_n$, where $a_i \in A_i$. If in addition each A_i also contains e, then the factorization $G = A_1 \cdots A_n$ is said to be a normalized factorization. A subset A of G is said to be periodic if there is a non-identity element x in G such that $xA = A$. Such an element x when it exists is called a period for A. A subset A of G of the form $A = \{e, a, a^2, \ldots, a^k\}$ is called cyclic; here k is an integer with $k < |a|$. A subset A of G is called simulated if $A = \{e, a, a^2, \ldots, a^{|a|-1}d\}$. We observe that if $d = e$, then $A = < a >$ the subgroup generated by a. Otherwise, A differs from the subgroup $< a >$ generated by a in the element $a^{|a|-1}d$. The subgroup $< a >$ is referred to as a corresponding subgroup of A. If A and A' are subsets of G such that for every subset B of G, whenever $AB = G$ is a factorization of G, then so is $A'B$, then we say that A is replaceable by A'. A group G is said to be n-good if from each factorization $G = A_1 \cdots A_n$ it follows that at least one of the A_i is periodic. Otherwise G is said to be n-bad. Furthermore, we will say G is totally-good if it is n-good for all possible values of n.

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1. Introduction

A famous theorem of Hajós [4] is the following: If $G = A_1 \dots A_n$ is a factorization of a group G, where each of the subsets A_i is cyclic, then one of the A_i is a subgroup of G. In [5], Rédei generalized this theorem to: If $G = A_1 \cdots A_n$ is a normalized factorization of G, where each of the A_i contains a prime number of elements, then one of the A_i is a subgroup of G. This gave rise to the following question: Suppose $G = A_1 \cdots A_n$ is a factorization of G, does it follow that one of A_i is periodic? Or, in our terminology, is every group n-good? The literature is abundant with results for the case $n = 2$. Indeed, in this case all good-bad groups have been classified. In [1] it is shown that p- groups are n-bad for all $n > 1$, if $p \geq 5$, except when G is cyclic. It is peculiar that the cases of 2-groups and 3-groups are more difficult to handle as we shall see in this paper. This is probably due to the fact that the techniques used there, do not generalize. In this paper, we shall study some 3-groups. We will show that the groups $(3^2, 3), (3, 3, 3)$ are totally-good, while the groups $(3^3, 3), (3^2, 3^2), (3^2, 3, 3)$ and $(3, 3, 3, 3)$ are 3-good but 2-bad. For 3-groups of order 3^5, I am only able to show that if G is elementary, then G is 2-bad, and 3-bad but 4-good. At present, I cannot contribute anything to the case $n = 4$, for the rest of the groups of order 3^5; namely the groups $(3^4, 3), (3^3, 3, 3), (3^2, 3, 3, 3), (3^2, 3^2, 3)$, and $(3^3, 3^2)$.

2. Preliminaries

We shall use the following results:

Theorem 2.1. [3] *Suppose G has a proper subgroup H of type $(3, 3)$. If $|G/H|$ is composite, then G is 2-bad.*

Theorem 2.2. [8] *Suppose $|G| = p^4$, where p is any prime. Then in any factorization $G = ABC$ of G where $|A| = p^2$ and B and C are either cyclic or simulated, then one of A, B or C is periodic.*

Lemma 2.3. [7] *Let $G = AB$ be a normalized factorization of G, with B simulated, say, $B = \{e, b_1, b_2, \dots, b_{k-1}, b_k d\}$, where $d \neq e$, and $k \geq 3$, and let $H = \{e, b_1, b_2, \dots, b_s\}$ be the subgroup of G. Then B is periodic with period d.*

Lemma 2.4. [7] *Let $G = AB$ be a factorization of G, where $B = \{e, b_1, b_2, \dots, b_{k-1}, b_k d\}$ is simulated. Then B can be replaced by the subgroup $H = \{e, b_1, b_2, \dots, b_s\}$.*

Lemma 2.5. [9] *If a simulated subset is periodic, then it can be replaced by a corresponding subgroup.*

Lemma 2.6. [2] *If G has a proper subgroup H which is n-bad, then G is both n and $(n+1)$-bad.*

3. The results

3.1. **3-groups of order 3^3.** In this section we will show that 3-groups of order 3^3 are totally- good. We observe that we need only detail the case $n = 2$. This is a consequence of the following lemma.

Lemma 3.1.1. *Suppose $G = AB$ is a factorization of G, where $|A| = 3$, then either A or B is periodic.*

Proof. Let $A = \{e, a, b\}$, where $a \neq e$, $b \neq e$. Now

(1) $$G = AB = B \cup aB \cup bB$$

Multiplying (1) by a, gives

(2) $$G = aB \cup a^2B \cup abB$$

Comparing (1) and (2) gives

(3) $$B \cup bB = a^2B \cup abB$$

Using (3) and the fact that $G = AB$ is a factorization, we get that $B = abB$ and $bB = a^2B$. This implies that either B is periodic or A is periodic. □

3.2. **3-groups of order 3^4.**

3.2.1. **3**-*groups of order* $\mathbf{3^4}$ *are* **2**-*bad.* Observe that in this case the group types are $(3^3, 3, 3), (3, 3, 3, 3), (3^3, 3), (3^2, 3^2)$. The fact that these groups are 2-bad follows from Theorem 2.1.

3.2.2. **3**-*groups of order* $\mathbf{3^4}$ *are* **3**-*good.* In this case, we see that in any factorization $G = ABC$, one of the factors, say A is of order 9 while the factors B and C must be of order 3. Hence B and C are either cyclic or simulated and the result follows from Theorem 2.3.

3.3. **3-groups of order 3^5.** In this section we will show that an elementary 3-group of order 3^5 is 4-good. We start with the following lemmas, whose proof are clear.

Lemma 3.3.1. *Suppose $G = AB$ is a factorization of G, K a subgroup of G, and $B \subset K$. Then $K = (A \cap K)B$ is a factorization of K.* □

Lemma 3.3.2. *Let $G = ABH$ be a factorization of G, where $A, B \subset G$, and H is a subgroup of G. Then $G/H = (AH/H).(BH/H)$ is a factorization of the quotient group G/H.* □

Theorem 3.3.1. *If G is of type $(3, 3, 3, 3, 3)$, then G is 4-good.*

Proof. Observe that G is 2-bad and 3-bad by the results in 3.2.1 and Lemma 2.6. We will show that G is 4-good. This will take several steps:

(*) Let $G = < x, y, u, v, w >$, where $|x| = |y| = |u| = |v| = |w| = 3$ and suppose $G = ABCD$ is a factorization of G, where, $|A| = 3^2$ and $|B| = |C| = |D| = 3$. Clearly B, C and D are simulated. Say $B = \{e, b, b^2 r\}$, $C = \{e, c, c^2 s\}$ and $D = \{e, d, d^2 t\}$. It is clear that if one of r, s, or t is the identity element e of G, then there is nothing to do. Therefore, we assume that none of r, s, and t is e.

(**) Since B, C, and D are simulated, we can use Lemma 2.6 to get the factorization $G = A < b > < c > < d > = A < b, c, d >$. Hence, $3^3 \leq | < b, c, d, r, s, t > | \leq 3^5$. If $| < b, c, d, r, s, t > | = 3^3$. That is, if $r, s, t \in \; < b, c, d > = H$, the subgroup generated by b, c and d, then $H = BCD$ is a factorization of H and so by Rédei's Theorem, one of the subsets B, C, or D is a subgroup of H. So we are done. There remain two cases that we have to distinguish.

(I) $\qquad | < b, c, d, r, s, t > | = 3^4; \; s, t \in \; < b, c, d, r > .$

(II) $\qquad | < b, c, d, r, s, t > | = 3^5; \; t \in \; < b, c, d, r, s > .$

Case (I)

(1) In this case, we may choose b, c, d, and r to be u, v, w, and x respectively. Now, $B = \{e, u, u^2 x\}, C = \{e, v, v^2 s\}$ and $D = \{e, w, w^2 t\}$. From (**), we have $G = A < u > < v > < w > = < u, v, w >$. It follows that A is a complete set of representatives *modulo* $< u, v, w >$. Thus A consists of the following elements:

$$\begin{array}{lll} eu^{\alpha_{00}}v^{\beta_{00}}w^{\gamma_{00}} & yu^{\alpha_{01}}v^{\beta_{01}}w^{\gamma_{01}} & y^2u^{\alpha_{02}}v^{\beta_{02}}w^{\gamma_{02}} \\ xu^{\alpha_{10}}v^{\beta_{10}}w^{\gamma_{10}} & xyu^{\alpha_{11}}v^{\beta_{11}}w^{\gamma_{11}} & xy^2u^{\alpha_{12}}v^{\beta_{12}}w^{\gamma_{12}} \\ x^2u^{\alpha_{20}}v^{\beta_{20}}w^{\gamma_{20}} & x^2yu^{\alpha_{21}}v^{\beta_{21}}w^{\gamma_{21}} & x^2y^2u^{\alpha_{22}}v^{\beta_{22}}w^{\gamma_{22}} \end{array}$$

Here $\alpha_{00} = \beta_{00} = \gamma_{00} = 0$, since A is normalized.

(2) Using Lemma 3.3.1, we obtain the factorization $K = (A \cap K)BCD$. Note that $A \cap K$ now consists of the following elements: $A \cap K = \{e, xu^{\alpha_{10}}v^{\beta_{10}}w^{\gamma_{10}}, x^2u^{\alpha_{20}}v^{\beta_{20}}w^{\gamma_{20}}\}$. By Rédei's Theorem, one of $A \cap K, B, C$ or D is a subgroup of K. If this is B or C or D, we are done. So assume that $A \cap K$ is a subgroup of K. Now, we can rewrite the elements of $A \cap K$ as:

$$\{e, xu^{\alpha_0+\alpha_0'}v^{\beta_0+\beta_0'}w^{\gamma_0+\gamma_0'}, x^2u^{2\alpha_0+\alpha_0'}v^{2\beta_0+\beta_0'}w^{2\gamma_0+\gamma_0'}\}$$

Restricting the factorization $G = [(\, yu^{\alpha_{01}}v^{\beta_{01}}w^{\gamma_{01}})^{-1}A]BCD$ to $K = \; < x, u, v, w >$ and arguing as above gives that $[(\, yu^{\alpha_{01}}v^{\beta_{01}}w^{\gamma_{01}})^{-1}A] \cap K$

will be the subgroup $[(yu^{\alpha_{01}}v^{\beta_{01}}w^{\gamma_{01}})^{-1}A] \cap K = < xu^{\alpha_1}v^{\beta_1}w^{\gamma_1} >$ of K. Thus, $A \cap K$ consists of the following elements:

$$A \cap K = \{yu^{\alpha_1'}v^{\beta_1'}w^{\gamma_1'}, xyu^{\alpha_1+\alpha_1'}v^{\beta_1+\beta_1'}w^{\gamma_1+\gamma_1'}, x^2yu^{2\alpha_1+\alpha_1'}v^{2\beta_1+\beta_1'}w^{2\gamma_1+\gamma_1'}\}$$

Finally, restricting the factorization $G = [(y^2u^{\alpha_{02}}v^{\beta_{02}}w^{\gamma_{02}})^{-1}A]BCD$,to $K = < x, u, v, w >$ and arguing similarly again as above gives that $[(y^2u^{\alpha_{02}}v^{\beta_{02}}w^{\gamma_{02}})^{-1}A] \cap K$ will be the subgroup of K that consists of the following elements:

$$A \cap K = \{y^2u^{\alpha_2'}v^{\beta_2'}w^{\gamma_2'}, xy^2u^{\alpha_2+\alpha_2'}v^{\beta_2+\beta_2'}w^{\gamma_2+\gamma_2'}, x^2y^2u^{2\alpha_2+\alpha_2'}v^{2\beta_2+\beta_2'}w^{2\gamma_2+\gamma_2'}\}$$

Putting all this together, we get that A consists of the following elements:

$$\begin{array}{lll}
eu^{\alpha_0'}v^{\beta_0'}w^{\gamma_0'} & yu^{\alpha_1'}v^{\beta_1'}w^{\gamma_1'} & y^2u^{\alpha_2'}v^{\beta_2'}w^{\gamma_2'} \\
xu^{\alpha_0+\alpha_0'}v^{\beta_0+\beta_0'}w^{\gamma_0+\gamma_0'} & xyu^{\alpha_1+\alpha_1'}v^{\beta_1+\beta_1'}w^{\gamma_1+\gamma_1'} & xy^2u^{\alpha_2+\alpha_1'}v^{\beta_2+\beta_1'}w^{\gamma_2+\gamma_2'} \\
x^2u^{2\alpha_0+\alpha_0'}v^{2\beta_0+\beta_0'}w^{2\gamma_0+\gamma_0'} & x^2yu^{2\alpha_1+\alpha_1'}v^{2\beta_1+\beta_1'}w^{2\gamma_1+\gamma_1'} & x^2y^2u^{2\alpha_2+\alpha_1'}v^{2\beta_2+\beta_1'}w^{2\gamma_2+\gamma_2'}
\end{array}$$

Here, $\alpha_0' = \beta_0' = \gamma_0' = 0$, since A is normalized.

(3) Consider the factorization $G = AB < v > < w > = AB < v, w >$. Forming the quotient group $\overline{G} = G/ < v, w >$, and using Lemma 3.3.2, we get that $\overline{G} = \overline{A}\ \overline{B}$ is a factorization of $\overline{G}$, where $\overline{A} = A < v, w > / < v, w >$, and $\overline{B} = B < v, w > / < v, w >$. Now, observe that $\overline{G}$ is of order 3^3. So by 3.1, either $\overline{A}$ or $\overline{B}$ is periodic. But $\overline{B} = \{H, uH, (uH)^2(xH)\}$, where $H = < v, w >$. $\overline{B}$ cannot be periodic, by Lemma 2.5. It follows that $\overline{A}$ is periodic. In fact, by Lemma 2.4, $\overline{A}$ is periodic with period $\overline{x} = xH$. Now, $\overline{A}$ looks like:

$$\begin{array}{lll}
eu^{\alpha_0'}H & yu^{\alpha_1'}H & y^2u^{\alpha_2'}H \\
xu^{\alpha_0+\alpha_0'}H & xyu^{\alpha_1+\alpha_1'}H & xy^2u^{\alpha_2+\alpha_1'}H \\
x^2u^{2\alpha_0+\alpha_0'}H & x^2yu^{2\alpha_1+\alpha_1'}H & x^2y^2u^{2\alpha_2+\alpha_1'}H
\end{array}$$

Since, $xH = (xH)(eu^{\alpha_0'}H) = (xu^{\alpha_0+\alpha_0'}H) = (xu^{\alpha_0'}H)$, this implies that $\alpha_0 = 0$. Similarly, one shows that $\alpha_1 = \alpha_2 = 0$. Thus the elements of A are the following:

$$\begin{array}{lll}
ev^{\beta_0'}w^{\gamma_0'} & yu^{\alpha_1'}v^{\beta_1'}w^{\gamma_1'} & y^2u^{\alpha_2'}v^{\beta_2'}w^{\gamma_2'} \\
xv^{\beta_0+\beta_0'}w^{\gamma_0+\gamma_0'} & xyu^{\alpha_1'}v^{\beta_1+\beta_1'}w^{\gamma_1+\gamma_1'} & xy^2u^{\alpha_1'}v^{\beta_2+\beta_1'}w^{\gamma_2+\gamma_2'} \\
x^2v^{2\beta_0+\beta_0'}w^{2\gamma_0+\gamma_0'} & x^2yu^{\alpha_1'}v^{2\beta_1+\beta_1'}w^{2\gamma_1+\gamma_1'} & x^2y^2u^{\alpha_1'}v^{2\beta_2+\beta_1'}w^{2\gamma_2+\gamma_2'}
\end{array}$$

Here as before, $\beta_0' = \gamma_0' = 0$, since A is normalized.

(4) Restrict the factorization $G = ABCD$ to $K = < x, u, v, w >$ again to get the factorization $K = (A \cap K)BCD$, so that one of $(A \cap K)B, (A \cap K)C, (A \cap K)D$, is a subgroup of K. Now observe the following: $B \subset (A \cap K)B$, which implies that $< u, x > = < B > \subset (A \cap K)B$. However, $|(A \cap K)B| = 9 = | < u, x > |$. Therefore $(A \cap K)B = < u, x >$. Similarly, $(A \cap K)C = < v, s >$ and $(A \cap K)D = < w, t >$. But, we also note that $xv^{\beta_0}w^{\gamma_0}$ belongs to $A \cap K$. This gives that $xv^{\beta_0}w^{\gamma_0}$ belongs to one of $< u, x >, < v, s >$ or $< w, t >$. Similarly, restricting the factorizations $G = [(yu^{\alpha_1'}v^{\beta_1'}w^{\gamma_1'})^{-1}A]BCD$, and $G = [(\ y^2u^{\alpha_2'}v^{\beta_2'}w^{\gamma_2'}\ )^{-1}A]BCD$ to $K = < x, u, v, w >$ gives that $xv^{\beta_i}w^{\gamma_i}$ belongs to one of $< u, x >, < v, s >$or $< w, t >$, for each i, where $0 \leq i \leq 2$.

(5) From the factorization $G = ABC < w >$, we get the factorization $\overline{G} = \overline{A}\ \overline{B}\ \overline{C}$ of the quotient group $\overline{G}$, where $\overline{G} = G/ < w >$, $\overline{A} = A < w > / < w >$, $\overline{B} = B < w > / < w >$, and $\overline{C} = C < w > / < w >$. So one of $\overline{A}$, $\overline{B}$ or $\overline{C}$ is periodic. Since $\overline{B}$ cannot be periodic, it follows that either $\overline{A}$ or $\overline{C}$ is periodic. If $\overline{C}$ is periodic, then $\overline{C}$ is a subgroup of $\overline{G}$. Thus $\overline{C} = \{e < w >, v < w >, v^2 < w >\}$ which implies that $s \in W = < w >$. Hence, $C = \{e, v, v^2w^{\gamma}\}$, where $0 \leq \gamma \leq 2$. If $\overline{A}$ is periodic, then its period is either $\overline{x} = x < w >$ or $\overline{s'} = s' < w >$, where $s' \in < u, v, x >$. Now $\overline{A}$ looks as follows :

$$\begin{array}{lll} ev^{\beta_0'}W & yu^{\alpha_1'}v^{\beta_1'}W & y^2u^{\alpha_2'}v^{\beta_2'}W \\ xv^{\beta_0+\beta_0'}W & xyu^{\alpha_1'}v^{\beta_1+\beta_1'}W & xy^2u^{\alpha_1'}v^{\beta_2+\beta_1'}W \\ x^2v^{2\beta_0+\beta_0'}W & x^2yu^{\alpha_1'}v^{2\beta_1+\beta_1'}W & x^2y^2u^{\alpha_1'}v^{2\beta_2+\beta_1'}W \end{array}$$

where $W = < w >$.

— Firstly, assume that $\overline{A}$ is periodic with period $s' < w > = s'W$. Say $s' = x^{\epsilon}u^{\eta}v^{\delta}$, where $0 \leq \epsilon, \eta, \delta \leq 2$. Now $x^{\epsilon}u^{\eta}v^{\delta}W = xv^{\beta_0}W$. This gives $\beta_0 = \delta$. Similarly, $xyu^{\alpha_1'}v^{\beta_1+\beta_1'}W = (x^{\epsilon}u^{\eta}v^{\delta}W)(\ yu^{\alpha_1'}v^{\beta_1'}W)$. This gives $\beta_1 = \delta$. Finally, $xy^2u^{\alpha_1'}v^{\beta_2+\beta_1'}W = (x^{\epsilon}u^{\eta}v^{\delta}W)(\ y^2u^{\alpha_2'}v^{\beta_2'}W)$. This gives $\beta_2 = \delta$. Therefore, we conclude that $\beta_0 = \beta_1 = \beta_2$.

— Secondly, assume that $\overline{A}$ is periodic with period $\overline{x} = x < w > = xW$. Then we must have $(xW) = (xv^{\beta_0+\beta_0'}W)$, which gives $\beta_0 = \delta$. Similarly, we get that $\beta_1 = \beta_2 = 0$. Summing up all the above, we conclude that if $\overline{A}$ is periodic, then $\beta_0 = \beta_1 = \beta_2$.

(6) Consider the case when $C = \{e, v, v^2w^{\gamma}\}$, where $0 \leq \gamma \leq 2$. If $\gamma = 0$, then C is a subgroup of G. So we assume that $\gamma \neq 0$. In this case, we may take $\gamma = 1$. The factorization $G = A < u > CD$ gives that one of $\overline{A}$, $\overline{C}$, or $\overline{D}$ is a periodic subset of the quotient

group $\overline{G} = G/ < u > = G/U$, say. Again, it is clear that $\overline{C}$ cannot periodic. Hence, either $\overline{A}$ or $\overline{D}$ is periodic. If $\overline{D}$ is periodic, then $\overline{D}$ $= \{< u >, w < u >, w^2 < u >\}$, which implies that $t \in < u >$. Hence, $D = \{e, w, w^2u^{\delta}\}$, where $0 \leq \delta \leq 2$. If $\overline{A}$ is periodic, then its period must be either $\overline{w} = x < u > = xU$ or $\overline{t'} = t' < u >= t'U$, where $t' \in$ $< x, v, w >$. Clearly $\overline{A}$ cannot be periodic with period $\overline{w} = x < u >$ $= xU$. Therefore it must be periodic with period $\overline{t'} = t' < u >= t'U$. Now $t' \in < x, v, w >$, say $t' = x^{\epsilon}v^{\eta}w^{\gamma}$, where $0 \leq \epsilon, \eta, \gamma \leq 2$. From this, it follows that $\beta_0 = \beta_1 = \beta_2$ and $\gamma_0 = \gamma_1 = \gamma_2$. Thus the elements of A are the following:

$$\begin{array}{lll}
e & yu^{\alpha_1'}v^{\beta_1'}w^{\gamma_1'} & y^2u^{\alpha_2'}v^{\beta_2'}w^{\gamma_2'} \\
xv^{\beta_0}w^{\gamma_0} & xyu^{\alpha_1'}v^{\beta_1+\beta_1'}w^{\gamma_1+\gamma_1'} & xy^2u^{\alpha_1'}v^{\beta_2+\beta_1'}w^{\gamma_2+\gamma_2'} \\
x^2v^{2\beta_0}w^{2\gamma_0} & x^2yu^{\alpha_1'}v^{2\beta_1+\beta_1'}w^{2\gamma_1+\gamma_1'} & x^2y^2u^{\alpha_1'}v^{2\beta_2+\beta_1'}w^{2\gamma_2+\gamma_2'}
\end{array}$$

Hence A is periodic with period $xv^{\beta_0}w^{\gamma_0}$.

Suppose $D = \{e, w, w^2u^{\delta}\}$, where $0 \leq \delta \leq 2$. Now if $\delta = 0$, then D is a subgroup of G and hence periodic. Otherwise, we may assume $\delta = 1$. But then $t = u$, and $s = w$. Now $xv^{\beta_i}w^{\gamma_i} \notin < v, s >$, and $xv^{\beta_i}w^{\gamma_i} \notin$ $< w, t >$, and so $xv^{\beta_i}w^{\gamma_i} \in < u, x >$, for each i, $0 \leq i \leq 2$. Therefore $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = \gamma_2$. Hence A is periodic with period x.

(7) The factorization $G = AB < v > D$ gives that one of $\overline{A}$, $\overline{B}$, or $\overline{D}$ is a periodic subset of the quotient group $\overline{G} = G/ < v > = G/V$, say. Again, $\overline{B}$ cannot be periodic. Therefore, either $\overline{A}$ or $\overline{D}$ is periodic. If $\overline{D}$ is periodic, then $D = \{e, w, w^2v^{\eta}\}$, where $0 \leq \eta \leq 2$. If $\overline{A}$ is periodic, then it is periodic with period either $\overline{x} = x < v >= xV$ or $\overline{t'} = t' < v >= t'V$, where $t' \in\ < x, u, w >$.

— Firstly, assume that $\overline{A}$ is periodic with period $\overline{t'}$. Now $t' \in\ <$ $x, u, w >$. It follows that $\gamma_0 = \gamma_1 = \gamma_2$.

— Secondly, assume that $\overline{A}$ is periodic with period $\overline{x}$. Then it follows that $\gamma_0 = \gamma_1 = \gamma_2 = 0$. Summing up our considerations so far, we see that if $\overline{A}$ is periodic, then $\gamma_0 = \gamma_1 = \gamma_2$.

(8) Consider the case, when $D = \{e, w, w^2v^{\eta}\}$, where $0 \geq \eta \leq 2$. If $\eta = 0$, then D is a subgroup. Otherwise, we may assume $\eta = 1$. From the factorization $G = A < u > CD$, we get that one of $\overline{A}$, $\overline{C}$, $\overline{D}$ is a periodic subset of the quotient group $\overline{G} = G/ < u >= G/U$, say. But $\overline{D}$ cannot be periodic. Hence either $\overline{A}$ or $\overline{C}$ is periodic. If $\overline{C}$ is periodic, then $C = \{e, v, v^2u^{\nu}\}$, where $0 \leq \nu \leq 2$. If $\overline{A}$ is periodic, then it is periodic with period $\overline{v} = v < u > = vU$ or $\overline{s'} = s' < u > = s'U$, where $s' \in\ < x, v, w >$. However, $\overline{A}$ cannot be periodic with period $\overline{v}$. So it must be periodic with period $\overline{s'}$. It follows that $\beta_0 = \beta_1 = \beta_2$ and

$\gamma_0 = \gamma_1 = \gamma_2$. Therefore, we conclude that A is periodic with period $xv^{\beta_0}w^{\gamma_0}$. Now, $C = \{e, v, v^2u^\nu\}$, where $0 \leq \nu \leq 2$. If $\nu = 0$, then C is a subgroup of G, and hence periodic. Otherwise, we may assume that $\nu = 1$. But, then $s = u$, and $t = v$. Now $xv^{\beta_i}w^{\gamma_i} \notin < v, s >$, and $xv^{\beta_i}w^{\gamma_i} \notin < w, t >$, and so $xv^{\beta_i}w^{\gamma_i} \in < u, x >$, for each $i, 0 \leq i \leq 2$. Therefore $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = \gamma_2 = 0$. Hence A is periodic with period x. Summing up (7) and (9) , we may assume $\gamma_0 = \gamma_1 = \gamma_2$.

(10) The case when $\beta_0 = \beta_1 = \beta_2$ and $\gamma_0 = \gamma_1 = \gamma_2$ are left uncovered yet. But in this case A is periodic with period $xv^{\beta_0}w^{\gamma_0}$.

Case (II)

(1) In this case, we may choose b, c, d, r, s to be u, v, w, x, y respectively. Now $B = \{e, u, u^2x\}$, $C = \{e, v, v^2y\}$ and $D = \{e, w, w^2t\}$. As in case (I), A is a complete set of representatives *modulo* $< u, v, w >$. Thus, here again A consists of the following elements:

$$\begin{array}{ccc} eu^{\alpha_{00}}v^{\beta_{00}}w^{\gamma_{00}} & yu^{\alpha_{01}}v^{\beta_{01}}w^{\gamma_{01}} & y^2u^{\alpha_{02}}v^{\beta_{02}}w^{\gamma_{02}} \\ xu^{\alpha_{10}}v^{\beta_{10}}w^{\gamma_{10}} & xyu^{\alpha_{11}}v^{\beta_{11}}w^{\gamma_{11}} & xy^2u^{\alpha_{12}}v^{\beta_{12}}w^{\gamma_{12}} \\ x^2u^{\alpha_{20}}v^{\beta_{20}}w^{\gamma_{20}} & x^2yu^{\alpha_{21}}v^{\beta_{21}}w^{\gamma_{21}} & x^2y^2u^{\alpha_{22}}v^{\beta_{22}}w^{\gamma_{22}} \end{array}$$

(2) From the factorization $G = AB < v >< w >$, we get that either $\overline{A}$ or $\overline{B}$ is periodic with period $\overline{x} = xH$, where $H = < v, w >$. As before, $\overline{B}$ cannot be periodic and so $\overline{A}$ is periodic with period $\overline{x} = xH$. From this it follows that:

$$\begin{array}{ccccc} \alpha_{00} = & \alpha_{10} = & \alpha_{20} = & \alpha_0 = & 0 \\ \alpha_{01} = & \alpha_{11} = & \alpha_{21} = & \alpha_1 = & 0 \\ \alpha_{02} = & \alpha_{12} = & \alpha_{22} = & \alpha_2 = & 0 \end{array}$$

(3) Similarly from the factorization $G = A < u > C < w >$, we get that either $\overline{A}$ or $\overline{C}$ is periodic with period $\overline{y} = yH$, where $H = < u, w >$. As before, $\overline{C}$ cannot be periodic, and so $\overline{A}$ is periodic with period $\overline{y} = yH$. From this it follows that:

$$\begin{array}{ccccc} \beta_{00} = & \beta_{10} = & \beta_{20} = & \beta_0 = & 0 \\ \beta_{01} = & \beta_{11} = & \beta_{21} = & \beta_1 = & 0 \\ \beta_{02} = & \beta_{12} = & \beta_{22} = & \beta_2 = & 0 \end{array}$$

Therefore, the elements of A are the following:

$$\begin{array}{ccc} eu^{\alpha_0}w^{\gamma_{00}} & yu^{\alpha_1}w^{\gamma_{01}} & y^2u^{\alpha_2}w^{\gamma_{02}} \\ xu^{\alpha_0}w^{\gamma_{10}} & x\ yu^{\alpha_1}w^{\gamma_{11}} & xy^2u^{\alpha_2}w^{\gamma_{12}} \\ x^2u^{\alpha_0}w^{\gamma_{20}} & x^2y^2u^{\alpha_1}w^{\gamma_{21}} & x^2y^2u^{\alpha_2}w^{\gamma_{22}} \end{array}$$

Here $\alpha_0 = \beta_0 = \gamma_0 = 0$, since A is normalized.

(4) Consider the factorization $G = ABC < w >$. This gives that one of $\overline{A}$, $\overline{B}$ or $\overline{D}$ is periodic with period $\overline{x} = x < w > = xW$ or

$\overline{y} = y < w > = yW$. If $\overline{D}$ is periodic with period $\overline{x}$, then $\beta_0 = \beta_1 = \beta_2 = 0$, and so v is missing from A. If $\overline{A}$ is periodic with period $\overline{y}$, then $\alpha_0 = \alpha_1 = \alpha_2 = 0$, and so u is missing from A.

(5) Assume v is missing from A. Consider the factorization $G = AB < v > D$. This gives that one of $\overline{A}$, $\overline{B}$ or $\overline{D}$ is periodic. Now, $\overline{B}$ cannot be periodic. This leaves the possibilities that either $\overline{A}$ is periodic with period $\overline{x} = x < v > = xV$ or $\overline{t'} = t' < v >= t'V$, where $t' \in \ < x, u, w >$ or that $\overline{D}$ is periodic which in turn implies that $D = \{e, w, w^2v^\eta\}$, where $0 \leq \eta \leq 2$. If $\overline{A}$ is periodic, then A is periodic since v is missing from A. For the case $D = \{e, w, w^2v^\eta\}$, we may assume as in the previous cases that $D = \{e, w, w^2v\}$. Consider the factorization $G = A < u > CD$. From this, we get that one of $\overline{A}$, $\overline{C}$ or $\overline{D}$ is periodic. Only $\overline{A}$ can be periodic. Now the only candidates for a period of $\overline{A}$ are $\overline{y}$ and $\overline{v}$. However, it is clear that $\overline{v}$ cannot be a period for $\overline{A}$. Hence $\overline{A}$ is periodic with period $\overline{y}$. From this, it follows that $\alpha_0 = \alpha_1 = \alpha_2 = 0$, and

$$\begin{array}{lllll}
\gamma_{00} = & \gamma_{10} = & \gamma_{20} = & \gamma_0 = & 0 \\
\gamma_{01} = & \gamma_{11} = & \beta_{21} = & \gamma_1 = & 0 \\
\gamma_{02} = & \gamma_{12} = & \gamma_{22} = & \gamma_2 = & 0
\end{array}$$

Therefore, in this case A is periodic with period y.

(6) Assuming u is missing from A, the factorization $G = A < u > CD$ gives that one of one of $\overline{A}$, $\overline{C}$, or $\overline{D}$ is periodic. It is clear that $\overline{C}$ cannot be periodic. So either $\overline{A}$ is periodic with period $\overline{y} = y < u > = yU$ or $\overline{t'} = t' < u >= t'U$ where $t' \in \ < x, v, w >$ or that $\overline{D}$ is periodic, which gives that $D = \{e, w, w^2u^\delta\}$, where $0 \leq \delta \leq 2$. If $\overline{A}$ is periodic, then A is periodic since u is missing from A. For the case $D = \{e, w, w^2u^\delta\}$, we may assume that $D = \{e, w, w^2u\}$. Consider the factorization $G = AB < v > D$. From this, we get that one of $\overline{A}$, $\overline{B}$, or $\overline{D}$ is is periodic. Only $\overline{A}$ can be periodic. The only candidates for a period of $\overline{A}$ are $\overline{x}$ and $\overline{u}$. However, it is clear that $\overline{u}$ cannot be a period for $\overline{A}$. Hence $\overline{A}$ is periodic with period $\overline{x}$. From this, it follows that $\beta_0 = \beta_1 = \beta_2 = 0$ and

$$\begin{array}{lllll}
\gamma_{00} = & \gamma_{10} = & \gamma_{20} = & \gamma_0 = & 0 \\
\gamma_{01} = & \gamma_{11} = & \beta_{21} = & \gamma_1 = & 0 \\
\gamma_{02} = & \gamma_{12} = & \gamma_{22} = & \gamma_2 = & 0
\end{array}$$

Therefore in this case, A is periodic with period x. $\square$

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